Solutions: Introduction to Queueing Networks 2018

1. (a) (seen in homework)
The generating function of X_1 is

$$G_X(z) = \mathbb{E}\big[z^{X_1}\big] = 1 - p + pz.$$

Using the independence of the X_i and the tower rule for expectations, the generating function of Y is

$$G_Y(z) = \mathbb{E}\mathbb{E}\big[z^Y|N\big] = \mathbb{E}\big[(1-p+pz)^N\big].$$

Now, using the fact that N has a Poisson(λ) distribution, we get

$$G_Y(z) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda}(1-p+pz)^n = e^{-\lambda+\lambda(1-p+pz)} = e^{-\lambda p(1-z)},$$

which we recognise as the generating function of a Poisson(λp) random variable. By the uniqueness of the generating function, we conclude that Y has a Poisson(λp) distribution.

(b) (seen in homework)
By the independence of X and Y,

$$G_Z(z) = G_X(z)G_Y(z) = e^{\lambda(z-1)}e^{\mu(z-1)} = e^{(\lambda+\mu)(z-1)},$$

which is the generating function of a Poisson random variable with mean $\lambda+\mu$. Again invoke uniqueness of the generating function.

(c) (seen similar in homework)
Let $\lambda = 3$ denote the mean number of admissions per day. From the answer to part (a), the number of patients who were operated on five days ago and are sitill in the ward is Poisson (0.1λ). Moreover, it is independent of the number operated on four days ago and still in the ward, which is Poisson($(0.1 + 0.2)\lambda$). These are mutually independent of the numbers operated on three, two and one days ago and still in the ward, which are Poisson with means 0.6λ, 0.9λ and λ respectively. Finally, these are all independent of the number due to be operated on that day, which is Poisson(λ). Using the answer to part (b), this tells us that the total number in the ward is Poisson, with mean $\lambda(1 + 1 + 0.9 + 0.6 + 0.3 + 0.1)$, which works out to $3.9\lambda = 11.7$ people.

2. (completely unseen, fairly challenging)
Recall that if cars pass by according to a PP of intensity λ, then gaps between cars have an Exp(λ) distribution. Following the hint, the expected number of cars that get through a gap is

$$\int_{t=0}^{\infty} \lfloor t \rfloor \lambda e^{-\lambda t} dt = \sum_{n=0}^{\infty} \int_{t=n}^{n+1} n\lambda e^{-\lambda t} dt$$

$$= \sum_{n=0}^{\infty} n(e^{-\lambda n} - e^{-\lambda(n+1)}) = (e^{-\lambda} - e^{-2\lambda}) + 2(e^{-2\lambda} + e^{-3\lambda}) + \dots$$

$$= e^{-\lambda} + e^{-2\lambda} + \dots = \frac{e^{-\lambda}}{1 - e^{-\lambda}}.$$

Also, the typical gap between cars, $\mathbb{E}[T]$ is just $1/\lambda$, using known properties of the exponential distribution.

Hence, the throughput from the slip road is the ratio,

$$\frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}.$$

As a sanity check, this tends to zero as λ tends to infinity, and to 1 as λ tends to zero, as expected.

3. (a) (unseen but straightforward, seen similar)

   The system can be modelled as a Jackson network consisting of an $M/M/1$ queue (denoted Node 1) served by the web server, and an $M/G/\infty$ queue (denoted Node 2) corresponding to the time spent reading an article. (Clearly, all readers can read in parallel, so this is an infinite server queue.)

   The model parameters are as follows: external arrivals to node 1 follow a Poisson process of rate $\eta_1 = 2$ (per second), and there are no external arrivals directly to node 2, i.e., $\eta_2 = 0$. The service rates are $\mu_1 = 10$ and $\mu_2 = 1/200$ per second. The routing probabilities are $r_{12} = 1$ (everyone served an article reads it) and $r_{21} = 1/4$. Hence, the traffic equations are

   $$\lambda_1 = 2 + \frac{\lambda_2}{4}, \quad \lambda_2 = \lambda_1,$$

   which have the solution $\lambda_1 = \lambda_2 = 8/3$.

   i. We have an $M/M/1$ queue with arrival rate $\lambda_1 = 8/3$ and service rate $\mu_1 = 10$. The mean sojourn time is $1/(\mu_1 - \lambda_1) = 3/22$ seconds.
      If this expression is not remembered, then the mean occupancy N_1 of the first queue is $\rho_1/(1-\rho_1)$, where $\rho_1 = \lambda_1/\mu_1 = 8/30$; hence, $N_1 = 8/22$. Then, using Little's law, the sojourn time is N_1/λ_1, which again yields $3/22$.
      If neither expression is remembered, the student has to calculate the mean of a geometric random variable from first principles.

   ii. As this is an $M/G/\infty$ queue, the number of people in the queue in stationarity has a Poisson distribution. The mean of this Poisson is $\lambda_2/\mu_2 = 1600/3 \approx 533$.

   (b) (unseen but straightforward, seen similar)

   This is an $M/G/1$ queue, with arrival rate $\lambda = 1/4$ per minute, mean service time $\mathbb{E}[S] = 3$ minutes, and second moment of service time equal to $\mathbb{E}[S^2] = 9$ square minutes, as S is deterministic. Hence, using the Pollaczek-Khinchin formula, the mean waiting time (excluding service time) is

   $$\mathbb{E}[W] = \frac{\lambda \mathbb{E}[S^2]}{2(1-\rho)} = \frac{9/4}{2(1/4)} = \frac{9}{2} \text{ minutes},$$

   where $\rho = \lambda/\mu$ is the load on the queue. Hence, by Little's formula, the mean number of customers waiting for service (excluding the one currently in service, if any) is

   $$\mathbb{E}[N] = \lambda \mathbb{E}[W] = 9/8.$$