Introduction to Queuing Networks

Problem Sheet 5

$**$ **Please hand in solutions to questions 4 and 5 on this sheet.** $**$

1. A ski resort rents out skis for a whole number of days only. The number of people wanting to rent out skis on any given day is random, and has a Poisson distribution with a mean of 100. Each person renting out a pair of skis keeps them for a random number of days uniformly distributed between 3 and 7 (inclusive) before returning them, independent of all other people.

   (a) Let N be a Poisson random variable with mean λ, and let $X_1, X_2, \ldots$ be iid Bernoulli(p) random variables independent of N. Let $Y = X_1 + X_2 + \cdots + X_N$. (Equivalently, we could say that Y is Binomial with parameters N and p). Using generating functions, or properties of Poisson processes, show that Y is a Poisson random variable with mean λp.

   (b) Use the answer to the last part to show that the number of skis outstanding on any day (rented out on that day or any previous day - assume all rentals happen at the beginning of the day) is a Poisson random variable, and compute its mean.

   Hint. Express the number of skis outstanding as a sum of the number of skis rented out on that day, and on any previous day but not yet returned.

2. Consider a single independent delay queueing system (i.e. an $M/M/\infty$ system) with two different job classes. Jobs of each class arrive as independent Poisson processes with rates λ_1 and λ_2 respectively. The number of active servers at any time is assumed to be equal to the number of jobs present, with jobs of each class independently completing service at rates μ_1 and μ_2 respectively. The state of the system is described by $\boldsymbol{n} = (n_1, n_2)$, where n_i is the numbers of class i jobs in the system.

   (a) Write down the possible transitions and transition rates for this two-class system.

   (b) Show that, in equilibrium, the process satisfies the detailed balance equations (and so is reversible) with equilibrium distribution

   $$\pi(n_1, n_2) = \rho_1^{n_1} \frac{e^{-\rho_1}}{n_1!} \rho_2^{n_2} \frac{e^{-\rho_2}}{n_2!} \quad n_1 = 0, 1, 2, \ldots, \quad n_2 = 0, 1, 2, \ldots$$

   where $\rho_i = \lambda_i / \mu_i,\ i = 1, 2$.

3. Consider a simple open Jackson network consisting of J linked single server queueing systems $Q_1, \ldots, Q_J$ each with the same service rate μ. Jobs arrive to Q_1 from outside as a Poisson process, rate $\gamma_1 = 1$, and there are no other external arrivals. For $j = 2, \ldots, J-1$,

jobs departing Q_j are routed to Q_{j+1} with probability $r_{jj+1} = (J-j)/J$ and to Q_{j-1} with probability $r_{jj-1} = j/J$. Jobs departing Q_1 are routed to Q_2 with probability $(J-1)/J$ and leave the network with probability $r_{10} = 1/J$. Jobs departing Q_J are routed back to Q_{J-1} with probability $r_{JJ-1} = 1$. All other routing probabilities are zero.

(a) Sketch the network as a series of linked nodes and arcs, marking in the external arrival rates and the routing probabilities at each node.

(b) Considering each node in turn, write down the traffic equations for the effective arrival rates $\lambda_1, \ldots, \lambda_J$. Show that these equations are satisfied by taking

$$\lambda_j = \binom{J}{j}, \quad j = 1, \ldots, J.$$

(c) Show that, for J odd, the maximum value of λ_j occurs at $j = (J-1)/2$, and hence find a lower bound on μ for the network to be stable. Show that for $J = 5$ this means the network is stable if and only if $\mu > 10$.

4. Consider a simple open Jackson network with J single server queueing nodes $Q_1, \ldots, Q_J$. Assume each node has the same service rate μ and the network satisfies the standard Markov arrival and routing properties. For $j = 1, \ldots, J-1$, assume the arrival rates and routing parameters are $\gamma_j = 1/2$; $r_{jj+1} = 3/4$; $r_{jj-1} = 1/4$. For $j = J$, $\gamma_J = (J+3)/4$, $r_{JJ-1} = 1/4$ and $r_{J0} = 3/4$. All other routing parameters are zero.

(a) Write down the traffic equations for the effective arrival rates λ_j, $j = 1, \ldots, J$. You are *given* that $\lambda_1 = 1$. Use this fact, and the traffic equation for Q_1, to determine λ_2, and then use the traffic equation for λ_2 to determine λ_3.

(b) Guess the general form for λ_j and check that your guess satisfies both the general equation for λ_j, $j = 1, \ldots, J-1$ and the equation for λ_J. Hence find a lower bound on the value of μ for the network as a whole to be stable.

5. Consider a queueing network with two independent delay (i.e., infinite server) nodes Q_1 and Q_2, both with infinite waiting room and Exponential service distribution with rate μ. Jobs arrive to Q_1 from outside the network as a Poisson process with rate λ; a job that completes service in Q_1 is routed to Q_2 with probability r or leaves the network with probability $(1-r)$; and a job that completes service in Q_2 leaves the network. There are no other arrivals or departures. Denote the state of the network by $\boldsymbol{n} = (n_1, n_2)$, where n_j is the number of jobs in Q_j.

(a) Identify the arrival and routing parameters, the possible transitions and the transition rates $q(\boldsymbol{n}, \boldsymbol{m})$ for the queue size process.

(b) Sketch the network as a series of linked nodes and arcs, marking in the external arrival rates, the effective arrival and departure rates and the routing probabilities at each node. Use this to identify the corresponding diagram for the reversed process. Hence or otherwise identify the arrival and routing parameters, the possible transitions and the transition rates $q'(\boldsymbol{n}, \boldsymbol{m})$ for the reversed process.

(c) Guess the invariant distribution for this network and show that q and q' satisfy the relevant condition of Kelly's lemma, for each state $\boldsymbol{n}$.