Complex Networks
Problem Sheet 8

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1. Compute the generating function on $[0, \infty)$, and the mean for the following distributions. In each case, state whether the generating function is differentiable at 1.

   (a) The Poisson(λ) distribution,

   $$\mathbb{P}(\xi = n) = \frac{\lambda^n}{n!}e^{-\lambda}, \quad n = 0, 1, 2, \dots$$

   (b) The Geometric(p) distribution,

   $$\mathbb{P}(\xi = n) = (1-p)^{n-1}p, \quad n = 1, 2, 3, \dots$$

   (c) A modified Pareto distribution, with

   $$\mathbb{P}(\xi = n) = \frac{1}{(n+1)(n+2)}, \quad n = 0, 1, 2, \dots$$

   Hint. In order to compute $\sum_{n=0}^{\infty} x^n/((n+1)(n+2))$, first note that $\sum_{n=0}^{\infty} x^n = 1/(1-x)$ if $|x| < 1$. Integrate both sides of the expression twice. You don't need to justify interchanging the order of integration and summation on the left; you can do this for power series in the interior of their region of convergence.

2. Jeeves and Wooster visit a casino which has infinitely many roulette tables. The roulette wheels are independent and, due to a design flaw, end up red with probability $p > 1/2$. Jeeves and Wooster both know this, and always bet on red. If you bet £1 on red and the outcome is red, then you receive £2; otherwise you lose your bet.

   Both of them are compulsive gamblers and will not stop while they have money left, but they follow different strategies. Wooster always bets his entire fortune on a single table, whereas Jeeves splits his in each round, betting £1 on each table. They both start the evening with £1 each.

   (a) How many rounds does it take until Wooster has no money left? Specify both the distribution and the mean of this random variable.

   (b) What is the (approximate) probability that Jeeves is still playing at closing time?

3. A Hamiltonian path in a graph is a path which contains all nodes, and in which no node is repeated. In other words, if the graph has n nodes, it is a permutation $v_1, v_2, \dots, v_n$ of the nodes such that the edge (v_i, v_{i+1}) is present in the graph for each i between 1 and $n-1$. Less formally, it is a route for visiting each node once, and exactly once.

   Let $G(n, p)$ be the Erdős-Rényi random graph on n nodes with edge probability p.

(a) Compute the expected number of Hamiltonian paths in $G(n, p)$. Show that this number tends to infinity as $n \to \infty$ if $p = \lambda/n$ for some constant $\lambda > e$. *Hint.* Stirling's formula might be useful.

(b) From your lecture notes, write down a condition for p, in terms of n, such that $G(n, p)$ is disconnected with high probability.

(c) Briefly explain what is paradoxical about the answers to the two parts above, and what the resolution of this paradox is. You don't need to do detailed calculations, but just provide a verbal explanation.

4. Let $G(n, p)$ be the Erdős-Rényi random graph on n nodes with edge probability p.

(a) Using Cayley's formula, write down an expression for the expected number of spanning trees present in $G(n, p)$.

(b) From your lecture notes, write down a condition for p, in terms of n, such that $G(n, p)$ is disconnected with high probability.

(c) From the answers to the last two parts, identify a range of p over which $G(n, p)$ is disconnected with high probability, but the expected number of spanning trees it contains tends to infinity!