POORLY CONNECTED GROUPS

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Abstract. We investigate groups whose Cayley graphs have poorly connected subgraphs. We prove that a finitely generated group has bounded separation in the sense of Benjamini–Schramm–Timár if and only if it is virtually free. We then prove a gap theorem for connectivity of finitely presented groups, and prove that there is no comparable theorem for all finitely generated groups. Finally, we formulate a connectivity version of the conjecture that every group of type F with no Baumslag-Solitar subgroup is hyperbolic, and prove it for groups with at most quadratic Dehn function.

1. Introduction

When studying an infinite group through the geometry of its Cayley graphs, a natural question to ask is: If the Cayley graph is poorly connected, what does this imply about the structure of the group?

If we interpret this question as asking about disconnecting the Cayley graph by sets of finite diameter, we arrive at the theory of ends as explored by Freudenthal, Hopf, Stallings and others. However, we can also vary the question by instead asking about disconnecting the Cayley graph, or all its subgraphs, by sets of finite, or at least relatively small, *volume*.

The invariant we use to make this precise is the separation profile, which was introduced by Benjamini, Schramm and Timár [BST12] as a measurement of how hard it can be to cut subgraphs of X into components of at most half the size.

In this paper we study groups where the separation profile is small: we characterise those groups with bounded separation profile, find a gap theorem for finitely presented groups, and explore connections with Gromov hyperbolicity.

We begin by recalling the definition of the separation profile.

Definition 1.1. Fix $\varepsilon \in (0,1)$. A subset S of the vertex set $V\Gamma$ of a finite graph Γ is an ε-**cut set** of Γ, if $\Gamma - S$ has no connected component with more than $\varepsilon\,|\Gamma|$ vertices. The ε-**cut size** of Γ, $\mathrm{cut}^{\varepsilon}(\Gamma)$, is the minimal cardinality of an ε-cut set of Γ. The ε-**separation profile** of

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an infinite graph X is the function $\mathrm{sep}^{\varepsilon}_X : \mathbb{N} \to \mathbb{N}$ defined by

$$\mathrm{sep}^{\varepsilon}_X(n) = \max\left\{\mathrm{cut}^{\varepsilon}(\Gamma) \ \middle|\ \Gamma \subset X,\ |\Gamma| \leq n\right\}.$$

We consider separation profiles up to the equivalence $\simeq$ defined by $f \simeq g$ if $f \lesssim g$ and $g \lesssim f$, where $f \lesssim g$ if there exists a constant $C > 0$ such that $f(n) \leq Cg(Cn + C) + C$ for all n.

As an invariant, the separation profile enjoys the following robustness properties [BST12]:

(1) for all graphs X and all $\varepsilon, \varepsilon' \in (0, 1)$, $\mathrm{sep}^{\varepsilon}_X(n) \simeq \mathrm{sep}^{\varepsilon'}_X(n)$,
(2) if X, Y are bounded degree graphs and $f : X \to Y$ is a Lipschitz map such that $\sup_{y\in VY} |f^{-1}(y)| < \infty$, then $\mathrm{sep}_X \lesssim \mathrm{sep}_Y$.

We call a map f **regular** if it satisfies the two properties given in (2). Unless explicitly stated, we always assume $\varepsilon = \frac{1}{2}$.

In particular, the separation profile of a finitely generated group is independent of the choice of Cayley graph, and for any finitely generated subgroup H of a finitely generated group G we have $\mathrm{sep}_H \lesssim \mathrm{sep}_G$.

To give a flavour of potential separation profiles, we note that $\mathbb{Z}^d$ has $\mathrm{sep}_{\mathbb{Z}^d}(n) \simeq n^{(d-1)/d}$, cocompact Fuchsian groups have separation $\simeq \log n$, and (virtually) free groups have bounded separation profiles [BST12]; there are also examples of hyperbolic groups with separation $\simeq n^{\alpha}$ for a dense set of $\alpha \in (0, 1)$ [HMT18]. Separation profiles are always at most linear, the case where a graph has linear separation is completely explained in [Hum17]. The goal of this paper is to look at the other extreme.

First we observe that groups with bounded separation have a simple characterisation.

Theorem 1.2. *A vertex transitive, bounded degree, connected graph X has bounded separation if and only if X is quasi-isometric to a tree.*

In particular, a finitely generated group G has bounded separation if and only if G is virtually free.

This follows by combining work of Benjamini, Schramm and Timár with results of Kuske and Lohrey on graphs with "bounded treewidth", see Section 2. Note that the first claim of Theorem 1.2 fails for general bounded degree graphs: as observed in [BST12], the Sierpiński triangle graph has bounded separation but is not quasi-isometric to a tree.

Theorem 1.2 raises a natural question: if a group is not virtually free, how poorly connected can it be? In the case of finitely presented groups, we find a gap in the spectrum of possible separation profiles. We use the notation B_r for a closed ball of radius r in a metric space, or $B_r(x)$ if the centre x of the ball is important.

Theorem 1.3. *A finitely presented group G which is not virtually free satisfies*

$$\mathrm{sep}_G(n) \gtrsim \kappa_G(n),$$

*where κ_G is the **inverse growth function** of the Cayley graph of G:*

$$\kappa_G(n) = \max\{r \in \mathbb{N} \mid |B_r| \leq n\}.$$

In particular, if G is finitely presented, either

- $\mathrm{sep}_G(n) \simeq 1$ *and G is virtually free, or*
- $\mathrm{sep}_G(n) \gtrsim \log n$ *and G is not virtually free.*

The example of cocompact Fuchsian groups shows that the $\log n$ bound is sharp. In the special case that G is assumed to be hyperbolic (and hence finitely presented), Theorem 1.2 and the log-gap of Theorem 1.3 were shown by Benjamini–Schramm–Timár [BST12, Theorem 4.2].

Theorem 1.3 is proven in Section 3 by showing that in a one-ended finitely presented group it is always possible to connect annuli of bounded radius, implying that to cut a ball of radius r requires (at least) a set of size proportional to r. We then use in a crucial way the accessibility of finitely presented groups to extend the gap theorem from one-ended finitely presented groups to all finitely presented groups. Given this use of accessibility, one may wonder what the separation of an inaccessible group can be.

For finitely generated groups we show that there is no gap like that of Theorem 1.3 in the possible separation profiles.

Theorem 1.4. *Let $\rho : \mathbb{N} \to \mathbb{N}$ be an unbounded non-decreasing function. There is a finitely generated group G such that*

$$1 \not\simeq \mathrm{sep}_G(n) \quad \text{and} \quad \mathrm{sep}_G(n) \not\gtrsim \rho(n).$$

The groups we use are the elementary amenable lacunary hyperbolic groups constructed in [OOS09], and the key property we require of them is that they are not virtually free, but are limits of virtually free groups (see Section 4). We note that these are the first examples of amenable groups whose separation profile is not $n/\kappa(n)$ where κ is the inverse growth function.

Finally, we consider the following question, to which no counterexample is currently known.

Question 1.5. If G is a finitely presented group, and $\mathrm{sep}_G(n) = o(n^{1/2})$, then must G be hyperbolic?

As some weak evidence for this conjecture, note that such a G cannot contain a subgroup isomorphic to $\mathbb{Z}^2$ (with separation $\simeq n^{1/2}$) or more generally a Baumslag–Solitar group (which have separation $n^{1/2}$ or $n/\log n$ by Hume–Mackay–Tessera [HMT19]), and it is a well-known question whether such groups of type F must necessarily be hyperbolic.

Here we present a step towards a positive answer to Question 1.5.

Theorem 1.6. *Let G be a finitely presented group with (exactly) quadratic Dehn function. Then there is an infinite subset $I \subseteq \mathbb{N}$ such that $\mathrm{sep}_G(n) \gtrsim n^{1/2}$ for all $n \in I$.*

Thus, if a finitely presented group G has Dehn function $\lesssim n^2$, and separation function $o(n^{1/2})$, it must be hyperbolic.

The class of groups with at-most-quadratic Dehn function is rich, including: CAT(0) groups, automatic and more generally combable groups [ECH+92], and free-by-cyclic groups [BG10].

The main step of the proof of Theorem 1.6 is the following result, which may be of independent interest.

Proposition 5.1. *Let X be a connected graph. X is not hyperbolic if and only if X admits arbitrarily long* 18*-biLipschitz embedded cyclic subgraphs.*

To show this we use Papozoglou's criterion for hyperbolicity of graphs in terms of thin bigons [Pap95]. One may like to view this result in the context of shortcut graphs considered in [Hod18].

We construct grid-like families of paths in the Cayley graph of G crossing fillings of the cyclic subgraphs of Proposition 5.1; when G has quadratic Dehn function these paths give the desired $n^{1/2}$ lower bound on separation by an argument similar to that for balls in $\mathbb{Z}^2$.

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2. Bounded separation

In this section we characterise groups with bounded separation.

Theorem 1.2. *A vertex transitive, bounded degree, connected graph X has bounded separation if and only if X is quasi-isometric to a tree.*

In particular, a finitely generated group G has bounded separation if and only if G is virtually free.

Proof. Let X be a vertex transitive, bounded degree, connected graph. By [BST12, Lemma 2.3], if X has bounded separation then all finite subgraphs of X have uniformly "bounded treewidth". Thus by [BST12, Proof of Theorem 2.1] (see also [KL05, Theorem 3.3, Lemma 3.2]) X itself has "bounded strong treewidth", namely there is a tree T and a map $f : X \to T$ sending VX to VT so that if $x, y \in VX$ are adjacent then $f(x)$ and $f(y)$ are equal or adjacent in T, and moreover $\sup_{z \in VT} |f^{-1}(z)| < \infty$.

Using [KL05, Theorem 3.7], this map $f : X \to T$ can be chosen so that $\sup_{z \in VT} \operatorname{diam} f^{-1}(z) < \infty$. Therefore X satisfies Manning's "Bottleneck Property" and so is quasi-isometric to a tree [Man05, Theorem 4.6].

Conversely, if X is quasi-isometric to a tree it certainly has bounded separation since the separation profile of any tree is equal to 1.

Finally, a finitely generated group is quasi-isometric to a tree if and only if it is virtually free as a consequence of work of Stallings and Dunwoody, see e.g. [DK18, Theorem 20.45]. □

3. A gap between constant and logarithmic separation

As stated in the introduction, we claim the following gap theorem for separation.

Theorem 1.3. *A finitely presented group G which is not virtually free satisfies*

$$\operatorname{sep}_G(n) \gtrsim \kappa_G(n),$$

where κ_G is the inverse growth function $\kappa_G(n) = \max\{r \mid |B_r| \leq n\}$.

In particular, if G is finitely presented, either

- $\operatorname{sep}_G(n) \simeq 1$ *and G is virtually free, or*
- $\operatorname{sep}_G(n) \gtrsim \log n$ *and G is not virtually free.*

Proof of Theorem 1.3. We begin by using the accessibility of finitely presented groups to prove the theorem, assuming that it is true in the case G is one-ended.

The group G is accessible so can be written as a graph of groups, where each edge group is finite and each vertex group has at most one end [Dun85]. Each vertex group H is finitely presentable: recall that a group is finitely presentable if and only if it is coarsely simply connected (e.g. [DK18, Corollary 9.55]). It follows that as G is finitely presented and H is a vertex group in a splitting of G over finite edge groups, H is finitely presentable too. Also, each vertex group H is undistorted in G, so $\kappa_H(n) \gtrsim \kappa_G(n)$.

Now since G is not virtually free, some vertex group H must be one-ended, and by the discussion above it is finitely presentable and undistorted in G, so applying the result in the case of one-ended groups we have:

$$\operatorname{sep}_G(n) \gtrsim \operatorname{sep}_H(n) \gtrsim \kappa_H(n) \gtrsim \kappa_G(n).$$

Finally, as balls in G grow at most exponentially, $\kappa_G(n) \gtrsim \log n$.

It remains to show that $\operatorname{sep}_G(n) \gtrsim \kappa_G(n)$ when G is a finitely presented, one-ended group. This follows from the following proposition:

Proposition 3.1. *Let G be a one-ended, finitely presented group where all relations have length at most M, and let X be the corresponding Cayley graph. Then $\mathrm{cut}(B_r) \geq r/400M$, where B_r denotes the ball of radius r about the identity in X.*

We defer the proof of this proposition until later, but observe that for any n, if $r = \kappa_G(n)$ we have $|B_r| \leq n$, so for X the Cayley graph of G the proposition gives us:

$$\mathrm{sep}_X(n) \geq \mathrm{cut}(B_r) \geq \frac{r}{400M} \simeq \kappa_G(n). \qquad \square$$

Before proving the proposition, we give a lemma which allows us to avoid connected sets in X. We denote open and closed r-neighbourhoods of a set $V \subset X$, for $r \geq 0$, as $N(V,r) = \{z \in X : d(z,V) < r\}$ and $\overline{N}(V,r) = \{z \in X : d(z,V) \leq r\}$, respectively. We denote closed annuli around V as $\overline{A}(V,r,R) = \overline{N}(V,R) \setminus N(V,r)$ for $0 \leq r \leq R$.

Lemma 3.2. *Let X be the Cayley graph of a one-ended group, where all relations have length at most M. Let T be a bounded subset of X which is $8M$-coarsely connected, i.e. for any $x, y \in T$ there exists a chain of points $x = x_0, x_1, \ldots, x_n = y$ with each $d(x_i, x_{i+1}) \leq 8M$.*

Suppose we have points $x, y \in X$ with $d(x,T), d(y,T) = 4M$, and so that x and y can be connected to $X \setminus N(T, 8M + \frac{1}{2}\mathrm{diam}(T))$ inside $\overline{A}(T, 4M, 8M + \frac{1}{2}\mathrm{diam}(T))$ by paths γ_x and γ_y, respectively.

Then there exists a path joining x to y in $\overline{A}(T, M, 4M)$.

The proof of this lemma follows [MS11, Lemma 6.6] quite closely.

Proof. Let x', y' be the other endpoints of γ_x, γ_y, with $d(x',T), d(y',T) = 8M + \frac{1}{2}\mathrm{diam}(T)$.

As X is vertex transitive and bounded degree, there exists an infinite geodesic line $\alpha : \mathbb{R} \to X$ through x', with $\alpha(0) = x'$. We claim that either $\alpha|_{(-\infty,0]}$ or $\alpha|_{[0,\infty)}$ gives a geodesic ray from x' to infinity outside $N(T, 4M)$. If not, we have $z, z' \in \alpha$ on either side of x' with $d(z,T), d(z',T) < 4M$, so $d(z,z') < 8M + \mathrm{diam}(T)$. On the other hand, $d(z,z') = d(z,x') + d(x',z') \geq 2(d(x',T) - 4M)$, thus $d(x',T) < 8M + \frac{1}{2}\mathrm{diam}(T)$, a contradiction.

Now let α_x, α_y be the geodesic rays from x', y' which do not enter $N(T, 4M)$. Let x'', y'' be the last time these rays leave $N(T, 8M + \frac{1}{2}\mathrm{diam}(T))$. By one-endedness, we can join x'', y'' by a simple path β' outside $N(T, 8M + \frac{1}{2}\mathrm{diam}(T))$.

Let β_1 be the path outside $N(T, 4M)$ which starts at x, then follows γ_x to x', α_x to x'', β' to y'', α_y to y', γ_y to y. Remove cycles from β_1 to make it simple, keeping the same endpoints.

Let β_2 be a path inside $\overline{N}(T, 4M)$ which starts at x, then follows a geodesic of length $4M$ to T, then follows geodesics of length $\leq 8M$ from point to point in T, then follows a geodesic of length $4M$ to y. Again, remove cycles to make β_2 simple with the same endpoints. If having done so β_2 does not enter $N(T, M)$, then $\beta_2 \subset \overline{A}(T, M, 4M)$ serves as our desired path, so we may assume that $\beta_2 \cap N(T, M) \neq \emptyset$; let z be the last vertex of β_2 with $d(z, T) < M$.

Together, $\beta = \beta_1 \cup \beta_2$ give a cycle in X. To prove the Lemma it suffices to consider the case when $\beta_1 \cap \beta_2 = \{x, y\}$, i.e., this cycle is simple. Indeed, assume this case is known, and consider the situation when β_2, after leaving x, next meets β_1 at a point $\hat{x} \neq y$. Necessarily $d(\hat{x}, T) = 4M$, and the segments of β_1 and β_2 between x and $\hat{x}$ form a simple cycle, so we can find a path from x to $\hat{x}$ in $A(T, M, 4M)$. Replacing x by $\hat{x}$ we can then continue this argument, and in the end find a concatentated path from x to y in $A(T, M, 4M)$.

We continue considering the simple cycle β. Since β represents the identity in G, there is a van Kampen diagram D for β, that is, a contractible 2-complex D in the plane labelled by a combinatorial map φ from D into the Cayley 2-complex of G, so that the boundary ∂D of D maps to β. In this case, as β is simple, D is a topological disc.

Consider the function $f(\cdot) := d(\varphi(\cdot), T)$ defined on the 1-skeleton $D^{(1)}$ of D, which φ maps into X. On ∂D, we have $f(x) = f(y) = 4M$, and $f(z) < M$ for $z \in \beta_2 \cup \partial D$ given above. We consider ∂D as split into three subarcs, γ_{xz} between x and z, γ_{zy} between z and y, and γ_{yx} between y and x; note that $\varphi(\gamma_{yx}) = \beta_1$ and $\varphi(\gamma_{xz} \cup \gamma_{zy}) = \beta_2$. On $\gamma_{xz} \cup \gamma_{zy}$ we have $f \leq 4M$, and on γ_{yx} we have $f \geq 4M$.

Let $D' \subset D$ be the union of closed 2-cells $F \subset D$ which have a point $u \in F \cap D^{(1)}$ with $f(u) \geq 2M$.

Let D'' be the connected component of x in D'. Let $\partial_O D''$ be the outer boundary path of D'', considering D'' as a subcomplex of the plane (and ignoring any bounded regions it encloses). Every point p in $\partial_O D''$ satisfies $p \in \partial D$ or $f(p) < 2M$, or both. Moreover, every point $p \in \partial_O D''$ has $f(p) \geq 2M - M = M$, so $z \notin \partial_O D''$.

Consider the path that follows $\partial_O D''$ from x starting along γ_{xz} and continues until it first hits $\gamma_{zy} \cup \gamma_{yx}$ at some point p; as $\gamma_{yx} \subset \partial_O D''$ such a point exists. Since $p \neq z$, just before p the path is not in ∂D, thus has $f < 2M$, so by continuity $f(p) \leq 2M$. Therefore $p \in \gamma_{zy}$, and we can continue from p along γ_{zy} to y. Along this entire path $f \in [2M - M, 4M]$, i.e. we have found our path in $\overline{A}(T, M, 4M)$. □

We can now show that balls in one-ended groups are at least a little hard to cut.

Proof of Proposition 3.1. Let $S \subset B_r$ be given with $|S| < r/400M$. We will show that $B_r \setminus S$ must have a connected component of size $> |B_r|/2$.

Let $\sim$ be the equivalence class on S generated by requiring $p \sim q$ if $d(p, q) \leq 8M$. Let $S = S_1 \sqcup \cdots \sqcup S_k$ be the decomposition of S into equivalence classes. Let $V_1 = \overline{N}(S_1, 4M), \ldots, V_k = \overline{N}(S_k, 4M)$, and observe that S_i is $8M$–coarsely connected in V_i. Note too that for $i \neq j$, $V_i \cap V_j = \emptyset$.

Let $U_1, \dots, U_k$ be given by $U_i = N(S_i, 12M + \mathrm{diam}(S_i)))$. We claim that given p, q in B_r outside $\bigcup_i U_i$, we can join p to q in $B_r \setminus S$:

Consider the oriented path $\gamma_0 = [p, 1] \cup [1, q]$. We modify γ_0 by following along γ_0 and considering each V_i which it meets. Observe that every V_i which it meets lies in B_r, for, supposing $V_i \cap [1, p] \neq \emptyset$,

$$\begin{aligned} d(1, V_i) + \mathrm{diam}(V_i) &\leq d(1, p) - d(p, V_i) + \mathrm{diam}(V_i) \\ &\leq d(1, p) - (d(p, S_i) - 4M) + (\mathrm{diam}(S_i) + 8M) \\ &\leq d(1, p) \leq r. \end{aligned}$$

Suppose γ_0 first meets V_{i_1}. Using Lemma 3.2 applied to $T = S_{i_1}$, reroute γ_0 in $\overline{A}(S_i, M, 4M) \subset V_{i_1}$ from the first time x it reaches $\overline{N}(S_i, 4M)$ to the last time y it leaves $\overline{N}(S_i, 4M)$. Continue for the next V_{i_2} which it reaches, all the way until one reaches q, and call this new path γ_1, which has our desired property: since γ_1 avoids every S_i, it avoids S.

It remains to show that $\bigcup_i U_i$ is a small set. Each U_i lives in a ball of radius $r_i = 12M + 2\,\mathrm{diam}(S_i) \leq 12M + 16M|S_i|$. The total diameter of these balls is

$$\leq 2 \sum_i \left(12M + 16M|S_i|\right) \leq 24M|S| + 32M|S| < 56M \cdot \frac{r}{400M} \leq \frac{r}{6}.$$

Take a geodesic segment γ' in B_r from 1 of length r. We can lay out three disjoint copies of these balls along the segments $[0, r/6]$, $[r/3, r/2]$, $[2r/3, 5r/6]$, and so $|\bigcup_i U_i| \leq \frac{1}{3}|B_r|$. □

Remark 3.3. A variation of the proof of Theorem 1.3 shows that the bound

$$\mathrm{sep}_X(n) \gtrsim \kappa_X(n) := \max\{k \mid \exists x : |B_k(x)| \leq n\}$$

holds for any one-ended, vertex transitive graph X that is coarsely simply connected and of bounded degree. It is quite conceivable that the 'vertex transitive' and 'one-ended' assumptions can be weakened.

4. No gap for finitely generated groups

Here we prove that there cannot be a gap theorem near bounded separation for finitely generated, infinitely presented groups. The key ingredient is families of epimorphisms

$$\langle \alpha_0, t \mid \, \rangle \to G_1 \to G_2 \to \ldots \to G_n \to \ldots G$$

satisfying the following three properties:

- for each $i \in \mathbb{N}$, G_i is virtually free,
- G is not virtually free,

- having already fixed $\langle \alpha_0, t \mid \rangle \to \ldots \to G_n$ for any r we may choose G_{n+1} so that the homomorphism $G_n \to G_{n+1}$ is injective on balls of radius r measured with respect to the generating set (the image of) $\{\alpha_0, t\}$.

Such a construction appears in [OOS09, Lemma 3.24]. The elementary amenable groups constructed are denoted $G(p, \mathbf{c})$ where p is a prime and $\mathbf{c}$ is an infinite sequence of natural numbers which grows sufficiently quickly. The intermediate groups G_n are determined uniquely by p and the finite subsequence $(c_1, \ldots, c_n)$. We now show that within this collection of groups one can construct groups with unbounded but arbitrarily small separation profile.

Theorem 1.4. *Let $\rho : \mathbb{N} \to \mathbb{N}$ be an unbounded non-decreasing function. There is a sequence $\mathbf{c} = (c_i)_{i\in\mathbb{N}}$ such that $G(p, \mathbf{c})$, which is not virtually free, satisfies*

$$1 \not\simeq \mathrm{sep}_{G(p,\mathbf{c})}(n) \quad \textit{and} \quad \mathrm{sep}_{G(p,\mathbf{c})}(n) \not\gtrsim \rho(n).$$

Proof. We will build the desired group by constructing a sequence $\mathbf{c}$ which grows sufficiently quickly. The choice of prime will not matter in our construction. Throughout we consider groups as metric spaces with respect to the generating set $\{\alpha_0, t\}$ (strictly speaking, the image of $\{\alpha_0, t\}$ in each group $G_k, G(p, \mathbf{c})$).

Fix $c_1 = 1$. The corresponding group G_1 is virtually free, so $\mathrm{sep}_{G_1} \leq M_1$ for some constant M_1. Choose c_2 sufficiently large for the construction [OOS09, Lemma 3.24] and also large enough so that $G_1 \to G_2$ is injective on balls of radius $2l_1$ where $\rho(l_1) \geq M_1^2$.

For each $k \geq 2$ in turn, G_k is virtually free, so $\mathrm{sep}_{G_k} \leq M_k$ for some constant M_k. Choose $l_k > l_{k-1}$ so that $\rho(l_k) \geq M_k^2$, then choose c_{k+1} sufficiently large for the construction [OOS09, Lemma 3.24] and also large enough so that $G_k \to G_{k+1}$ is injective on balls of radius $2l_k$. We now bound $\mathrm{sep}_{G(p,\mathbf{c})}$.

Let Γ be a connected subgraph of $G(p, \mathbf{c})$ with at most l_k vertices, so it has diameter at most l_k. The map $G_k \to G(p, \mathbf{c})$ is injective on balls of radius $2l_k$ so Γ is a connected subgraph of G_k. Thus

$$\mathrm{sep}_{G(p,\mathbf{c})}(l_k) = \mathrm{sep}_{G_k}(l_k) \leq M_k \leq \rho(l_k)^{\frac{1}{2}}.$$

Hence $\mathrm{sep}_{G(p,\mathbf{c})}(n) \not\gtrsim \rho(n)$. The fact that $\mathrm{sep}_{G(p,\mathbf{c})}(n) \not\simeq 1$ is immediate from Theorem 1.2 because $G(p, \mathbf{c})$ is not finitely presentable, and therefore not virtually free. □

5. Small separation and hyperbolicity

In this section we show the following.

Theorem 1.6 *Let G be a finitely presented group with (exactly) quadratic Dehn function. Then there is an infinite subset $I \subseteq \mathbb{N}$ such that $\mathrm{sep}_G(n) \gtrsim n^{1/2}$ for all $n \in I$.*

Thus, if a finitely presented group G has Dehn function $\lesssim n^2$, and separation function $o(n^{1/2})$, it must be hyperbolic.

One of the main steps is the following result which may be of independent interest.

Proposition 5.1. *Let X be a connected graph. X is hyperbolic if and only if there is some N such that every 18-bi-Lipschitz embedded cyclic subgraph in X has length at most N.*

By an 18-bi-Lipschitz embedded cyclic subgraph of length N we mean a cycle α in X so that for any $x, y \in \alpha$, $\frac{1}{18} d_\alpha(x,y) \leq d_X(x,y) \leq d_\alpha(x,y)$, where d_α and d_X are the distances in α and X respectively.

Proof. Firstly, if there exist arbitrarily long 18-bi-Lipschitz embedded cyclic subgraphs in X then it is not hyperbolic, by the Morse Lemma. To complete the proof we will show that any non-hyperbolic graph contains arbitrarily large 18-biLipschitz embedded geodesic quadrilaterals.

We use Papazoglou's criterion for hyperbolicity of graphs, namely, a graph is hyperbolic if and only if every geodesic bigon is thin [Pap95, Theorem 1.4].

Assume X is not hyperbolic, so for every M there exist finite geodesics γ, γ' with common endpoints such that the Hausdorff distance between them equals some $n \geq M$. Fix k such that $d_X(\gamma(k), \gamma') = n$, swapping γ, γ' if necessary.

Choose l, l' infimal such that

$$\frac{l}{d_X(\gamma(k-l), \gamma')} \geq 2, \quad \frac{l'}{d_X(\gamma(k+l'), \gamma')} \geq 2. \tag{5.2}$$

Let γ_1 be the subarc of γ between $\gamma(k-l)$ and $\gamma(k+l')$. Let β_1 be a geodesic from $\gamma(k-l)$ to a closest point in γ', and let β_2 be a geodesic from $\gamma(k+l')$ to a closest point in γ'. Let γ_2 be the subarc of γ' between the endpoints of β_1 and β_2.

Since the Hausdorff distance between γ, γ' is n we have $l - \varepsilon < 2d_X(\gamma(k-l+\epsilon), \gamma') \leq 2n$ for all $\varepsilon > 0$ by (5.2), so $l \leq 2n$, and likewise $l' \leq 2n$. As $d_X(\gamma(k), \gamma') = n$ we have $l \geq 2n/3$, else a contradiction follows from

$$2d_X(\gamma(k-l), \gamma') > 2(n - 2n/3) > l;$$

likewise $l' \geq 2n/3$. As the lengths $|\beta_1|$, $|\beta_2|$ of β_1, β_2 satisfy $2|\beta_1| \leq l, 2|\beta_2| \leq l'$, we have

$$d_X(\beta_1, \beta_2) \geq l + l' - |\beta_1| - |\beta_2| \geq \frac{1}{2}(l + l') \geq \frac{2n}{3}.$$

Now we provide a lower bound on $d_X(\gamma_1, \gamma_2)$. For $a \in [0, l]$ we have $d_X(\gamma(k-a), \gamma') \geq d_X(\gamma_1(k), \gamma') - a = n - a$. On the other hand, by

(5.2) we have $d_X(\gamma(k-a),\gamma') \geq a/2$, so combining these cases with the similar calculation for $d_X(\gamma(k+a),\gamma')$, we find

$$d_X(\gamma_1,\gamma_2) \geq \min_a \max\{n-a, \frac{a}{2}\} = \frac{n}{3}.$$

Let α be the quadrilateral $\gamma_1,\beta_2,\gamma_2,\beta_1$ with distance d_α. As α has length at most $12n$, if x,y are in γ_1,γ_2, or in β_1,β_2, we have

$$d_X(x,y) \geq \frac{n}{3} \geq \frac{1}{18} d_\alpha(x,y).$$

Suppose now $x \in \beta_1$ and $y \in \gamma_1$; reparametrize β_1 and γ_1 so that $\beta_1(0) = \gamma_1(0)$, and fix a,b so that $x = \beta_1(a), y = \gamma_1(b)$, so $d_\alpha(x,y) = a+b$. If $b \geq l$ then $d(x,y) \geq l - |\beta_1| \geq l/2 \geq n/3$, so we have the lower bound as before. Thus we may assume $b < l$. Let $c = d_X(x,y)$. Suppose for a contradiction that $c < \frac{1}{8}(a+b)$. By (5.2) applied to y we have

$$l - b < 2d_X(\gamma(k-l+b),\gamma') \leq 2(d_X(y,x) + d_X(x,\gamma')) = 2(c + |\beta_1| - a).$$

As $2|\beta_1| \leq l$, we have $-b < 2c - 2a$, thus

$$2a < b + 2c < b + \frac{a+b}{4} \Rightarrow a < \frac{5}{7}b$$

so

$$c \geq b - a \geq \frac{2}{7}b = \frac{2b}{7b+7a}(a+b) > \frac{2b}{12b}(a+b) = \frac{1}{6}(a+b),$$

contradicting $c < \frac{1}{8}(a+b)$.

The final case is $x \in \beta_1$ and $y \in \gamma_2$. Parametrise β_1, γ_2 so that $\beta_1(0) = \gamma_2(0)$, $x = \beta_1(a), y = \gamma_2(b)$, and let $c = d_X(x,y)$. If $a \geq b/2$ then as β_1 is a shortest path to γ', $c \geq a = \frac{a}{3} + \frac{2a}{3} \geq \frac{1}{3}(a+b)$. If $a < b/2$, then by the triangle inequality $c \geq b - a = \frac{b}{3} + \frac{b/2}{3} + (\frac{b}{2} - a) \geq \frac{1}{3}(a+b)$. □

Remark 5.3. Since a geodesic metric space is hyperbolic if and only if every 3-biLipschitz geodesic is uniformly close to any geodesic with the same endpoints [CH17, Proposition 3.2], the above proof can easily be adapted to the setting of general geodesic metric spaces again producing N-biLipschitz embedded cycles in any non-hyperbolic space with N some universal constant.

Fix a triangular presentation of G and let X be the Cayley graph of G with respect to this presentation, where G satisfies the assumptions of Theorem 1.6.

By Proposition 5.1 there exists an unbounded set $J \subset \mathbb{N}$ so that we can find, for each $n \in J$, an 18-biLipschitz embedded cycle γ_n in X of length n. For $n \in J$, let D_n be a diagram with boundary γ_n with at most Cn^2 vertices. This can always be done: the Dehn function guarantees the existence of diagrams with quadratic area, and the number of vertices is at most the area times the length of the longest relator in the presentation as the boundary is an embedded cycle in X.

Let $p_n : D_n^{(1)} \to X$ be the map from the 1-skeleton of D_n into X which extends the inclusion of $\gamma_n = \partial D_n$ into X, preserving edge labels and directions. Set $\Gamma_n = p_n(D_n^{(1)})$.

Split γ_n into four subpaths $\gamma_n^1, \ldots, \gamma_n^4$ of equal length and consider the $\sqrt{2}$-Lipschitz map

$$\pi_n : p_n(D_n^{(1)}) \to \left[0, \frac{n}{72}\right]^2$$

given by

$$(\pi_n(x)_1, \pi_n(x)_2) = \left(\min\left\{\frac{n}{72}, d_X(x, \gamma_n^1)\right\}, \min\left\{\frac{n}{72}, d_X(x, \gamma_n^2)\right\}\right).$$

Note that since γ_n is 18-biLipschitz embedded in X, $\pi_n(x)_i = \frac{n}{72}$ for all $x \in \gamma_n^{i+2}$, where $i = 1, 2$. In what follows we fix an $n \in J$ and drop the subscript n's from the notation.

Lemma 5.4. *For each $k \in (0, \frac{n}{72} - 1) \cap 3\mathbb{Z}$ the set*

$$p^{-1}\pi^{-1}\left([k-1, k+1] \times \left[0, \frac{n}{72}\right]\right)$$

contains a path H'_k in $D^{(1)}$ connecting γ^2 to γ^4. Similarly, for each $l \in (0, \frac{n}{72} - 1) \cap 3\mathbb{Z}$

$$p^{-1}\pi^{-1}\left(\left[0, \frac{n}{72}\right] \times [l-1, l+1]\right)$$

contains a path V'_l in $D^{(1)}$ connecting γ^1 to γ^3.

Proof. Suppose no such H'_k exists. Then the induced subgraph E_k of $D^{(1)}$ containing $p^{-1}\pi^{-1}\left([k - \frac{1}{2}, k + \frac{1}{2}] \times [0, \frac{n}{72}]\right)$ contains no path joining γ^2 to γ^4. Thus by planarity, there is a path α in D which joins γ^1 to γ^3 in $D \setminus E_k$; we can assume α meets the interiors of 2–cells in D finitely many times. Any 2–cell F in D encloses a simply connected planar region (filling in any holes); let F_0 be the interior of that region. Each time α meets any such F_0, E_k can contain at most two vertices and one edge of ∂F, so we can replace the part of α in F_0 by a path in $\partial F \cap (D \setminus E_k)$. Doing this for each 2–cell F, we can assume that α lies in $D^{(1)}$.

The image $\pi(p(\alpha))$ is a path in $[0, \frac{n}{72}]^2$ joining points with first coordinate 0 to those with first coordinate $\frac{n}{72}$. Since $\pi \circ p$ is 1-Lipschitz in the first coordinate, there must be a vertex v of α with $\pi(p(v))_1 \in [k - \frac{1}{2}, k + \frac{1}{2}]$, thus $v \in E_k$, a contradiction.

The proof proceeds similarly for V'_l. □

Define $H_k = p(H'_k)$ for each $k \in (0, \frac{n}{72} - 1) \cap 3\mathbb{Z}$, and $V_l = p(V'_l)$ for each $l \in (0, \frac{n}{72} - 1) \cap 3\mathbb{Z}$. Since p is a combinatorial map, these are combinatorial paths in Γ.

Observe that by construction, if $k \neq k'$ then $H_k \cap H_{k'} = \emptyset$, since $\pi(H_k) \subset [0, \frac{n}{72}] \times [k-1, k+1]$ and similarly for k', and $[k-1, k+1] \cap [k'-1, k'+1] = \emptyset$. In addition, since the vertex path in H_k gives a sequence of points $\pi(H_k)$ in $[0, \frac{n}{72}]^2$ whose second coordinates jump by at most 1 each time, there must be at least $\frac{n}{72}$ vertices in H_k.

Lemma 5.5. *For every k, l, $H_k \cap V_l \neq \emptyset$.*

Proof. By planarity, H'_k and V'_l must intersect in $D^{(1)}$, so their images intersect in $p(D^{(1)}) = \Gamma$. □

Our constructed paths together are well-connected.

Lemma 5.6. *Suppose $0 < \delta \leq \frac{1}{2000}$ and suppose $n \in J$ satisfies $n \geq 1000$. Let S be a set of vertices in $\Gamma_n = p_n(D_n)$ containing at most δn vertices. Then there is a connected component of $\Gamma_n \setminus S$ containing at least $\frac{1}{144000}n^2$ vertices.*

Proof. Note that the set K_S of $k \in (0, \frac{n}{72} - 1) \cap 3\mathbb{Z}$ such that $H_k \cap S = \emptyset$ contains at least $\frac{n}{1000} - \delta n \geq \frac{n}{2000}$ elements. Likewise, for many choices of l we have $V_l \cap S = \emptyset$; fix one such l. Now the set

$$\bigcup_{k \in K_S} H_k \cup V_l$$

is disjoint from S (by construction), connected (by Lemma 5.5, since each $H_k \cup V_l$ is connected), and contains more than $\frac{1}{144000}n^2$ vertices, since the H_k are disjoint and each contains at least $\frac{n}{72}$ vertices. □

Proof of Theorem 1.6. We want to show that $\text{sep}_G(n) \gtrsim n^{1/2}$ for all n in an infinite set $I \subset \mathbb{N}$; that is, we want to find an infinite set $I \subset \mathbb{N}$ and a nondecreasing function $\rho : \mathbb{N} \to \mathbb{N}$ with $\rho(n) \geq n^{1/2}$ for all $n \in I$ and $\text{sep}_G(n) \gtrsim \rho(n)$. For each $n \in J$ as above, Γ_n has at most Cn^2 vertices. Set $\varepsilon = \frac{1}{144000C}$, then by Lemma 5.6,

$$\text{sep}^{\varepsilon}_X(Cn^2) \geq \frac{n}{2000} \quad \text{for all} \quad n \in J,$$

where $\text{sep}^{\varepsilon}_X$ is as defined in the introduction. If we now set $I = \{Cn^2 \mid n \in J, n \geq 1000\}$ and $\rho(n) = \max\{m^{1/2} \mid m \in I, m \leq n\}$, this gives

$$\text{sep}_X(n) \simeq \text{sep}^{\varepsilon}_X(n) \geq \frac{\rho(n)}{2000C^{\frac{1}{2}}}.$$ □

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