EXISTENCE OF QUASI-ARCS

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ABSTRACT. We show that doubling, linearly connected metric spaces are quasi-arc connected. This gives a new and short proof of a theorem of Tukia.

1. Introduction

It is a standard topological fact that a complete metric space which is locally connected, connected and locally compact is arc-wise connected. Tukia [6] showed that an analogous geometric statement is true: if a complete metric space is linearly connected and doubling, then it is connected by quasi-arcs, quantitatively. In fact, he proved a stronger result: any arc in such a space may be approximated by a local quasi-arc in a uniform way. In this note we give a new and more direct proof of this fact.

This result is of interest in studying the quasisymmetric geometry of metric spaces. Such geometry arises in the study of the boundaries of hyperbolic groups; Tukia's result was used in this context by Bonk and Kleiner [1], and also by the author [5]. (Bonk and Kleiner use Assouad's embedding theorem to translate Tukia's result from its original context of subsets of $\mathbb{R}^n$ into our setting of doubling and linearly connected metric spaces.)

Before stating the theorem precisely, we recall some definitions. A metric space (X, d) is said to be *doubling* if there exists a constant N such that every ball can be covered by at most N balls of half the radius. Note that any complete, doubling metric space is proper: all closed balls are compact.

We say (X, d) is *L-linearly connected* for some $L \geq 1$ if for all $x, y \in X$ there exists a compact, connected set $J \ni x, y$ of diameter less than or equal to $Ld(x, y)$. (This is also known as bounded turning or LLC(1).) We can actually assume that J is an arc, at the cost of increasing L by an arbitrarily small amount. To see this, note that X is locally connected, and so the connected components of an open set are open. Thus, for any open neighborhood U of J, the connected component of U that contains J is an open set. We can replace J inside U by an arc with the same endpoints, since any open, connected subset of a locally compact, locally connected metric space is arc-wise connected [3, Corollary 32.36].

For any x and y in an embedded arc A, we denote by $A[x, y]$ the closed, possibly trivial, subarc of A that lies between them. We say that an arc A in a doubling and complete metric space is an *ϵ-local λ-quasi-arc* if $\operatorname{diam}(A[x, y]) \leq \lambda d(x, y)$ for all $x, y \in A$ such that $d(x, y) \leq \epsilon$. (This terminology is explained by Tukia and

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Väisälä's characterization of quasisymmetric images of the unit interval as those metric arcs that are doubling and bounded turning [7].)

One non-standard definition will be useful in our exposition. We say that an arc B *ϵ-follows* an arc A if there exists a coarse map $p : B \to A$, sending endpoints to endpoints, such that for all x, $y \in B$, $B[x, y]$ is in the ϵ-neighborhood of $A[p(x), p(y)]$; in particular, p displaces points at most ϵ. (We call the map p *coarse* to emphasize that it is not necessarily continuous.)

The condition that B ϵ-follows A is stronger than the condition that B is contained in the ϵ-neighborhood of A. It says that, coarsely, the arc B can be obtained from the arc A by cutting out 'loops.' (Of course, A contains no actual loops, but it may have subarcs of large diameter whose endpoints are 2ϵ-close.)

We can now state the stronger version of Tukia's theorem precisely, and as an immediate corollary our initial statement [6, Theorem 1B, Theorem 1A]:

Theorem 1.1 (Tukia)**.** *Suppose (X, d) is a L-linearly connected, N-doubling, complete metric space. For every arc A in X and every $\epsilon > 0$, there is an arc J that ϵ-follows A, has the same endpoints as A, and is an $\alpha\epsilon$-local λ-quasi-arc, where $\lambda = \lambda(L, N) \geq 1$ and $\alpha = \alpha(L, N) > 0$.*

Corollary 1.2 (Tukia)**.** *Every pair of points in a L-linearly connected, N-doubling, complete metric space is connected by a λ-quasi-arc, where $\lambda = \lambda(L, N) \geq 1$.*

Our strategy for proving Theorem 1.1 is straightforward: find a method of straightening an arc on a given scale (Proposition 2.1), then apply this result on a geometrically decreasing sequence of scales to get the desired local quasi-arc as a limiting object. The statement of this proposition and the resulting proof of the theorem essentially follow Tukia [6], but the proof of the proposition is new and much shorter. We include a complete proof for convenience to the reader.

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2. Main Results

Given any arc A and $\iota > 0$, the following proposition allows us to straighten A on a scale ι inside the ι-neighborhood of A.

Proposition 2.1. *Given a complete metric space X that is L-linearly connected and N-doubling, there exist constants $s = s(L, N) > 0$ and $S = S(L, N) > 0$ with the following property: for each $\iota > 0$ and each arc $A \subset X$, there exists an arc J that ι-follows A, has the same endpoints as A, and satisfies*

$$(*) \qquad \forall x, y \in J,\ d(x, y) < s\iota \implies \operatorname{diam}(J[x, y]) < S\iota.$$

We will apply this proposition on a decreasing sequence of scales to get a local quasi-arc in the limit. The key step in proving this is given by the following lemma.

Lemma 2.2. *Suppose (X, d) is a L-linearly connected, N-doubling, complete metric space, and let s, S, ϵ and δ be fixed positive constants satisfying $\delta \leq \min\{\frac{s}{4+2S}, \frac{1}{10}\}$. Now, if we have a sequence of arcs $J_1, J_2, \ldots, J_n, \ldots$ in X, such that for every $n \geq 1$*

- *J_{n+1} $\epsilon\delta^n$-follows J_n, and*
- *J_{n+1} satisfies $(*)$ with $\iota = \epsilon\delta^n$ and s, S as fixed above,*

then the Hausdorff limit $J = \lim_{\mathcal{H}} J_n$ exists, and is an $\epsilon\delta^2$-local $\frac{4S+3\delta}{\delta^2}$-quasi-arc.

Moreover, the endpoints of J_n converge to the endpoints of J, and J ϵ-follows J_1.

We shall need some standard definitions. The (infimal) distance between two subsets $U, V \subset X$ is defined as $d(U,V) = \inf\{d(u,v) : u \in U, v \in V\}$. If $U = \{u\}$, then we set $d(u,V) = d(U,V)$.

The r-neighborhood of U is the set $N(U,r) = \{x : d(x,U) < r\}$, and the Hausdorff distance between U and V, $d_{\mathcal{H}}(U,V)$, is defined to be the infimal r such that $U \subset N(V,r)$ and $V \subset N(U,r)$. For more information, see [2, Chapter 7].

We will now prove Theorem 1.1.

Proof of Theorem 1.1. Let s and S be given by Proposition 2.1, and set $\delta = \min\{\frac{s}{4+2S}, \frac{1}{10}\}$.

Let $J_1 = A$ and apply Proposition 2.1 to J_1 and $\iota = \epsilon\delta$ to get an arc J_2 that $\epsilon\delta$-follows J_1. Repeat, applying the lemma to J_n and $\iota = \epsilon\delta^n$, to get a sequence of arcs J_n, where each J_{n+1} $\epsilon\delta^n$-follows J_n, and satisfies $(*)$ with $\iota = \epsilon\delta^n$.

We can now apply Lemma 2.2 to find an $\alpha\epsilon$-local λ-quasi-arc J that ϵ-follows A, where $\alpha = \delta^2$ and $\lambda = \frac{4S+3\delta}{\delta^2}$. Every J_n has the same endpoints as A, so J will also have the same endpoints. □

The proof of Lemma 2.2 relies on some fairly straightforward estimates and a classical characterization of an arc.

Proof of Lemma 2.2. For every $n \geq 1$, J_{n+1} $\epsilon\delta^n$-follows J_n. We denote the associated coarse map by $p_{n+1} : J_{n+1} \to J_n$.

In the following, we will make frequent use of the inequality $\sum_{n=0}^{\infty} \delta^n < \frac{11}{9}$.

We begin by showing that the Hausdorff limit $J = \lim_{\mathcal{H}} J_n$ exists. The collection of all compact subsets of a compact metric space, given the Hausdorff metric, is itself a compact metric space [2, Theorem 7.3.8]. Since $\{J_n\}$ is a sequence of compact sets in a bounded region of a proper metric space, to show that the sequence converges with respect to the Hausdorff metric, it suffices to show that the sequence is Cauchy.

One bound follows by construction: $J_{n+m} \subset N(J_n, \frac{11}{9}\epsilon\delta^n)$ for all $m \geq 0$. For the second bound, take J_{n+m} and split it into subarcs of diameter at most $\epsilon\delta^n$, and write this as $J_{n+m} = J_{n+m}[z_0, z_1] \cup \cdots \cup J_{n+m}[z_{k-1}, z_k]$ for some $z_0, \ldots, z_k$ and some $k > 0$. Our coarse maps compose to give $p : J_{n+m} \to J_n$, showing that J_{n+m} $\frac{11}{9}\epsilon\delta^n$-follows J_n. Furthermore, since $d(z_i, z_{i+1}) \leq \epsilon\delta^n$, we have $d(p(z_i), p(z_{i+1})) \leq 4\epsilon\delta^n \leq s\epsilon\delta^{n-1}$. Combining this with the fact that p maps endpoints to endpoints, for $n \geq 2$ we have

$$\begin{aligned} J_n &= J_n[p(z_0), p(z_1)] \cup \cdots \cup J_n[p(z_{k-1}), p(z_k)] \subset N(\{p(z_0), \ldots, p(z_k)\}, S\epsilon\delta^{n-1}) \\ &\subset N\left(J_{n+m}, \frac{11}{9}\epsilon\delta^n + S\epsilon\delta^{n-1}\right). \end{aligned}$$

Taken together, these bounds give $d_{\mathcal{H}}(J_n, J_{n+m}) \leq \frac{11}{9}\epsilon\delta^n + S\epsilon\delta^{n-1}$, so $\{J_n\}$ is Cauchy and the limit $J = \lim_{\mathcal{H}} J_n$ exists. Moreover, J is compact (by definition) and connected (because each J_n is connected).

Now we let a_n, b_n denote the endpoints of J_n. Since $p_n(a_n) = a_{n-1}$, and p_n displaces points at most $\epsilon\delta^n$, the sequence $\{a_n\}$ is Cauchy and hence converges to some point $a \in J$. Similarly, $\{b_n\}$ converges to a point $b \in J$.

There are two cases to consider. If $a = b$, then $d(a_n, b_n) \le 2\frac{11}{9}\epsilon\delta^n \le s\epsilon\delta^{n-1}$. Consequently, $\mathrm{diam}(J_n) \le S\epsilon\delta^{n-1}$, $J = \lim_{\mathcal{H}} J_n$ has diameter zero, and thus $J = \{a\}$. Otherwise, $a \ne b$ and so J is non-trivial. We claim that in this case J is a local quasi-arc.

To show J is an arc with endpoints a and b it suffices to demonstrate that every point $x \in J \setminus \{a, b\}$ is a cut point [4, Theorems 2-18 and 2-27]. The topology of J_n induces an order on J_n with least element a_n and greatest b_n. Given $x \in J$, we define three points $h_n(x)$, x_n and $t_n(x)$ that satisfy $a_n < h_n(x) < x_n < t_n(x) < b_n$, where x_n is chosen such that $d(x, x_n) \le \frac{11}{9}\epsilon\delta^n$, and $h_n(x)$ and $t_n(x)$ are the first and last elements of J_n at distance $(S+1)\epsilon\delta^{n-1}$ from x. As long as x is not equal to a or b, for n greater than some n_0 these points will exist and this definition will be valid.

We shall denote the $\frac{11}{9}\epsilon\delta^n$-neighborhoods of $J_n[a_n, h_n(x)]$ and $J_n[t_n(x), b_n]$ by $H_n(x)$ and $T_n(x)$ respectively, and define $H(x) = \cup\{H_n(x) : n \ge n_0\}$ (the Head) and $T(x) = \cup\{T_n(x) : n \ge n_0\}$ (the Tail). By definition, $H(x)$ and $T(x)$ are open. We claim that, in addition, they are disjoint and cover $J \setminus \{x\}$, and so x is a cut point.

Fix $y \in J$, and suppose $y \notin H(x) \cup T(x)$. We want to show that $y = x$. To this end, we bound the diameter of $J_n[h_n(x), t_n(x)]$ using J_{n-1}. Because $d(p_n(h_n(x)), p_n(t_n(x))) \le 2\epsilon\delta^{n-1} + 2(S+1)\epsilon\delta^{n-1} \le s\epsilon\delta^{n-2}$, we know that the diameter of $J_{n-1}[p_n(h_n(x)), p_n(t_n(x))]$ must be less than $S\epsilon\delta^{n-2}$. Thus the diameter of $J_n[h_n(x), t_n(x)]$ is less than $S\epsilon\delta^{n-2} + 2\epsilon\delta^{n-1}$, as J_n $\epsilon\delta^{n-1}$-follows J_{n-1}.

For every $n \ge n_0$, y is $\frac{11}{9}\epsilon\delta^n$ close to some $y_n \in J_n$. Since $y \notin H(x) \cup T(x)$, y_n must lie in $J_n[h_n(x), t_n(x)]$, so

$$\begin{aligned} d(x,y) &\le d(x, J_n[h_n(x), t_n(x)]) + \mathrm{diam}(J_n[h_n(x), t_n(x)]) + d(y_n, y) \\ &\le 2\frac{11}{9}\epsilon\delta^n + (S+2\delta)\epsilon\delta^{n-2} = \left(2\frac{11}{9}\delta^2 + S + 2\delta\right)\epsilon\delta^{n-2}, \end{aligned}$$

therefore $d(x, y) = 0$ and $J \setminus (H(x) \cup T(x)) = \{x\}$.

We now show that $H(x)$ and $T(x)$ are disjoint. If not, then $H_n(x) \cap T_m(x) \ne \emptyset$ for some n and m. It suffices to assume $n \le m$. Now $T_m(x) \subset N(J_m[x_m, b_m], \frac{11}{9}\epsilon\delta^m)$ by definition. We send J_m to J_n using $f = p_{n+1} \circ \cdots \circ p_m : J_m \to J_n$, to get that $T_m(x) \subset N(J_n[f(x_m), b_n], 3\epsilon\delta^n)$. Since

$$d(f(x_m), x_n) \le d(f(x_m), x_m) + d(x_m, x) + d(x, x_n) < 4\epsilon\delta^n < s\epsilon\delta^{n-1}$$

we have, even for $n = m$,

$$T_m(x) \subset N(J_n[x_n, b_n], 3\epsilon\delta^n) \cup B(x_n, (S+3\delta)\epsilon\delta^{n-1}).$$

Since $(S+3\delta)\epsilon\delta^{n-1} + \frac{11}{9}\epsilon\delta^n < (S+\frac{1}{2})\epsilon\delta^{n-1}$, $H_n(x)$ cannot meet $T_m(x)$ in the ball $B(x_n, (S+3\delta)\epsilon\delta^{n-1})$. Thus $H_n(x) \cap T_m(x) \ne \emptyset$ implies that there exist points p and q in J_n such that $a_n \le p \le h_n(x) < x_n \le q \le b_n$ and $d(p, q) < 3\epsilon\delta^n < s\epsilon\delta^{n-1}$. But then we know that $J_n[p, q]$ has diameter less than $S\epsilon\delta^{n-1}$, while containing both $h_n(x)$ and x_n. This contradicts the definition of $h_n(x)$, so $H(x) \cap T(x) = \emptyset$.

We have shown that J is an arc with endpoints a and b; it remains to show that J is a local quasi-arc with the required constants.

Say we are given x and y in J, with x_n and y_n as before. Our argument shows that the segments $J_n[x_n, y_n]$ converge to some arc $\tilde{J}[x, y]$, because $J_{n+1}[x_{n+1}, y_{n+1}]$ $(\epsilon\delta^n + S\epsilon\delta^{n-1})$-follows $J_n[x_n, y_n]$ for all $n \ge 2$. This arc $\tilde{J}[x, y]$ must lie in J, therefore $\tilde{J}[x, y]$ must equal $J[x, y]$. Now, suppose that $d(x, y) \in (\epsilon\delta^{n+1}, \epsilon\delta^n]$ holds for some $n \geq 2$. Then $d(x_n, y_n) \leq 3\epsilon\delta^n + \epsilon\delta^n < s\epsilon\delta^{n-1}$, and so the subarc $J[x, y]$, which lies in $N(J_n[x_n, y_n], \frac{11}{9}\epsilon(\delta^n + S\delta^{n-1}))$, has diameter less than $S\epsilon\delta^{n-1} + 3\epsilon(\delta^n + S\delta^{n-1}) \leq \frac{4S+3\delta}{\delta^2} d(x, y)$, as desired.

Furthermore, this same argument gives that, for all $n \geq 2$, J $\frac{11}{9}\epsilon(\delta^n + S\delta^{n-1})$-follows J_n, which itself $\frac{11}{9}\epsilon\delta$-follows $J_1 = A$. Taking n sufficiently large, we have that J ϵ-follows A. □

3. Discrete paths and the proof of Proposition 2.1

The proof of Proposition 2.1 is based on a quantitative version of a simple geometric result. Before we state this result, recall that a maximal r-separated set $\mathcal{N}$ is a subset of X such that for all distinct x, $y \in \mathcal{N}$ we have $d(x, y) \geq r$, and for all $z \in X$ there exists some $x \in \mathcal{N}$ with $d(z, x) < r$.

Now suppose that we are given a maximal r-separated set $\mathcal{N}$ in an L-linearly connected, N-doubling, complete metric space X. Then we can find a collection of sets $\{V_x\}_{x\in\mathcal{N}}$ so that each V_x is a connected union of finitely many arcs in X, and for all x, $y \in \mathcal{N}$:

(1) $d(x, y) \leq 2r \implies y \in V_x$.
(2) $\mathrm{diam}(V_x) \leq 5Lr$.
(3) $V_x \cap V_y = \emptyset \implies d(V_x, V_y) > 0$.

For $x \in \mathcal{N}$, we can construct each V_x by defining it to be the union of finitely many arcs joining x to each $y \in \mathcal{N}$ with $d(x, y) \leq 2r$. By linear connectedness, we can ensure that $\mathrm{diam}(V_x) \leq 4Lr$. Condition (3) is trivially satisfied for compact subsets of a metric space, but we will strengthen it to the following:

(3′) $V_x \cap V_y = \emptyset \implies d(V_x, V_y) > \delta r$.

Lemma 3.1. *We can construct the sets V_x satisfying (1), (2) and (3′) for $\delta = \delta(L, N)$.*

Proof. Without loss of generality, we can rescale the metric to set $r = 1$.

Since X is doubling, the maximum number of 1-separated points in a $20L$-ball is bounded by a constant $M = M(20L, N)$. We can label every point of $\mathcal{N}$ with an integer between 1 and M, such that no two points have the same label if they are separated by a distance less than $20L$.

To find this labelling, consider the collection of all such labellings on subsets of $\mathcal{N}$ under the natural partial order. A Zorn's Lemma argument gives the existence of a maximal element: our desired labelling. So $\mathcal{N}$ is the disjoint union of $20L$-separated sets $\mathcal{N}_1, \ldots, \mathcal{N}_M$.

Now let $\mathcal{N}_{\leq n} = \cup_{k=1}^n \mathcal{N}_k$, and consider the following

Claim $\Delta(n)$**.** *We can find V_x for all $x \in \mathcal{N}_{\leq n}$, such that for all $x, y \in \mathcal{N}_{\leq n}$ (1), (2) and (3′) are satisfied with $\delta = \frac{1}{2}(5L)^{-n}$.*

$\Delta(0)$ holds trivially, and Lemma 3.1 immediately follows from $\Delta(M)$, with $\delta = \delta(L, N) = \frac{1}{2}(5L)^{-M}$. So we are done, modulo the statement that $\Delta(n) \implies \Delta(n + 1)$ for $n < M$. □

Proof that $\Delta(n) \implies \Delta(n + 1)$, for $n < M$. By $\Delta(n)$, we have sets V_x for all x in $\mathcal{N}_{\leq n}$.

As $\mathcal{N}_{n+1}$ is $20L$-separated we can treat the constructions of V_x for each $x \in \mathcal{N}_{n+1}$ independently. We begin by creating a set $V_x^{(0)}$ that is the union of finitely many arcs joining x to its 2-neighbors in $\mathcal{N}$. We can ensure that $\operatorname{diam}(V_x^{(0)}) \leq 4L$.

Now construct $V_x^{(i)}$ inductively, for $1 \leq i \leq n$. $V_x^{(i-1)}$ can be $5L$-close to at most one $y \in \mathcal{N}_i$. If $d(V_x^{(i-1)}, V_y) \in (0, \frac{1}{2}(5L)^{-i})$, then define $V_x^{(i)}$ by adding to $V_x^{(i-1)}$ an arc of diameter at most $L(5L)^{-i}$ joining $V_x^{(i)}$ to V_y. Otherwise, let $V_x^{(i)} = V_x^{(i-1)}$. Continue until $i = n$ and set $V_x = V_x^{(n)}$.

Note that V_x satisfies (1) and (2) by construction. The only non-trivial case we need to check for ($3'$) is whether $d(V_x, V_y) \in (0, \frac{1}{2}(5L)^{-n})$ for some $y \in \mathcal{N}_i$, $i < n$. (The $i = n$ case follows from the last step of the construction.) Then, since $V_x = V_x^{(n)} \supset V_x^{(i)}$, $V_x^{(i)} \cap V_y \neq \emptyset$, and $d(V_x^{(i)}, V_y) \geq \frac{1}{2}(5L)^{-i}$. The construction then implies that

$$\begin{aligned} d(V_x, V_y) &\geq \frac{1}{2}(5L)^{-i}(1 - (2L)(5L)^{-1} - (2L)(5L)^{-2} - \cdots - (2L)(5L)^{-(n-i)}) \\ &> \frac{1}{2}(5L)^{-n}(5L)\left(1 - \frac{2/5}{1-(1/(5L))}\right) \geq \frac{5}{2}\left(\frac{1}{2}(5L)^{-n}\right), \end{aligned}$$

contradicting our assumption, so $\Delta(n+1)$ holds. □

We now finish by using this construction to prove our proposition.

Proof of Proposition 2.1. By rescaling the metric, we may assume that $\iota = 20L$. If $d(a,b) \leq 20 = \frac{\iota}{L}$, then join a to b by an arc of diameter less than ι. This arc will, trivially, satisfy our conclusion for any $S \geq 1$.

Otherwise, $d(a,b) > 20$. In the hypotheses for Lemma 3.1, let $r = 1$ and let $\mathcal{N}$ be a maximal 1-separated set in X that contains both a and b. Now apply Lemma 3.1 to get $\{V_x\}_{x \in \mathcal{N}}$ satisfying (1), (2) and ($3'$) for $\delta = \delta(L, N) > 0$.

We want to 'discretize' A by finding a corresponding sequence of points in $\mathcal{N}$. Now, every open ball in X meets the arc A in a collection of disjoint, relatively open intervals. Since $\mathcal{N}$ is a maximal 1-separated set, the collection of open balls $\{B(x,1) : x \in \mathcal{N}\}$ covers X; in particular, it covers A. By the compactness of A, we can find a finite cover of A by connected, relatively open intervals, each lying in some $B(x,1)$, $x \in \mathcal{N}$.

Using this finite cover, we can find points $x_i \in \mathcal{N}$ and $y_i \in A$ for $0 \leq i \leq n$, such that $a = y_0 < \cdots < y_n = b$ in the order on A, and $A[y_i, y_{i+1}] \subset B(x_i, 1)$ for each $0 \leq i < n$. Since a, $b \in \mathcal{N}$, we have that $x_0 = a$ and $x_n = b$. The sequence $(x_0, \ldots, x_n)$ is a discrete path in $\mathcal{N}$ that corresponds naturally to A.

We now find a subsequence (x_{r_j}) of (x_i) such that the corresponding sequence of sets $(V_{x_{r_j}})$ forms a 'path' without repeats. Let $r_0 = 0$, and for $j \in \mathbb{N}^+$ define r_j inductively as $r_j = \max\{k : V_{x_k} \cap V_{x_{r_{j-1}}} \neq \emptyset\}$, until $r_m = n$ for some $m \leq n$. The integer r_j is well defined since $d(y_{(r_{j-1}+1)}, x_k) \leq 1$ for $k = r_{j-1}$ and $k = r_{j-1}+1$, so $V_{x_{(r_{j-1}+1)}} \cap V_{x_{r_{j-1}}} \neq \emptyset$. Note that if $i+2 \leq j$ then $V_{x_{r_i}} \cap V_{x_{r_j}} = \emptyset$, and thus $d(V_{x_{r_i}}, V_{x_{r_j}}) > \delta$.

Let us construct our arc J in segments. First, let $z_0 = x_{r_0}$. Second, for each i from 0 to $m-1$, let $J_i = J_i[z_i, z_{i+1}]$ be an arc in $V_{x_{r_i}}$ that joins $z_i \in V_{x_{r_i}}$ to some $z_{i+1} \in V_{x_{r_{i+1}}}$, where z_{i+1} is the first point of J_i to meet $V_{x_{r_{i+1}}}$. (In the case $i = m-1$, join z_{m-1} to $x_{r_m} = z_m$.) Set $J = J_0 \cup \cdots \cup J_m$.

This path J is an arc since each $J_i \subset V_{x_{r_i}}$ is an arc, and if there exists a point $p \in J_i \cap J_j$ for some $i < j$, then $j = i+1$ and $p = z_{i+1} = z_j$. This is true because $V_{x_{r_i}} \cap V_{x_{r_j}} \neq \emptyset$ implies that $j = i+1$, and the definition of z_{i+1} implies that $J_i \cap V_{x_{r_{i+1}}} = \{z_{i+1}\}$. Any finite sequence of arcs that meet only at consecutive endpoints is also an arc, so we have that J is an arc.

In fact, J satisfies $(*)$. For any y, $y' \in J$, $y < y'$, we can find $i \leq j$ such that $z_i \leq y < z_{i+1}$, $z_j \leq y' < z_{j+1}$. (If $y = z_m$, set $i = m$; likewise for y'.) If $d(y, y') \leq \delta$ then, because $y \in V_{x_{r_i}}$ and $y' \in V_{x_{r_j}}$, we have $d(V_{x_{r_i}}, V_{x_{r_j}}) \leq \delta$, so either $j = i$ or $j = i+1$. This gives that $J[y, y'] \subset V_{x_{r_i}} \cup V_{x_{r_j}}$, and so $\operatorname{diam}(J[y, y'])$ is bounded above by $10L$.

Furthermore, J ι-follows A. There is a coarse map $f : J \to A$ defined by the following composition: first map J to $\mathcal{N}$ by sending $y \in J[z_i, z_{i+1}) \subset J$ to $x_{r_i} \in \mathcal{N}$, and sending x_{r_m} to itself. Second, map each x_{r_i} to the corresponding y_{r_i} in A. Taking arbitrary $y < y'$ in J as before, we see that

$$\begin{aligned} J[y, y'] &\subset J[z_i, z_{j+1}] \subset N(\{x_{r_i}, \ldots, x_{r_j}\}, 5L) \subset N(\{y_{r_i}, \ldots, y_{r_j}\}, 5L+1) \\ &\subset N(A[y_{r_i}, y_{r_j}], 5L+1) \subset N(A[f(y), f(y')], \iota). \end{aligned}$$

We let $s = \frac{1}{20L}\delta$ and $S = \frac{1}{20L} 10L$, and have proven the Proposition. $\square$

Remark: This method of proof allows one to explicitly estimate the constants given in the statements of Theorem 1.1 and Corollary 1.2, but for most applications this is not necessary.

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