Introduction to Queueing Networks

Problem Sheet 2

$**$ **Please hand in solutions to questions 3 and 5 on this sheet.** $**$

1. Consider a continuous-time Markov chain X_t on a finite state space S with infinitesimal generator Q, and suppose that $X_0 = i$. There are two ways to describe the first jump out of this state.

   - For each $j \neq i$ with $q_{ij} > 0$, let T_j be an exponential random variable with parameter q_{ij}, and assume these random variables are mutually indepenent. Let $T = \min_j T_j$. Then the first jump occurs at the random time T, and $X_T = k$ if $T = T_k$. (This defines X_T unambiguously because the probability that $T = T_j$ and $T = T_k$ for distinct j and k is zero.)
   - Let T be exponential with parameter $-q_{ii}$. Then the first jump occurs at the random time T and $X_T = k$ with probability $q_{ik}/(-q_{ii})$, independent of the value of T.

   Use the answer to Question 5 from Homework 1 to explain why these two descriptions are equivalent.

2. Let $X_t^1, t \geq 0$ and $X_t^2, t \geq 0$ be independent Poisson processes of rate λ_1 and λ_2 respectively. Let $X_t = X_t^1 + X_t^2$ denote their superposition. Use the answer to Question 4 in Homework 1 to show that $X_t, t \geq 0$ is a Poisson process of rate $\lambda = \lambda_1 + \lambda_2$.

3. Let $X_t^1, t \geq 0$ and $X_t^2, t \geq 0$ be independent Poisson processes of rate λ_1 and λ_2 respectively. Let $X_t = X_t^1 + X_t^2$ denote their superposition. Use the answer to Question 5 in Homework 1 to show that $X_t, t \geq 0$ is a Poisson process of rate $\lambda = \lambda_1 + \lambda_2$ by showing that the times between successive events of the X_t process are iid Exp(λ) (and any other properties required for it to be a Poisson process).

4. We say that a random variable N has a Geometric distribution with parameter p, written $N \sim Geom(p)$ if
   $$P(N = k) = p(1-p)^{k-1}, \quad k = 1, 2, 3, \ldots$$
   Let $N \sim Geom(p)$, and let $T_1, T_2, T_3, \ldots$ be iid Exp(λ) random variables, independent of N. Let $T = \sum_{k=1}^{N} T_k$. Using moment generating functions or otherwise, show that T is exponentially distributed with parameter λp. (***Hint.*** Recall that the moment generating function of T is defined as $M(\theta) = \mathbb{E}[\exp(\theta T)]$. First compute $\mathbb{E}[\exp(\theta T)|N = n]$ and then average over N to obtain the unconditional expectation.)

5. Let $X_t, t \geq 0$ be a Poisson process of rate λ_1, and let $Y_1, Y_2, Y_3, \ldots$ be iid Bernoulli(p) random variables. Recall that this means that $Y_i = 1$ with probability p and $Y_i = 0$ with probability $1 - p$.

   Let $X_t^1 = \sum_{i=1}^{X_t} Y_i$ be the process obtained by retaining each point of the Poisson process X_t independently with probability p and discarding it with probability $1 - p$. It is called the Bernoulli(p) thinning of the Poisson process X_t.

   Using the answer to question 4, show that $X_t^1, t \geq 0$ is a Poisson process of rate λp by showing that the times between successive events are Exp(λp), and any other properties required.

6. Suppose that the rate matrix Q of a Markov chain is diagonalisable, and can be written as ADA^{-1}, where D is a diagonal matrix. Show that for $t > 0$, the transition probability matrix $P(t)$ can be written as

$$P(t) = Ae^{Dt}A^{-1}.$$