Introduction to Queuing Networks

Solutions to Problem Sheet 3

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1. (a) The state space is the whole numbers $\{0, 1, 2, \ldots\}$. The transition rates are $q_{i,i+1} = \lambda$ for all $i \geq 0$ and $q_{i,0} = \mu$ for all $i \geq 1$ since, when a bus arrives, the bus stop empties. Thus, the transition rate matrix is given by

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \ldots \\ \mu & -(\lambda+\mu) & \lambda & 0 & \ldots \\ \mu & 0 & -(\lambda+\mu) & \lambda & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(b) The Markov chain is not reversible since there is a transition from 2 to 0, but not from 0 to 2, for example. In more detail, the local balance equation corresponding to those two states reads $\pi_0 \cdot 0 = \mu\pi_2$, which implies that $\pi_2 = 0$. (We assume that $\mu > 0$.) Similarly, $\pi_3, \pi_4, \ldots$ are all zero.

Next, there is a transition from 1 to 2, not the other way round. Writing down the corresponding local balance equation, we get $\lambda\pi_1 = 0\pi_2$, which implies that $\pi_1 = 0$. Finally, looking at the local balance equation for states 0 and 1, we have $\lambda\pi_0 = \mu\pi_1$ which implies that $\pi_0 = 0$ because $\pi_1 = 0$. Hence, all π_n are zero. But there is no such probability distribution!

(c) As the Markov chain isn't reversible, we need to solve the global balance equations $\pi Q = \mathbf{0}$ to find the invariant distribution. Writing out the components of this vector equality, we get

$$\begin{aligned} \lambda\pi_0 &= \mu(\pi_1 + \pi_2 + \pi_3 + \ldots) \\ \lambda\pi_j &= (\lambda+\mu)\pi_{j+1}, \quad j = 0, 1, 2, \ldots, \end{aligned}$$

from which it follows that $\pi_j = \left(\frac{\lambda}{\lambda+\mu}\right)^j \pi_0$ for all $j \geq 0$. Combining this with the condition that $\sum_{j=0}^{\infty} \pi_j = 1$, we get

$$\pi_j = \frac{\mu}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^j, \quad j \geq 0. \tag{1}$$

The only condition required for there to be an invariant distribution is $\mu > 0$; since buses have infinite capacity, the queue is stable so long as buses arrive at any positive rate.

(d) Notice that the invariant distribution given in (1) is geometric, of the form $(1-\rho)\rho^j$ with $\rho = \lambda/(\lambda+\mu)$. Recall that the mean queue length in such a queue is given by $E[N] = \rho/(1-\rho)$. (This formula can be derived using generating functions but may be worth remembering.)

Thus, $E[N] = \lambda/\mu$. Hence, by Little's law, the mean sojourn time is given by $E[W] = E[N]/\lambda = 1/\mu$. The direct way to reach this answer is to note that the times between buses are exponentially distributed with mean $1/\mu$ and so, by the memoryless property of the exponential distribution (established in Problem 1), the typical customer has to wait for a mean time of $1/\mu$ irrespective of how long it was since the last bus arrived.

2. Let us condition on the event $W = w$. Since the arrival process $N(t)$ is a Poisson process of rate λ, the number of arrivals in a period of length w is a Poisson random variable with mean λw. In other words,

$$P(A = k | W = w) = \frac{(\lambda w)^k}{k!} e^{-\lambda w}.$$

Now, we are told that W has an Exp(μ) distribution, i.e., W has density $f(w) = \mu e^{-\mu w}$ on $w \geq 0$. Hence, we get

$$\begin{aligned} P(A=k) &= \int_0^\infty P(A=k|W=w)f(w)dw = \int_0^\infty \frac{(\lambda w)^k}{k!}e^{-\lambda w}\mu e^{-\mu w}dw \\ &= \frac{\mu}{\lambda+\mu}\Big(\frac{\lambda}{\lambda+\mu}\Big)^k \int_0^\infty (\lambda+\mu)\frac{((\lambda+\mu)w)^k}{k!}e^{-(\lambda+\mu)w}dw \\ &= \frac{\mu}{\lambda+\mu}\Big(\frac{\lambda}{\lambda+\mu}\Big)^k. \end{aligned}$$

The last equality holds because the integrand is the density of a Gamma random variable with shape parameter k and scale parameter $\lambda+\mu$. Alternatively, to evaluate this integral, make the change of variables $x = (\lambda+\mu)w$ and rewrite it as

$$I_k = \int_0^\infty \frac{x^k}{k!}e^{-x}dx.$$

Integrating by parts, it is easy to verify that $I_k = I_{k-1}$. By induction, $I_k = I_0$. But $I_0 = \int_0^\infty e^{-x}dx$, and so $I_0 = 1$. Therefore, $I_k = 1$ for all $k = 1, 2, \ldots$

We have thus shown that

$$P(A=k) = \frac{\mu}{\lambda+\mu}\Big(\frac{\lambda}{\lambda+\mu}\Big)^k,$$

for all $k = 0, 1, 2, \ldots$, which is what we were asked to show.

Compare this with (1) in the solution to the last problem. Do you see why they are the same?

3. (a) The transition rates are given by

$$q_{i,i+1} = \frac{\lambda}{1+i},\ q_{i,i-1} = \mu 1(i \geq 1).$$

Hence, the jump probabilities are given by

$$p_{i,i+1} = \frac{q_{i,i+1}}{q_i} = \frac{\lambda}{\lambda + (i+1)\mu 1(i \geq 1)} = 1 - p_{i,i-1}.$$

In order to compute the stationary distribution π, we use the fact that the Markov process is a birth-death process. Hence, it is reversible if it has an invariant distribution. Assuming that there is an invariant distribtuion π, it must solve the detailed balance equations $\pi_i q_{ij} = \pi_j q_{ji}$ for all i, j in the state space, i.e.,

$$\frac{\pi_i\lambda}{i+1} = \pi_{i+1}\mu \Rightarrow \pi_{i+1} = \frac{\rho}{i+1}\pi_i,$$

where ρ is defined as λ/μ. Iterating the above equation, we get $\pi_i = (\rho^i/i!)\pi_0$. The invariant distribution π must also be a probability distribution, i.e., it should sum to 1. It is clear that this can be achieved for any ρ by setting $\pi_0 = e^{-\rho}$. Hence,

$$\pi_i = \frac{\rho^i}{i!}e^{-\rho}, \quad i = 0, 1, 2, \ldots$$

is an invariant distribution for the reversible Markov process $X(t)$. Moreover, it is the unique invariant distribution since the chain is irreducible.

(b) i. We consider the queue in equilibrium and condition on there being an arrival in time $(t, t+dt)$ which decides to join the queue. Using Bayes formula, we get

$$\begin{aligned}
&P(X_t = i | \text{job arrives in } (t, t+dt) \text{ and decides to join queue}) \\
&= \frac{P(X_t = i, \text{ and job arrives in } (t, t+dt) \text{ and decides to join queue})}{P(\text{job arrives in } (t, t+dt) \text{ and decides to join queue})} \\
&= \frac{P(X_t = i) P(\text{job arrives in } (t, t+dt) \text{ and decides to join queue} | X_t = i)}{\sum_{j=0}^{\infty} P(X_t = j) P(\text{job arrives in } (t, t+dt) \text{ and decides to join queue} | X_t = j)} \\
&= \frac{\pi_i \lambda dt \frac{1}{1+i}}{\sum_{j=0}^{\infty} \pi_j \lambda dt \frac{1}{1+j}} = \frac{\rho^i/(i+1)!}{\sum_{j=0}^{\infty} \rho^j/(j+1)!}.
\end{aligned}$$

In order to obtain the third equality above, we have used the fact that the arrival process is Poisson of rate λ, and so the probability of an arrival in $(t, t+dt)$ is λdt, independent of the past (and hence of the current state X_t, which is a function of the past of the arrival and service processes). The probability that the arriving job decides to enter the queue does of course depend on X_t, which we have taken into account. Now,

$$\sum_{j=0}^{\infty} \frac{\rho^j}{(j+1)!} = \frac{1}{\rho} \sum_{k=1}^{\infty} \frac{\rho^k}{(k!} = \frac{1}{\rho} \Big(\sum_{k=0}^{\infty} \frac{\rho^k}{k!} - \frac{\rho^0}{0!} \Big) = \frac{e^{\rho} - 1}{\rho},$$

since $0! = 1$ by convention. (The empty product is taken to be the multiplicative identity, by convention, just as the empty sum is taken to be the additive identity, zero.) Substituting this above, we obtain

$$P(X_t = i | \text{job arrives in } (t, t+dt) \text{ and decides to join queue}) = \frac{1}{e^{\rho} - 1} \frac{\rho^{i+1}}{(i+1)!}.$$

This is the distribution of the number of customers already present in the system, as seen by a typical arrival who decides to join the queue. Note that it is different from the invariant distribution.

ii. The calculation here is very similar, except that we condition on there being an arrival in time $(t, t+dt)$ who decides not to join the queue. Using Bayes formula, we get

$$\begin{aligned}
&P(X_t = i | \text{job arrives in } (t, t+dt) \text{ and decides not to join queue}) \\
&= \frac{P(X_t = i, \text{ and job arrives in } (t, t+dt) \text{ and decides not to join queue})}{P(\text{job arrives in } (t, t+dt) \text{ and decides not to join queue})} \\
&= \frac{P(X_t = i) P(\text{job arrives in } (t, t+dt) \text{ and decides not to join queue} | X_t = i)}{\sum_{j=0}^{\infty} P(X_t = j) P(\text{job arrives in } (t, t+dt) \text{ and decides not to join queue} | X_t = j)} \\
&= \frac{\pi_i \lambda dt \frac{i}{1+i}}{\sum_{j=0}^{\infty} \pi_j \lambda dt \frac{j}{1+j}} = \frac{i\rho^i/(i+1)!}{\sum_{j=0}^{\infty} j\rho^j/(j+1)!}.
\end{aligned}$$

Now,

$$\sum_{j=0}^{\infty} \frac{j\rho^j}{(j+1)!} = \sum_{j=0}^{\infty} \frac{(j+1-1)\rho^j}{(j+1)!} = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} - \sum_{j=0}^{\infty} \frac{\rho^j}{(j+1)!}.$$

The first sum above is e^{ρ}, while the second sum was evaluated in the last part, and was found to be $(e^{\rho} - 1)/\rho$. Hence,

$$\sum_{j=0}^{\infty} \frac{j\rho^j}{(j+1)!} = \frac{(\rho - 1)e^{\rho} + 1}{\rho}.$$

Substituting this in the expression for the conditional probability of seeing i customers conditioned on the arrival deciding not to join the queue, we get

$$P(X_t = i | \text{job arrives in } (t, t+dt) \text{ and decides not to join queue}) = \frac{i\rho^{i+1}}{(i+1)!} \frac{1}{(\rho-1)e^{\rho}+1}.$$

This is the distribution as seen by a typical arrival who decides to balk. Note that it is different both from the invariant distribution, and from the distribution seen by an arrival who decides to join the queue.

4. (a) Let $X(t)$ denote the number of customers in the system at time t. Then $X(t)$ is a CTMC with transition rates given by

$$q_{i,i+1} = \begin{cases} \alpha, & i = 0 \\ \lambda, & i \geq 1, \end{cases} \quad q_{i,i-1} = \begin{cases} 0, & i = 0, \\ \mu, & i = 1, \\ 2\mu, & i \geq 2. \end{cases}$$

Since this is a birth and death process, it is reversible if it has an invariant distribution. Assuming that there is one, and denoting it by π, it must solve the detailed balance equations

$$\pi_i q_{i,i+1} = \pi_{i+1} q_{i+1,i}$$

for all $i \geq 0$. Thus,

$$\pi_1 = \pi_0 \frac{q_{01}}{q_{10}} = \frac{\alpha}{\mu}\pi_0, \quad \pi_{i+1} = \frac{\lambda}{2\mu}\pi_i, \ i \geq 1.$$

Hence, $\pi_i = \frac{\alpha}{\mu}\left(\frac{\lambda}{2\mu}\right)^{i-1}$ for $i \geq 1$. Since π has to be a probability vector, we have

$$\sum_{i=0}^{\infty} \pi_i = \pi_0\Big(1 + \frac{\alpha}{\mu}\frac{1}{1-\frac{\lambda}{2\mu}}\Big) = \pi_0\Big(1 + \frac{2\alpha}{2\mu-\lambda}\Big) = 1,$$

which implies that $\pi_0 = \frac{2\mu-\lambda}{2\alpha+2\mu-\lambda}$, as required.

(b) We have by Bayes' theorem that

$$\begin{aligned} P(X_A = 0) &= \frac{P(X(t) = 0 \text{ and arrival in } (t, t+dt))}{P(\text{arrival in } (t, t+dt))} \\ &= \frac{\pi_0\alpha}{\pi_0\alpha + \sum_{i=1}^{\infty}\pi_i\lambda}, \end{aligned} \tag{2}$$

whereas, for $i \geq 1$,

$$\begin{aligned} P(X_A = i) &= \frac{P(X(t) = i \text{ and arrival in } (t, t+dt))}{P(\text{arrival in } (t, t+dt))} \\ &= \frac{\pi_i\lambda}{\pi_0\alpha + \sum_{i=1}^{\infty}\pi_i\lambda}. \end{aligned} \tag{3}$$

Now the denominator on the RHS in (2) and (3) is the same, but π is multiplied by different constants (α and λ) in the numerator depending on whether $i = 0$ or $i \geq 1$. Since π_0 is neither 0 nor 1, $P(X_A = i)$ can't be the same as π_i for all i. Thus, the distribution seen by arrivals is not the same as the invariant distribution.

5. (a) The stationary distribution is $\pi_n = (1-\rho)\rho^n$, $n \geq 0$.

(b) Let $X(t)$ denote the number of customers in the system at time t. Since the system is in equilibrium, $X(t)$ is a random variable with distribution π. Between time t and the arrival of C^*, which happens an Exp(λ) time after t, there are clearly no arrivals (since C^* is defined as the first arrival after t), but there may be some service completions. Let X^* denote the number of customers in the system at the arrival of C^*, excluding C^*. Then,

$$P(X^* = j) = \sum_{k=j}^{\infty} P(X(t) = k, X^* = j) = \sum_{k=j}^{\infty} \pi_k P(X^* = j | X(t) = k). \tag{4}$$

We evaluate the last conditional probability as follows. Suppose there are currently n jobs in the system. What is the probability that the next service completion happens before the next arrival? Since the time to the next arrival is Exp(λ), irrespective of how long it has been since the last arrival, and the time to the next service completion is Exp(μ), this probability is $\frac{\mu}{\mu+\lambda}$ (check this for yourself). Hence, for the event $\{X^* = j\}$ to occur, conditional on $X(t) = k$, there must be exactly $k - j$ service completions before the arrival of C^*. If $j \neq 0$, this means that the arrival must happen before the $(k - j + 1)^{\text{th}}$ service completion; if $j = 0$, all k customers should be served before the arrival of C^*. Thus, we have,

$$\begin{aligned} P(X^* = 0 | X(t) = k) &= \Big(\frac{\mu}{\mu+\lambda}\Big)^k, \\ P(X^* = j | X(t) = k) &= \Big(\frac{\mu}{\mu+\lambda}\Big)^{k-j} \frac{\lambda}{\mu+\lambda}, \quad j \geq 1. \end{aligned}$$

(Compare this with Problem 2. There, we calculated the number of arrivals during a service completion. Here, we are calculating the number of service completions during an inter-arrival period. Both are geometrically distributed, but the geometric distribution here is truncated at the number of customers in the system since there can't be more service completions than that.) Substituting the above formulae in (4), we get

$$\begin{aligned} P(X^* = 0) &= \sum_{k=0}^{\infty} (1-\rho)\rho^k \Big(\frac{\mu}{\mu+\lambda}\Big)^k = \sum_{k=0}^{\infty} (1-\rho)\Big(\frac{\rho}{1+\rho}\Big)^k \\ &= (1-\rho)\Big(1 - \frac{\rho}{1+\rho}\Big)^{-1} = (1-\rho)(1+\rho), \end{aligned}$$

whereas, for $j \geq 1$,

$$\begin{aligned} P(X^* = j) &= \sum_{k=j}^{\infty} (1-\rho)\rho^k \Big(\frac{\mu}{\mu+\lambda}\Big)^{k-j} \frac{\lambda}{\mu+\lambda} \\ &= \sum_{i=0}^{\infty} (1-\rho)\rho^{j+i} \frac{\rho}{1+\rho}\Big(\frac{1}{1+\rho}\Big)^i = (1-\rho)\rho^j \sum_{i=0}^{\infty} \Big(\frac{\rho}{1+\rho}\Big)^{i+1} \\ &= (1-\rho)\rho^j \Big[\Big(1 - \frac{\rho}{1+\rho}\Big)^{-1} - 1\Big] = (1-\rho)\rho^{j+1}, \end{aligned}$$

as we were asked to show.