1 Cheeger differentiation

after J. Cheeger [1] and S. Keith [3]
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Abstract

We construct a measurable differentiable structure on any metric measure space that is doubling and satisfies a Poincaré inequality.

1.1 Introduction

A key result of geometric function theory is Rademacher's theorem: any real-valued Lipschitz function on $\mathbb{R}^n$ is differentiable almost everywhere. In [1], Cheeger found a deep generalization of this result in the context of doubling metric measure spaces that satisfy a Poincaré inequality. We will outline the construction of a measurable differentiable structure on such spaces, following both his work and clarifications and extensions due to Keith [3]. We refer to these papers for further discussion of the background and consequences of this work.

Recall that a function $f : (X, d) \to (Y, \rho)$ between metric spaces is *(C-)Lipschitz* if for all $x, y \in X$,

$$\rho(f(x), f(y)) \leq C d(x, y). \tag{1}$$

The infimal value of C that satisfies this condition is denoted by $\mathrm{LIP} f$. We define $\mathrm{LIP}(X)$ to be the vector space of all real-valued Lipschitz functions on X.

We will measure the infinitesimal behavior of a real-valued function near a point in two different ways.

Definition 1. *Let $f : X \to \mathbb{R}$ be a function on a metric space X. For $x \in X$, let*

$$(\mathrm{lip} f)(x) := \liminf_{r \to 0} \sup_{y \in B(x,r)} \frac{|f(y) - f(x)|}{r} \tag{2}$$

and

$$(\mathrm{Lip} f)(x) := \limsup_{r \to 0} \sup_{y \in B(x,r)} \frac{|f(y) - f(x)|}{r}. \tag{3}$$

A *metric measure space* (X, d, μ) consists of a set X with a metric d and a Borel measure μ. When f is a Lipschitz function on (X, d, μ) both $\mathrm{lip} f$ and $\mathrm{Lip} f$ are Borel [3, Lemma 4.1.2]. In fact, for each $x \in X$,

$$|f|_x = (\mathrm{Lip} f)(x) \tag{4}$$

defines a semi-norm on $\mathrm{LIP}(X)$ [3, Definition 4.2.1].

The following definitions will give us a framework to talk about the infinitesimal behaviour of a collection of functions on a metric measure space.

Definition 2. *A function $f : X \to \mathbb{R}$* vanishes to first order *at $x \in X$ if $f(x) = 0$ and $|f|_x = 0$. (Equivalently, $f(\cdot) = o(d(x, \cdot))$ near x.)*

Definition 3. *An N-tuple of functions $\mathbf{f} = (f_1, \ldots, f_N)$, where $f_i : X \to \mathbb{R}$ for $1 \le i \le N$, is* dependent to first order *at $x \in X$ if there exists $\lambda \in \mathbb{R}^n \backslash \{0\}$ so that $\lambda \cdot \mathbf{f}(\cdot) - \lambda \cdot \mathbf{f}(x)$ vanishes to first order.*

Let $S(\mathbf{f})$ be the set of all points in X where $\mathbf{f}$ is not dependent to first order.

Definition 4. *We say that the* differentials have uniformly bounded dimension *if there exists $N \in \mathbb{N}$ so that any N-tuple of Lipschitz functions $(f_1, \ldots, f_N)$ is dependent to first order almost everywhere.*

As we shall see in Propositions 11 and 12, if our metric measure space is doubling, locally compact and satisfies a Poincaré inequality, then the differentials have uniformly bounded dimension.

Rademacher's theorem essentially states that the infinitesimal behaviour of any Lipschitz function on $\mathbb{R}^n$ is well approximated at almost every point by some linear function; that is, a linear combination of the coordinate functions.

Definition 5. *Say that an N-tuple of real-valued functions on X, $\mathbf{x} = (x_1, \ldots, x_N)$,* spans the differentials almost everywhere *if for any Lipschitz function $f : X \to \mathbb{R}$, for almost every $x \in X$ there exists $df(x) \in \mathbb{R}^N$ so that*

$$|f(\cdot) - df(x) \cdot \mathbf{x}(\cdot)|_x = 0. \tag{5}$$

Then Rademacher's theorem may be stated as

Theorem 6 (Rademacher)**.** *Denote the coordinate functions on $\mathbb{R}^n$ by $x_i : \mathbb{R}^n \to \mathbb{R}$, for $1 \le i \le n$, and let $\mathbf{x} = (x_1, \ldots, x_n)$. Then $\mathbf{x}$ spans the differentials almost everywhere for $\mathbb{R}^n$.*

Cheeger constructed such 'coordinates' for certain metric measure spaces. In this case, we may well require different tuples of coordinates (possibly of different cardinalities) in different locations.

Definition 7 (Cheeger, Keith)**.** *A* strong measurable differentiable structure *on a metric measure space* (X, d, μ) *is a countable collection of pairs* $\{(X_\alpha, \mathbf{x}_\alpha)\}$*, called* coordinate patches*, that satisfy the following conditions.*

Firstly, each X_α *is measurable with positive measure and the union of every* X_α *equals* X*.*

Secondly, each $\mathbf{x}_\alpha$ *is a* $N(\alpha)$*-tuple of Lipschitz functions. There exists some* N *so that* $N(\alpha) \leq N$ *for all* α*. The smallest such* N *is called the* dimension *of the differentiable structure.*

Finally, for each α*,* $\mathbf{x}_\alpha$ *spans the differentials almost everywhere for* X_α*. Moreover, for each* $f \in \mathrm{LIP}(X)$*, equation* (5) *defines the measurable function* $df^\alpha : X_\alpha \to \mathbb{R}^{N(\alpha)}$ *uniquely up to sets of measure zero.*

Remark 8. *If* $N(\alpha) = 0$ *for some* α *then the strong measurable differentiable structure is* degenerate*. For such* α *interpret* $\mathbf{x}_\alpha$ *to be the empty function. Equation* (5) *then means that for every Lipschitz function* f *and almost every* $x \in X_\alpha$*,* $|f|_x = 0$*.*

When X *is quasi-convex (for example, it supports a Poincaré inequality), distance functions* $d(z, \cdot)$ *violate this condition, and so any strong measurable differentiable structure will be* non-degenerate*.*

A strong measurable differentiable structure leads to a finite dimensional (L^∞) cotangent bundle on X. (The uniqueness of df^α implies that any 'transition functions' between patches are suitable well behaved.)

1.2 Main theorem and structure of proof

As stated in the abstract, our main theorem is the following [3, Theorem 2.3.1], [1, Theorem 4.38].

Theorem 9. *If* (X, d, μ) *is a metric measure space that is doubling and supports a* p*-Poincaré inequality (with constant* $L \geq 1$*) for some* $p \geq 1$*, then* X *admits a strong measurable differentiable structure with dimension bounded above by a constant depending only on* L *and the doubling constant.*

Recall that a measure μ on a metric space (X, d) is *doubling* if there exists a constant $C > 0$ such that for every ball $B = B(x, r) \subset X$ we have

$\mu(2B) \leq C\mu(B)$, where $2B$ denotes the ball $B(x, 2r)$ with the same center as B and twice the radius.

For a function $f : X \to \mathbb{R}$ and set $U \subset X$ with $0 < \mu(U) < \infty$, we define

$$f_U := \fint_U f d\mu := \frac{1}{\mu(U)} \int_U f d\mu. \tag{6}$$

Definition 10. *Fix $p \geq 1$. A metric measure space (X, d, μ) admits a p-Poincaré inequality (with constant $L \geq 1$) if every ball in X has positive and finite measure, and for every $f \in \mathrm{LIP}(X)$ and every ball $B = B(x, r)$*

$$\fint_B |f - f_B| d\mu \leq Lr \left(\fint_{LB} (\mathrm{lip} f)(x)^p d\mu \right)^{1/p}. \tag{7}$$

This is a different, but equivalent, definition to the usual definition of a Poincaré inequality [1, (4.3)]. (Note that $\mathrm{lip} f$ is an upper gradient for f.) Essentially, a space satisfies a Poincaré inequality if there are "lots of rectifiable curves" in every location and on every scale.

We will sketch a proof of Theorem 9 following Keith [3]. The three key propositions are as follows. In each case, (X, d, μ) is a locally compact, doubling, metric measure space.

Proposition 11 (Prop. 4.3.1, [3])**.** *Suppose X admits a p-Poincaré inequality (with constant $L \geq 1$) for some $p \geq 1$. Then there exists $K > 0$, depending only on L and the doubling constant, so that:*

$$\forall f \in \mathrm{LIP}(X),\ (\mathrm{Lip} f)(x) \leq K (\mathrm{lip} f)(x) \text{ for } \mu\text{-a.e. } x \in X. \tag{8}$$

Cheeger showed the much deeper result that $K = 1$, that is, $\mathrm{lip} f = \mathrm{Lip} f$ almost everywhere [1, Theorem 6.1].

Proposition 12 (Prop. 7.2.2, [3])**.** *Fix $K > 0$. Suppose that the Lip-lip bound* (8) *holds. Then for every N and every N-tuple $\mathbf{f}$ of Lipschitz functions, the set $S(\mathbf{f})$ is measurable. If $\mu(S(\mathbf{f}))$ is positive, then $N \leq N_0$, where N_0 depends only on K and the doubling constant.*

In other words, the differentials have uniformly bounded dimension.

Proposition 13 (Proof of Prop. 7.3.1, [3])**.** *If the differentials have uniformly bounded dimension N_0, and if $S(\mathbf{f})$ is measurable for every tuple of Lipschitz functions $\mathbf{f}$, then X admits a strong measurable differentiable structure whose dimension is at most N_0.*

By breaking down the proof into these steps we lose some information. In another paper [2], Keith gets a stronger result using the Poincaré inequality throughout. He shows that there exists a strong measurable differential structure where every coordinate function x^i_α is equal to some distance function $d(z, \cdot)$.

1.3 Outline of proofs

We now outline the main steps in the proof of each proposition.

Proof of Proposition 13. We have N_0 fixed by the hypotheses.

It suffices to show that given any measurable $A \subset X$ with positive measure, we can find a measurable $V \subset A$ with positive measure so that for each $f \in \mathrm{LIP}(X)$, equation (5) defines the measurable function $df : V \to \mathbb{R}^N$ uniquely up to sets of measure zero, for some $N \leq N_0$. This is because (X, d, μ) is a σ-finite measure space, and so a short argument using Zorn's lemma gives the required countable decomposition.

Now consider the maximal N so that there exists some positive measure set $V \subset A$, and some N-tuple of Lipschitz functions $\mathbf{f}$, so that $V \subset S(\mathbf{f})$. By Proposition 12 we have $0 \leq N \leq N_0$.

Take any Lipschitz function $g \in \mathrm{LIP}(X)$, and consider the $(N+1)$-tuple of functions $(\mathbf{f}, g)$. By the maximality of N this is dependent to first order almost everywhere in V, and by the assumption on $\mathbf{f}$ we can find some function $dg : V \to \mathbb{R}^N$ so that $|g(\cdot) - dg(x) \cdot \mathbf{f}(\cdot)|_x = 0$ for μ-almost every x in V.

It is not difficult to show that dg will be measurable and unique up to sets of measure zero. $\square$

The most difficult of the three steps is to show that the differentials have uniformly bounded dimension; for reasons of space we will be somewhat sketchy.

Proof of Proposition 12. The *variation* of a function $f : X \to \mathbb{R}$ on a ball $B(x, r)$ is defined to be

$$\mathrm{var}_{(x,r)} f := \sup \left\{ \frac{|f(y) - f(x)|}{r} : y \in B(x, r) \right\}. \tag{9}$$

A function $f \in \mathrm{LIP}(X)$ is K-quasi-linear if the variation on every ball is at least $\frac{1}{K} \mathrm{LIP} f$.

The first step in the proof is to show that a vector space of K-quasi-linear functions on a doubling metric measure space has dimension bounded from above by a constant dependent only on the data [3, Theorem 6.1.2].

Secondly, it is shown that for every $f \in \mathrm{LIP}(X)$, for almost every $x \in X$ there exists some tangent space $(X_\infty, d_\infty, x_\infty)$ so that every tangent function f_∞ satisfies

$$\mathrm{lip} f(x) \leq \mathrm{var} f_\infty \leq \mathrm{LIP} f_\infty \leq \mathrm{Lip} f(x). \tag{10}$$

(The variation is over any ball in the tangent space.) Combined with the $(K-)$Lip-lip bound, we see that tangent functions are K-quasi-linear [3, Proposition 6.2.1].

Thirdly, for an N-tuple of Lipschitz functions $\mathbf{f}$ we immediately obtain, almost everywhere, the K-quasi-linearity of tangent functions $(\lambda \cdot \mathbf{f})_\infty$, where λ lies in a countable dense subset of $\mathbb{R}^N$.

Fourthly, we show that, by choosing appropriate a priori bounds, on a set of positive measure the set of tangent functions $(\lambda \cdot \mathbf{f})_\infty$, $\lambda \in \mathbb{R}^N$, is a N-dimensional vector space of K-quasi-linear functions.

The proof follows. □

Finally, we show the existence of a Lip-lip bound on spaces satisfying a Poincaré inequality. (I thank Bruce Kleiner for explaining this proof of this proposition to me, as well as the terminology of Definitions 2–4.)

Proof of Proposition 11. A doubling and complete metric space (X, d, μ) is proper: closed balls are compact. Semmes [1, Appendix] showed that if X satisfies a Poincaré inequality then it is *quasi-convex*, that is, any two points can be joined by a rectifiable path of comparable length.

Since $\mathrm{lip} f$ is Borel it is approximately continuous almost everywhere. Let x be a point of approximate continuity and fix $\lambda \in (0, 1)$.

There exists a constant C depending only on the doubling constant so that if $r > 0$ is sufficiently small, for any two intersecting λr-balls $B_1, B_2 \subset B(x, r)$ we have

$$\left| \fint_{B_1} f - \fint_{B_2} f \right| \leq C \lambda r (\mathrm{lip} f)(x). \tag{11}$$

This follows from the Poincaré inequality (7), doubling, the approximate continuity of f at x and the fact that $(\mathrm{lip} f)(z) \leq \mathrm{LIP} f$ for every $z \in X$.

Chaining together λr-balls from x to $y \in B(x, r)$, we get the desired conclusion up to an additive error term comparable to $\lambda(\mathrm{LIP} f)$. Since $\lambda \in (0, 1)$ was arbitrary, we are done. □

References

[1] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. **9** (1999), no. 3, 428–517.

[2] S. Keith, *Modulus and the Poincaré inequality on metric measure spaces*, Math. Z. **245** (2003), no. 2, 255–292.

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