Applied Dynamical Systems Solution Sheet 1

1. Use the chain rule (and subscripts to denote derivatives)

$$\begin{aligned} \frac{d}{dt}H(\mathbf{q},\mathbf{p}) &= \frac{d\mathbf{q}}{dt}\cdot H_{\mathbf{q}} + \frac{d\mathbf{p}}{dt}\cdot H_{\mathbf{p}} \\ &= H_{\mathbf{p}}\cdot H_{\mathbf{q}} - H_{\mathbf{q}}\cdot H_{\mathbf{p}} \\ &= 0 \end{aligned}$$

   For the pendulum we have equations of motion

$$\dot{q} = H_p = p/m, \qquad \dot{p} = -H_q = -mgl\sin q$$

   Conservation of energy is

$$\begin{aligned} \frac{d}{dt}\left(\frac{p^2}{2m} - mgl\cos q\right) &= \dot{p}p/m + mgl\dot{q}\sin q \\ &= -pgl\sin q + pgl\sin q \\ &= 0 \end{aligned}$$

2. Separation of variables (ie divide by $\sqrt{x}$ and integrate) gives

$$\sqrt{x} = \frac{1}{2}(t - t_0)$$

   where t_0 is the constant of integration. Since (as usual) $\sqrt{x}$ must be positive, this solution is only valid for $t \geq t_0$. The separation of variables implicitly assumes that $x \neq 0$; if instead $x = 0$ we see that a constant $x = 0$ is also a solution. Thus the general solution is

$$x(t) = \begin{cases} 0 & t < t_0 \\ \frac{1}{4}(t - t_0)^2 & t \geq t_0 \end{cases}$$

   where t_0 can be $+\infty$ corresponding to the zero solution everywhere.

   If we consider initial data $x(0) = 0$ we could have any solution with $t_0 \geq 0$. Thus the solution is not unique. The Picard-Lindelöf theorem is not satisfied because $\sqrt{x}$ has unbounded derivative in the vicinity of $x = 0$ and hence is not Lipshitz continuous there.

3. Exponential sensitivity of initial conditions in chaotic systems suggests that the variance of errors should grow exponentially with prediction time and then saturate as prediction is effectively impossible. However there are many other factors such as uncertainty in some of the forcing terms, that complicate the analysis.

4. It is usually possible to distinguish periodic and chaotic regimes, but period doubling may require professional equipment.

5. The fixed points are solutions of $f(x) = x$ where $f(x) = rx(1-x)$. As discussed in lectures these are at $x = 0, (r-1)/r$. The period two points are the solutions to $f(f(x)) = x$. This is a quartic equation, but we can divide by the known (fixed point) solutions to get a quadratic; the answer is
$$x = \frac{r+1 \pm \sqrt{r^2-2r-3}}{2r}$$
These solutions are real and distinct (and hence the period 2 point exists) for $r > 3$ (also $r < -1$).

6. We have the damped oscillator $\dot{x} = v, \quad \dot{v} = -x - \alpha v$, $(\alpha > 0)$. This is linear with constant coefficients - we can either substitute to get a second order equation for x, or use matrix methods. The flow is
$$\begin{aligned} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} &= \Phi^t \begin{pmatrix} x(0) \\ v(0) \end{pmatrix} \\ &= e^{\frac{-\alpha t}{2}} \begin{pmatrix} \cos\omega t + \frac{\alpha}{2\omega}\sin\omega t & \frac{1}{\omega}\sin\omega t \\ \frac{-1}{\omega}\sin\omega t & \cos\omega t - \frac{\alpha}{2\omega}\sin\omega t \end{pmatrix} \begin{pmatrix} x(0) \\ v(0) \end{pmatrix} \end{aligned}$$
where $\omega = \sqrt{1-(\alpha/2)^2}$ is assumed to be real (that is, $\alpha < 2$). The time-one map is just Φ^1, that is, substitute $t = 1$ into the above. Return to the surface $x = 0$ occurs every time π/ω. We find
$$v(t + \pi/\omega) = e^{-\frac{\alpha\pi}{2\omega}} v(t)$$
so this is the Poincaré map. The flow is invertible: We can uniquely define the above for negative t.

The question of reversibility is rather difficult: It is clear that the usual $v \to -v$ does not reverse the dynamics. However, noting that in the variables $(x, v/\omega + (\alpha x)/(2\omega))$ the dynamics is just that of a focus —

rotation with exponential decay, and that a transformation of the form $v \to 1/v$ can reverse an exponential decay, we find that

$$(x, v) \to \frac{1}{r^2}(x, -v - \alpha x)$$

with

$$r^2 = x^2 + (v + \alpha x/2)^2/\omega^2$$

satisfies the conditions for a reversing involution, except for the fixed point itself. This is physically rather unintuitive, since the damped oscillator models dissipative processes. Note that there is no involution that takes account of the fixed point; it must reverse into an unstable focus but there is none. So it is correct in this sense to argue that the system is not reversible.

7. My code uses the change of variable to find a Poincare section where the surface is a level surface of one of the coordinates as here; times when $x = 1$ come to

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t=1.0492 x=1 y=-1.725
t=5.2565 x=1 y=1.6669
t=7.3138 x=1 y=-1.6536
t=11.559 x=1 y=1.5961
t=13.578 x=1 y=-1.5837
t=17.862 x=1 y=1.5266
t=19.841 x=1 y=-1.515
t=24.167 x=1 y=1.4583
t=26.103 x=1 y=-1.4476
t=30.472 x=1 y=1.3912
t=32.364 x=1 y=-1.3813
t=36.778 x=1 y=1.3251
t=38.624 x=1 y=-1.3159
t=43.086 x=1 y=1.2598
t=44.883 x=1 y=-1.2514
t=49.395 x=1 y=1.1954
t=51.14 x=1 y=-1.1876
t=55.705 x=1 y=1.1315
t=57.396 x=1 y=-1.1245
t=62.017 x=1 y=1.0681
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t=63.651 x=1 y=-1.0617
t=68.331 x=1 y=1.0049
t=69.903 x=1 y=-0.99913
t=74.646 x=1 y=0.94167
t=76.154 x=1 y=-0.93656
t=80.964 x=1 y=0.87823
t=82.403 x=1 y=-0.87372
t=87.284 x=1 y=0.81419
t=88.649 x=1 y=-0.81025
t=93.608 x=1 y=0.74911
t=94.892 x=1 y=-0.74572
t=99.935 x=1 y=0.68239
t=101.13 x=1 y=-0.67954
t=106.27 x=1 y=0.61319
t=107.37 x=1 y=-0.61085
t=112.6 x=1 y=0.54023
t=113.59 x=1 y=-0.53839
t=118.95 x=1 y=0.46138
t=119.81 x=1 y=-0.46002
t=125.31 x=1 y=0.3725
t=126.02 x=1 y=-0.3716
t=131.69 x=1 y=0.26308
t=132.21 x=1 y=-0.26262
t=138.17 x=1 y=0.063134
t=138.29 x=1 y=-0.063107
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