Open billiards and Applications

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“Mathematical billiards and their applications”
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Plan

1. Open dynamical systems

2. Open billiards

3. Applications

Part 1: Open dynamical systems

We consider a map $\Phi : \Omega \to \Omega$ and introduce a hole $H \subset \Omega$ so that if a trajectory $\Phi^t x$ reaches H it escapes and is no longer considered. Write $\Omega' = \Omega \backslash H$. We are interested in:

- The set that survives for some (possibly infinite) interval of time.

- The probability of surviving given a specified measure of initial conditions on Ω: Escape problem or on H: Recurrence problem. Choosing H to maximise or minimise these properties: Optimisation problem

- With more than one hole $H = \cup H_i$, the relation between the individual and combined survival probabilities: Interaction problem, and the (time-dependent) probability of reaching hole H_j from H_i: Transport problem

Hyperbolic example: Open Baker map

We define

$$\Phi(q,p) = \begin{cases} (3q, p/3) & 0 \le q < 1/3 \\ \text{escape} & 1/3 \le q < 2/3 \\ (3q-2, (p+2)/3) & 2/3 \le q < 1 \end{cases}$$

In ternary notation we have ($q_i, p_i \in \{0,1,2\}, q_1 \neq 1$)

$$\Phi(.q_1q_2q_3\ldots_3\,, .p_1p_2p_3\ldots_3) = (.q_2q_3q_4\ldots_3\,, .q_1p_1p_2\ldots_3)$$

Let $\Omega_{m,n}$ be the set in which q has no 1's in its first n ternary digits and p has no 1's in its first m ternary digits. $\Phi\Omega_{m,n} = \Omega_{m+1,n-1}$ if $n > 0$ so that $\Phi^t(q,p)\Omega_{m,n}$ is defined for $-m \le t \le n$. Infinite time limits in one or both directions lead to middle third Cantor sets.

Conditionally invariant measures

We define $\mu_{m,n}$ as the normalised uniform measure on $\Omega_{m,n}$, so that Φ maps $\mu_{m,n}$ to $\mu_{m+1,n-1}$ if $n > 0$. If however $n = 0$, some measure escapes so that $\mu_{m,0}$ is mapped to $2/3\mu_{m+1,0}$. Iterating this process, we find that $\mu_{\infty,0}$ is conditionally invariant: Given an initial point distributed with respect to this measure, the probability of surviving one iteration is 2/3, and the surviving points are distributed with respect to the same measure. This measure is smooth along the unstable manifold and fractal along the other direction. The repeller $\mu_{\infty,\infty}$ is (fully) invariant and fractal in both directions.

In general: A map $\Phi : \Omega' \to \Omega$ acts on a measure μ defined on Ω' as given by its action on (measurable) subsets $A \subset \Omega'$

$$(\Phi\mu)(A) = \mu(\Phi^{-1}A)$$

A measure μ is conditionally invariant if

$$\frac{(\Phi\mu)(A)}{(\Phi\mu)(\Omega')} = \mu(A)$$

Quantifying open dynamics

For the open Baker we have

- (Exponential) escape rate given uniform initial measure $\mu_{0,0}$

$$\gamma = -\lim_{t\to\infty} \frac{1}{t} \ln \mu_{0,0}(\Omega_{0,t}) = \ln(3/2)$$

- Lyapunov exponents on $\Omega_{\infty,\infty}$

$$\lambda_{\pm} = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\delta x(t)}{\delta x(0)} = \pm \ln 3$$

- Kolmogorov-Sinai entropy on $\Omega_{\infty,\infty}$

$$h = -\lim_{t\to\infty} \frac{1}{t} \sum_{q_1 \ldots q_t} (1/2)^t \ln(1/2)^t = \ln 2$$

- Partial Hausdorff dimension $\delta = \ln 2 / \ln 3$.

We note

$$\gamma = \lambda_+ - h, \qquad h = \delta \lambda_+$$

Calculation of γ: cycle expansions

See Dettmann & Howard, Physica D 2009 and refs. We consider a piecewise expanding map $\Phi : \mathbb{R}' \to \mathbb{R}$ and consider evolution of densities ($d\mu = \rho(x)dm$):

$$(\mathcal{L}_\Phi \rho)[y] = \int_{\mathbb{R}'} \delta(y - \Phi(x))\rho(x)dx$$

We expect that under iteration ρ will converge to a conditionally invariant density which is the eigenvector of $\mathcal{L}_\Phi$ with eigenvalue $z^{-1} = e^{-\gamma}$. We find this by expanding the characteristic equation in powers of z:

$$\begin{aligned} 0 &= \det(1 - z\mathcal{L}_\Phi) \\ &= \exp[\mathrm{tr}\ln(1 - z\mathcal{L}_\Phi)] \\ &= 1 - z\mathrm{tr}\mathcal{L}_\Phi - \frac{z^2}{2}\left[\mathrm{tr}\mathcal{L}_\Phi^2 - (\mathrm{tr}\mathcal{L}_\Phi)^2\right] + \dots \end{aligned}$$

with

$$\mathrm{tr}\mathcal{L}_\Phi^t = \int \delta(x - \Phi^t(x))dx = \sum_{x:\Phi^t(x)=x} \frac{1}{|1 - \Lambda_x|}, \qquad \Lambda_x = \frac{d}{dx}\Phi^t(x)$$

Truncating at z^t gives γ expressed in terms of non-escaping periodic orbits up to length t.

Local escape rates

See Keller & Liverani J Stat Phys 2009; Bunimovich & Yurchenko, Israel J Math (to appear). Still with a 1D piecewise expanding map, consider a sequence of holes of size h_n (calculated using the normalised invariant measure of Φ) shrinking to a point $x \in \mathbb{R}'$.

If x is not periodic, we expect to lose an amount h_n at each step:

$$\lim_{n\to\infty} \frac{\gamma_n}{h_n} = 1$$

If x has period t and stability factor Λ_x, then after t iterations a point starting less than $\epsilon/|\Lambda|$ from x will still be within ϵ of x. Thus the effective size of the hole is $h_n(1 - |\Lambda_x^{-1}|)$ and we expect

$$\lim_{n\to\infty} \frac{\gamma_n}{h_n} = 1 - |\Lambda_x^{-1}|$$

Not all is exponential...

Example: Single fixed point ($a > 0$, $\alpha \geq -1$)

$$\Phi(x) = x(1 + a|x|^{\alpha})$$

with escape for $|x| > 1$. We have

- Complete escape: $\alpha = -1$
- Superexponential escape: $-1 < \alpha < 0$
- Exponential escape: $\alpha = 0$
- Algebraic escape: $0 < \alpha < \infty$
- No escape: $\alpha = \infty$

More literature on open hyperbolic maps

Kac Bull AMS 1947 Exact formula for the mean recurrence time.

Pianigiani & Yorke, Trans AMS 1979 Escape problem: Convergence to conditionally invariant measures for expanding maps.

Hirata et al, CMP 1999 Small hole recurrence distribution Poissonian iff close to escape distribution; also follows from sufficiently strong mixing properties.

Demers & Young, Non 2006 Review (mathematics)

Altmann & Tél, Phys Rev E 2009 Review (physics)

Afraimovich & Bunimovich Non 2010 Topological approach, optimisation problem.

Bruin, Demers & Melbourne ETDS 2010; Christadoro et al Non 2010 1D maps with non-uniform hyperbolicity.

Part 2: Open billiards

Most of the discussion of general open dynamical systems applies, but...

- Billiards are more technically involved for various reasons including tangential orbits, intermittency, corners/cusps, infinite horizon.
- Holes in billiards tend to be small in one phase space direction and large in the other.
- Physically, we would like to treat the continuous time case.

Non-eclipsing case

Consider a billiard on $\mathbb{R}^d$ with at least three convex obstacles. The convex hull of any pair of obstacles does not intersect any of the others. In this case

Sjöstrand, Duke Math J 1990 "Fractal Weyl" bounds on the number of resonances in the quantum problem.

Cvitanović et al, "Pinball scattering" (book chapter) 1994 Cycle expansion calculation of the escape rate to 30 digits.

Lopez & Markarian, Siam J Appl Math 1996 Construction of the conditionally invariant measure in the 2D case.

Petkov & Stoyanov, Non 2009 Periodic orbit correlations in non-eclipsing billiards.

Chaotic billiards with small holes

See Bunimovich & Dettmann, EPL 2007. General idea: Use a characteristic function which is 0 on the hole and 1 elsewhere. Then surviving trajectories are identified by products

$$\prod_i \chi(\Phi^i x)$$

Continuous time effects are incorporated by a weighting e^{sT}, where T is the time between collisions, and the small hole and long time limit corresponds to an expansion in powers of s. A typical result is

$$\gamma_{AB} = \gamma_A + \gamma_B - \frac{1}{\langle T \rangle} \sum_{j=-\infty}^{\infty} \langle (u_A)(\Phi^j \circ u_B) \rangle + \ldots$$

where A, B label two holes, $\langle\rangle$ is an average computed using the standard billiard invariant measure $dsdp_{\|}$, and u is equal to -1 on the relevant hole and $hT/\langle T \rangle$ elsewhere. The neglected terms are expected to be small as long as the holes are not both covering points in the same short periodic orbit.

Integrable billiards

See Bunimovich & Dettmann, PRL 2005. In a circular billiard with a small hole in the boundary, surviving orbits are near periodic orbits, which are regular polygons and stars. These are enumerated using the Euler totient function $\varphi(n)$. We find for the survival probability $P(h,t)$

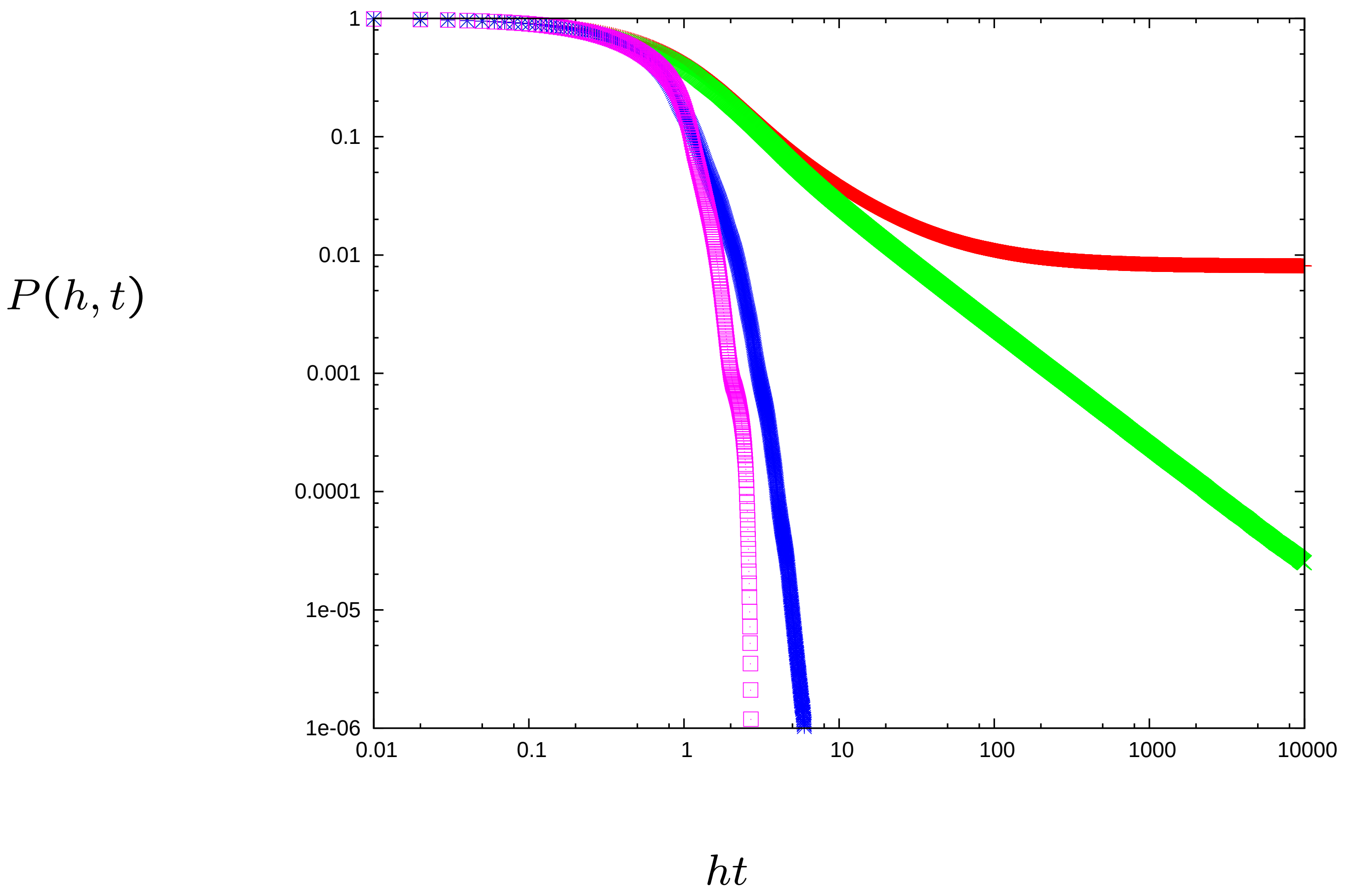
$$\lim_{t\to\infty} tP(h,t) = \frac{\pi}{2}\sum_{n=1}^{\lfloor h^{-1}\rfloor} \frac{\phi(n)-\mu(n)}{n}(1-nh)^2 = \frac{1}{\pi h} + o(h^{1/2-\delta})$$

with $\delta > 0$ determined by the Riemann Hypothesis.

In both the circle and ellipse, there is (numerically, at least) a nontrivial scaling limit:

$$\lim_{h\to 0} P(h,\tau/h) = f(\tau)$$

Elliptical billiard, different hole sizes

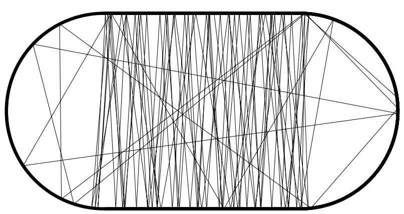


The stadium

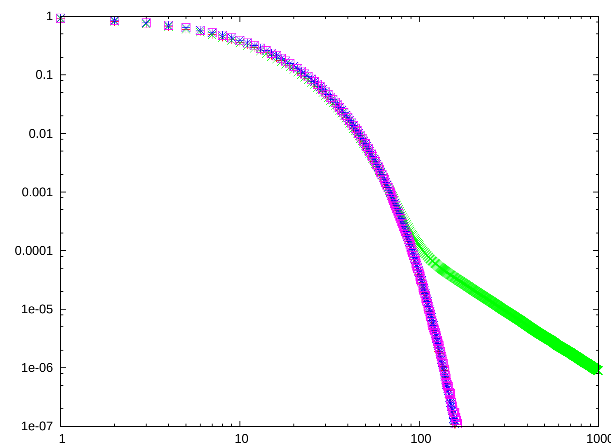
See Dettmann & Georgiou, Physica D 2009, arxiv 2010. For a hole in the straight segment, surviving orbits are mostly close to "bouncing ball" orbits. This means we can calculate the leading term in $P(t)$:

$$\lim_{t\to\infty} tP(t) = \frac{(3ln3+4)(L_1^2+L_2^2)}{4P}$$

with L_i the length of the straight segments to the left and right of the hole, and P the perimeter. This also leads to asymmetric transport:



$P_{ij}(t)$



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More literature related to open billiards

Dettmann & Cohen J Stat Phys 2000 Numerical results for some open polygonal billiards.

Tabachnikov "Geometry and billiards" 2005 General introduction to non-chaotic billiards

Chernov & Markarian "Chaotic billiards" 2006 Mathematical introduction to chaotic billiards

Bálint & Melbourne, J Stat Phys 2008 Statistical properties of billiard flows.

Demers et al, Commun Math Phys 2010 Convergence of measures in small hole limit, Sinai billiard.

Dettmann, "Recent advances in open billiards..." (book chapter) 2010 Review

Part 3: Applications

Open billiards model any physical system involving a container whose contents may escape. For example:

- Microwaves in metal cavities (cleanest experiments)
- Light in dielectric cavities (micro- and nano- technology)
- Cold atoms confined by lasers (allows time dependent cavities)
- Electrons in semiconductors (allows external forcing)
- Room acoustics (control of reflection/absorption effects)

There are also important applications in quantum chaos and statistical mechanics.

Modifications to billiard dynamics

- Soft potentials (eg electron and atom optics billiards).
- Bending of paths due to external electric and magnetic fields.
- Phase effects, eg from a weak magnetic field. These are proportional to the area enclosed by a trajectory.
- Goos-Hänchen shift, in which the reflection point shifts along the boundary by an amount proportional to the wavelength.
- Stochastic reflection, eg sound from a rough surface.

Modifications to the escape

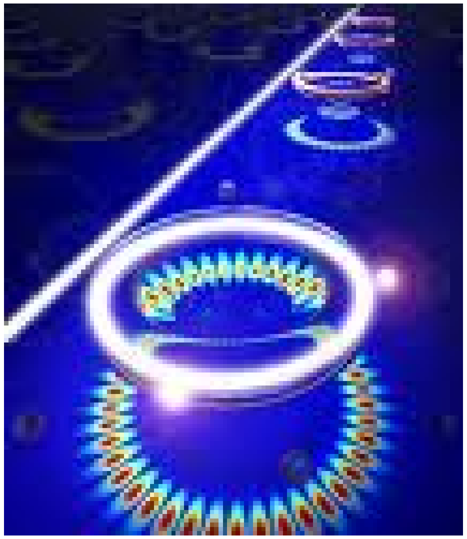
- Total internal reflection in dielectric cavities: escape when angle of incidence is small ($\sin\theta < n_2/n_1$ where n_i are refractive indices) independent of boundary position.
- Partial reflection/escape at small angles
- Partial absorption/amplification effects in the interior.
- Partial absorption effects at the boundary.

Optical microcavities

See Brambilla J Opt 2010; Xiao et al Front Optoelecton China 2010. These trap light using total internal reflection in a dielectric cavity a few microns in diameter, for example a circular "ring resonator".

Potential for: quantum computing, optical switching, microlasers for displays, biosensing.

Issues: We need high Q-factor (low escape), high directivity of escape, understanding of nonlinear effects, interaction between closely spaced cavities.



A ring resonator: J Zhu & J Gan, WUSTL

Quantum chaos

See Nonnemacher, Nonlinearity 2008. Open billiards are a mainstay of quantum chaos: The quantum version of a billiard is the Helmholtz equation with Dirichlet conditions at the boundary. Some important general aspects are

- Random matrix theory: How well do the correlations of the energy levels match those of ensembles of random matrices corresponding to the regular/chaotic dynamics and relevant symmetries?

- Fractal Weyl laws: Does the number of resonance states of open quantum chaotic systems correspond to the size of the repeller in phase space?

- Resonance eigenstates: How does the structure correspond to classical conditionally invariant measures?

Billiard singularities lead to “diffraction” effects.

Equilibrium statistical mechanics

Forces between atoms are very "steep", so hard ball models are used: These are equivalent to high dimensional semi-dispersing billiards.

Statistical mechanics can be justified by properties like ergodicity and mixing, which are proved for many hard ball systems: see Simányi Invent Math 2009.

Numerical simulations in large hard ball systems (and more recently soft potentials) exhibit "Lyapunov modes", a step structure in the Lyapunov exponents of small magnitude. Yang & Radons Phil Trans Roy Soc A 2009.

Warnings: We need to distinguish carefully between chaotic and multidimensional effects, between dynamics and stochastic forcing, and between physical and unphysical timescales. In addition, a slight softening of billiard potentials can break ergodicity: see Rapoport & Rom-Kedar, Phys Rev E 2008.

Nonequilibrium statistical mechanics

There are many dynamical approaches to nonequilibrium steady states, eg steady conduction of heat or elecricity, or Couette flow. Thermostats modify the equations of motion, often in a reversible and "hidden" Hamiltonian manner. Boundaries may be modified by stochastic laws or asymmetric collision laws. Some approaches use open billiards.

Escape rate formalism of Gaspard & Nicolis: Consider a large periodic array of circular scatterers (Lorentz gas) with finite horizon, and overall forming a square of size L. The diffusion equation $D\nabla^2\rho = \rho_t$ with Dirichlet boundary conditions has slowest decaying solution $\rho(x,y,t) = e^{-\gamma t}\sin(\pi x/L)\sin(\pi y/L)$ with $\gamma = 2\pi^2 D/L^2$. Thus the macroscopic diffusion coefficient D can be related to the escape rate γ. Gaspard "Chaos, Scattering and Statistical Mechanics" 1998.

Allowing particles exiting a hole to enter through another hole with scaled position and momentum corresponds to an infinite billiard with scale invariance; the particle preferentially moves toward the larger scales as described by Boltzmann's view of entropy. A 2D system of this kind can by a conformal transformation be shown equivalent to a periodic system with a thermostat. Barra et al, Nonlinearity 2007.

More references on applications

- Nakamura & Harayama "Quantum chaos and quantum dots" 2004. [Electron billiards]
- Kaplan "Atom optics billiards: Nonlinear dynamics with cold atoms in optical traps" (book chapter) 2005.
- Judd et al "Chaotic transport in semiconductor, optical and cold-atom systems" Prog Theor Phys Suppl 2007.
- Höhmann et al Phys Rev E 2009. [Microwave billiards]
- Weaver & Wright "New directions in linear acoustics and vibration" 2010.

Conclusion

Open billiards...

- are described as open dynamical systems with escape rates etc.

- differ from general open systems in the nature of the holes and singularities.

- are closely related to many important physical applications.