Complex Networks
Problem Sheet 7

$**$ **Please hand in solutions to questions 2 on this sheet** $**$

1. Let K_4 denote the complete graph on 4 vertices.

   (a) Draw K_4.

   (b) Compute exactly the expected number of copies of K_4 in $G(n, p)$, the Erdős-Rényi random graph on n vertices where each edge is present with probability p, independent of the others.

   (c) Compute the variance of the number of copies of K_4 in $G(n, p)$ from first principles. It is enough if your answer gets the correct scaling in n and p. You can ignore any constants that don't depend on n or p in your calculations. You can also ignore terms in n and p that grow more slowly than the dominant term.

   (d) Find an $\alpha_c \in (0, \infty)$ (or prove that none exists) such that the following is true:

$$\mathbb{P}(G(n, n^{-\alpha}) \text{ contains a copy of } K_4) \to \begin{cases} 0, & \text{if } \alpha > \alpha_c, \\ 1, & \text{if } \alpha < \alpha_c. \end{cases}$$

   Justify your answer fully.

2. Recall that $G = (V, E)$ is bipartite if there exist vertex sets X and Y such that $V = X \cup Y$, $X \cap Y = \emptyset$ and $E \subseteq X \times Y$. In words, X and Y partition the vertex set, and there is no edge between two elements of X or two elements of Y.

   The random bipartite graph $G(n, n, p)$ has $2n$ vertices which can be partitioned as $V = X \cup Y$, $X \cap Y = \emptyset$, with $|X| = |Y| = n$. Moreover, each edge in $X \times Y$ is present with probability p, independent of the others. There are no edges in $X \times X$ or $Y \times Y$.

   (a) Let $K_{2,2}$ denote the complete bipartite graph on 2+2 vertices. Draw $K_{2,2}$.

   (b) Compute exactly the expected number of copies of $K_{2,2}$ in $G(n, n, p)$.

   (c) Compute the variance of the number of copies of $K_{2,2}$ in $G(n, n, p)$ from first principles. It is enough if your answer gets the correct scaling in n and p. You can ignore any constants that don't depend on n or p in your calculations.

   (d) Find an $\alpha_c \in (0, \infty)$ (or prove that none exists) such that the following is true:

$$\mathbb{P}(G(n, n, n^{-\alpha}) \text{ contains a copy of } K_{2,2}) \to \begin{cases} 0, & \text{if } \alpha > \alpha_c, \\ 1, & \text{if } \alpha < \alpha_c. \end{cases}$$

   Justify your answer fully.

3. Let S_k denote the star graph on k nodes, consisting of a hub and $k-1$ leaves.

(a) Show that S_k is a balanced graph.

(b) Using the results in your notes for balanced graphs, find a value $\alpha_k \in (0, \infty)$ such that

$$\mathbb{P}(G(n, n^{-\alpha}) \text{ contains a copy of } S_k) \to \begin{cases} 0, & \text{if } \alpha > \alpha_k, \\ 1, & \text{if } \alpha < \alpha_k. \end{cases}$$

Clearly state the result you will use before using it.

(c) The chromatic number of a graph G, denoted $\chi(G)$, is defined as the minimum number of colours required to colour the nodes of the graph in such a way that no two nodes with an edge between them have the same colour. Show that $\chi(G) \leq d_{\max} + 1$, where $d_{\max}$ denotes the maximum degree of all nodes in G.

Hint. Consider a greedy algorithm that goes through the nodes in arbitrary order, assigning each node a colour distinct from that of all its already coloured neighbours. Show that, if $d_{\max} + 1$ colours are available, then the greedy algorithm will never get stuck.

(d) Provide either an upper or a lower bound on $\chi(G)$ that holds with high probability for G drawn from $G(n, p)$. You should state your result precisely, and justify it fully.