The Sine Map

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1 Introduction

Unimodal maps on the unit interval are among the most studied dynamical systems. Perhaps the two most frequently mentioned are the logistic map and the tent map. These two share many properties with each other, and can in fact be conjugated at certain parameter values via a homeomorphism

$$k(x) := 2\pi \arcsin(\sqrt{x}). \tag{1}$$

(I.e with logistic map f and tent map g, $f = k^{-1} \circ g \circ k$.) [Rauch,]. Furthermore, both of these maps can be topologically conjugated to the shift map on two symbols, which can be easily seen to have many interesting properties [Bhaumik and Choudhury, 2009], most of which are preserved by conjugacy. Many results are known concerning unimodal maps in general. Results had been proved in [Metropolis et al., 1973] concerning the so-called U-sequence of a map, that is, the order in which orbits of different periods become stable. It is shown to be 'universal' for a large family of maps. It is also shown in [Hussein and Abed, 2012] that all unimodal maps with negative Schwarzian derivative

$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2, \tag{2}$$

are chaotic (in many definitions on the word). The use of symbolic dynamics in analysis of maps on the unit interval can be seen in [Milnor and Thurston, 1988], where various results are proved using the *lap* number, which is defined to be the smallest number s such that f can be broken in to s monotone segments on the interval. We consider a similar quantity in Section 3 as a way of computing topological entropy.

In this project we look at the sine map given by

$$x_{n+1} = f_\mu(x_n) \tag{3}$$

$$f_\mu(x) = \mu sin(\pi x) \quad x \in [0,1], \mu > 0. \tag{4}$$

We can see in Figure 1.1 that it has some superficial similarities to the logistic map, but how do the dynamics compare? Computing the Schwarzian derivative yields

$$(Sf)(x) = -\pi^2(1 + \tfrac{3}{2}\tan^2(\pi x)) < 0, \tag{5}$$

so we expect chaotic behaviour. We focus on the analysis of fixed points, period points, local bifurcations and chaos.

2 Fixed and Periodic Points

We first consider the existence of fixed and periodic points of the sin map. There is one obvious fixed point at $x = 0$, and from the plot we can see there should be another which we call $x[\mu]$. We can find this numerically for given values of μ, for example we have that

$$x[1] \approx 0.7365 \tag{6}$$
$$x[2] \approx 0.8587 \tag{7}$$
$$x[100] \approx 0.9968. \tag{8}$$

Interestingly however we can see that for small μ, we have only one fixed point. This is when $x > \mu sin(\pi x)$ for all x or equivalently when $\frac{d}{dx}(\mu\sin(\pi x)) < 1$, which gives $\mu < \frac{1}{\pi}$. We can verify this graphically to see how the fixed point $x[\mu]$ behaves as μ increases, see Figure 1.2.

At $x = 0$, $x_{n+1} = f_\mu(x_n) \approx \mu\pi x_n$, hence the fixed point at $x = 0$ is stable in the regime $\mu < \frac{1}{\pi}$, marginal for $\mu = \frac{1}{\pi}$, and unstable otherwise. Expanding $\sin(\pi x)$ as a Taylor series about $x = 0$, setting $x = \frac{\sqrt{6}}{\pi}\delta$, and $\mu = \frac{1}{\pi}(1 + \nu)$ we obtain

$$\delta_{n+1} \approx (1 + \nu)(\delta_n - \delta_n^3), \tag{9}$$

which is the normal form for the supercritical pitchfork bifurcation. Indeed, we could have inferred this from Figure 1.2. This is different from the transcritical bifurcation at $x = 0$ on the logistic map covered in lectures as the sine map has symmetry that gives rise to an extra stable fixed point for negative x. The stability condition for the second fixed point is given by $|\pi\mu\cos(\pi x[\mu])| < 1$, and thus there is an area of stability which we can find numerically. We know analytically that it is stable for $\frac{1}{\pi} < \mu < c$ for some constant c which can be found as $c \approx 0.7200$. We can verify numerically that $f'_{0.72}(x[0.72]) = -1$, so we have a period doubling bifurcation at this point. This suggests we obtain a stable period two orbit that will then undergo period doubling itself. We can compute numerically the point (μ, x) at which the n-fold iterate map $f_\mu^n(x)$ is a fixed point, and simultaneously $[f_\mu^n]'(x) = -1$. These are listed in Table 1 and can be seen on the bifurcation diagram (see Figures 2.1,2.2).

We can use the values from Table 1 to give an estimate of the Feigenbaum constant, $\delta = 4.669\ldots$ [Briggs, 1991]. If μ_m is the location of the m^{th} bifurcation point we have that

$$\delta = \lim_{m\to\infty} \frac{\delta_m - \delta_{m-1}}{\delta_{m+1} - \delta_m}. \tag{10}$$

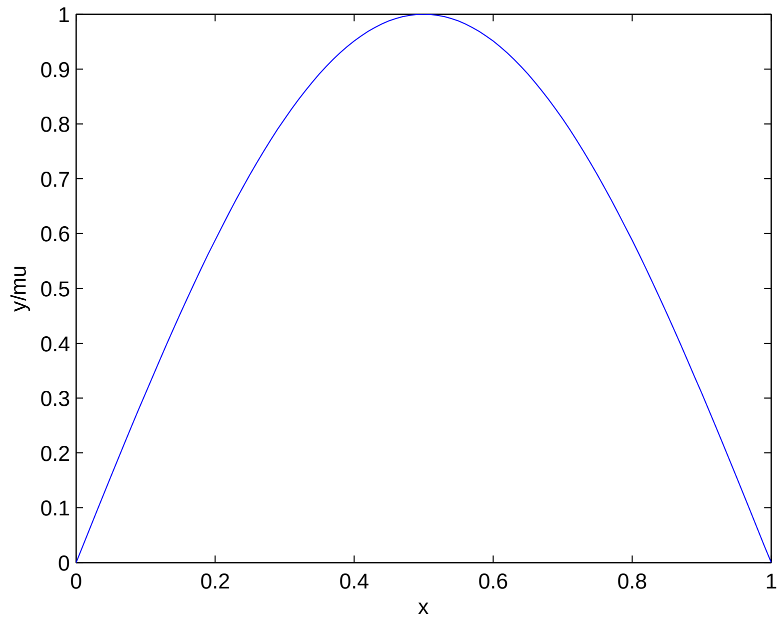

Figure 1.1: The sin map on the unit interval

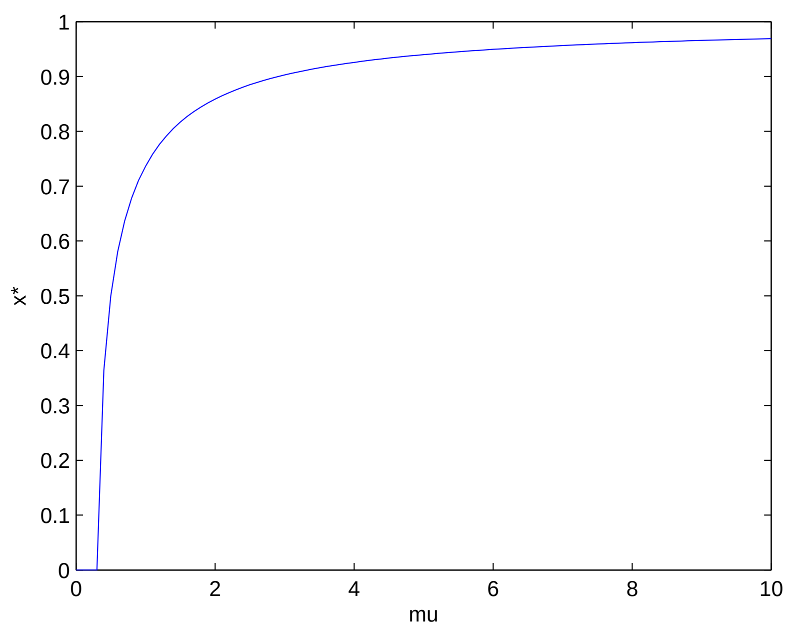

Figure 1.2: A plot of $x[\mu]$ against μ.

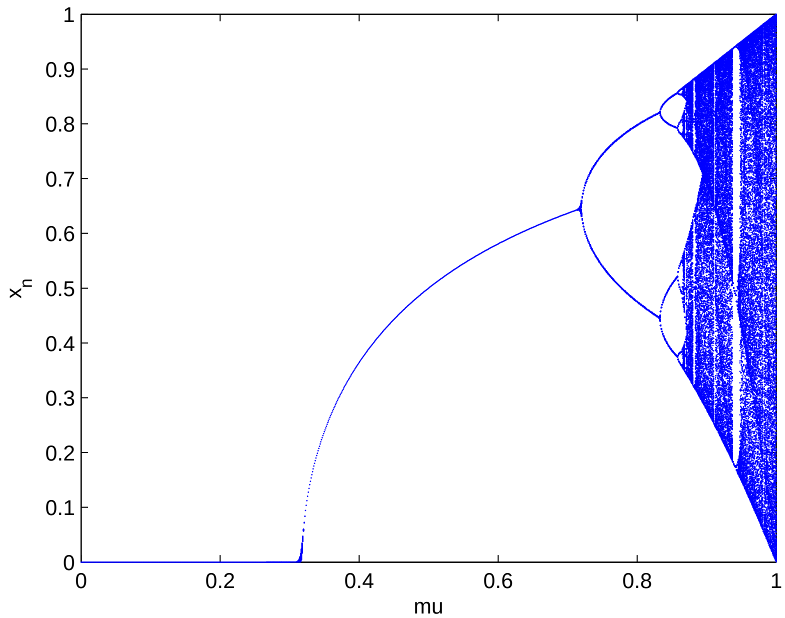

Figure 2.1: The bifurcation diagram for $\mu \in [0, 1]$.

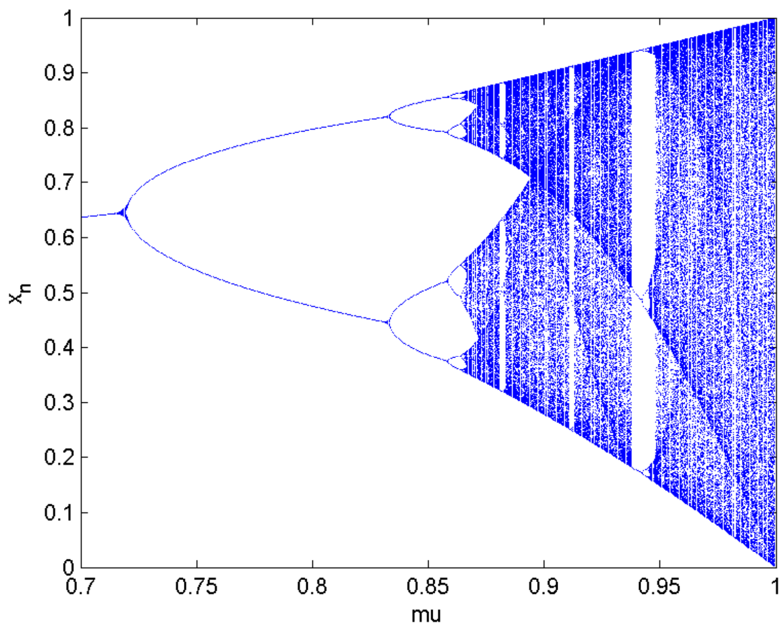

Figure 2.2: A close up of the chaotic region. Note the stable period 3 window around $\mu \approx 0.94$.

n	μ	x
1	0.7200	0.6458
2	0.8333	0.8208
4	0.8586	0.8566
8	0.8641	0.8638
16	0.8653	0.8517

Table 1: The bifurcation points and respective locations of stable fixed points for the map f_μ^n.

Substituting in our values for $m = 3$ gives us

$$\delta \approx \frac{0.8641 - 0.8586}{0.8653 - 0.8641} \approx 4.6605. \tag{11}$$

The period doubling bifurcation explains the orbits of even period, but not the period 3 orbit. We solve the equations $x = f_\mu^3(x)$ and $|[f_\mu^3]'(x)| = 1$ to find that the period three orbit is only stable for $\mu \in [0.9378, 0.9425]$. The stable period three orbit also has a matching unstable orbit which we can see in Figure 2.3.

The appearance and behaviour of the bifurcation diagram is very similar to that of the logistic map, albeit with different parameter values. There is a good reason for this. The following result was given by [Metropolis et al., 1973].

Theorem 2.1. *[Metropolis et al., 1973] Consider the map $x \mapsto \mu f(x)$ and suppose the following four properties hold:*

***A.1**: $f(x)$ is continuous, single-valued, and piecewise C^1 on $[0, 1]$, and strictly positive on the open interval, with $f(0) = f(1) = 0$.*

***A.2**: $f(x)$ has a unique maximum, $f_{max} \leq 1$, assumed either at a point or in an interval. To the left or right of this point (or interval), $f(x)$ is strictly increasing or strictly decreasing respectively.*

***A.3**: At any x such that $f(x) = f_{max}$, the derivative exists and is equal to zero.*

***B**: Let $\mu_{max} = 1/f_{max}$. Then there exists a μ_0 such that for $\mu_0 < \mu < \mu_{max}$, $\mu f(x)$ has only two fixed points, the origin and $x[\mu]$, say, both of which are repellent.*

Then , the order in which periodic orbits become stable (the U-sequence) is completely determined.

Note that these are sufficient conditions, but not necessary. It is trivial to check that the sine map satisfies these properties. So in fact the bifurcation diagram has precisely the same structure as that of the logistic map. The first few terms of this U-sequence up to period 7 are $2, 4, 6, 7, 5, 7, 3, 6 \ldots$

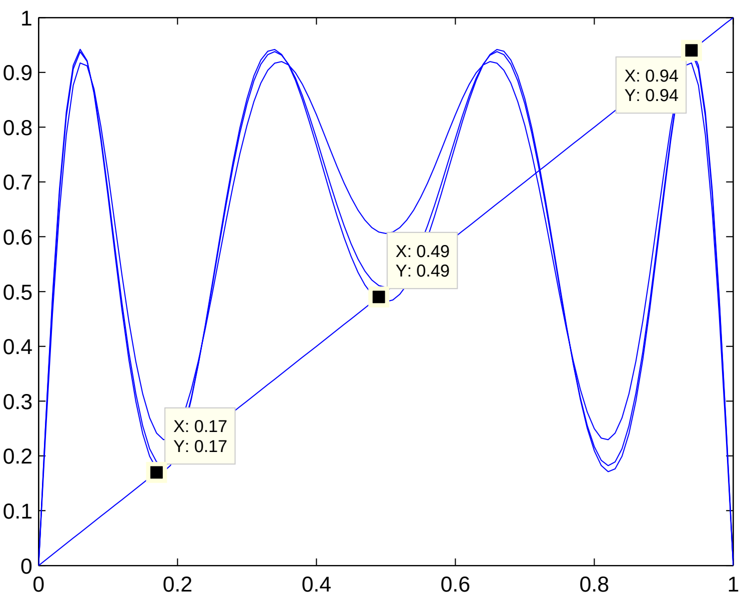

Figure 2.3: The point at which the period three orbits are generated, the stable one labelled.

3 Chaos and Entropy

We have seen from the bifurcation diagram (Fig. 2.1) that the sine map becomes chaotic as r approaches 1. We can quantify this chaos by computing the Lyapunov exponents for the sine map given by the formula

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |\mu\pi \cos(\pi x_i)|. \tag{12}$$

We plot the Lyapunov exponents against the parameter μ in Figure 3.5. Again, this image is very close to the equivalent plot for the logistic map given in [Hall and Wolff, 1995]. The areas in which the graph strays above the dotted line at zero are precisely those in which the map becomes chaotic. It is shown in [Misiurewicz, 1980] that the topological entropy of a piecewise monotone mapping, f, of an interval can be expressed as

$$h_{top}(f) = \lim_{n\to\infty} \frac{1}{n} \ln e_n, \tag{13}$$

where e_n is the number of critical points of the map f^n on the interval. A general method for computing the number of critical points is given in [Dilão and Amigó, 2010], but we can prove a result for the sine map directly.

Proposition 3.1. *Let $\mu = 1$, then the n-fold composition of the sine map $f_1^n(x)$ has $2^n - 1$ critical points on the unit interval.*

Let $E^n = \{x \in [0,1] \mid [f_1^n]'(x) = 0\}$. We seek to prove that $e_n = \|E^n\| = 2^n - 1$.

Proof. The result is clear when $n = 1$. Assume it is true for $n = k$, then

$$[f_1^{k+1}]'(x) = 0 \implies [f_1^k]'(f(x))f'(x) = 0 \tag{14}$$

$$\implies x \in \{\tfrac{1}{2}\} \cup f_1^{-1}(E^k). \tag{15}$$

Since $\mu = 1$, every $y \in [0,1)$ has precisely two preimages $x_1 \neq x_2 \neq \frac{1}{2}$. Furthermore

$$[f_1^k]'(1) \neq 0, \tag{16}$$

so $1 \notin E^k$. Hence $\|E^{k+1}\| = 2 \times \|E^k\| + 1 = 2^{k+1} - 1$. The result follows by induction. □

Hence, the topological entropy of the sine map when $\mu = 1$ is given by

$$h_{top}(f_1) = \lim_{n\to\infty} \frac{1}{n} \ln(2^n - 1) = \ln 2. \tag{17}$$

Again, this is the same as the maximal topological entropy for the logistic map [Froyland et al., 2001]. It seems then, that in all ways it has been examined,

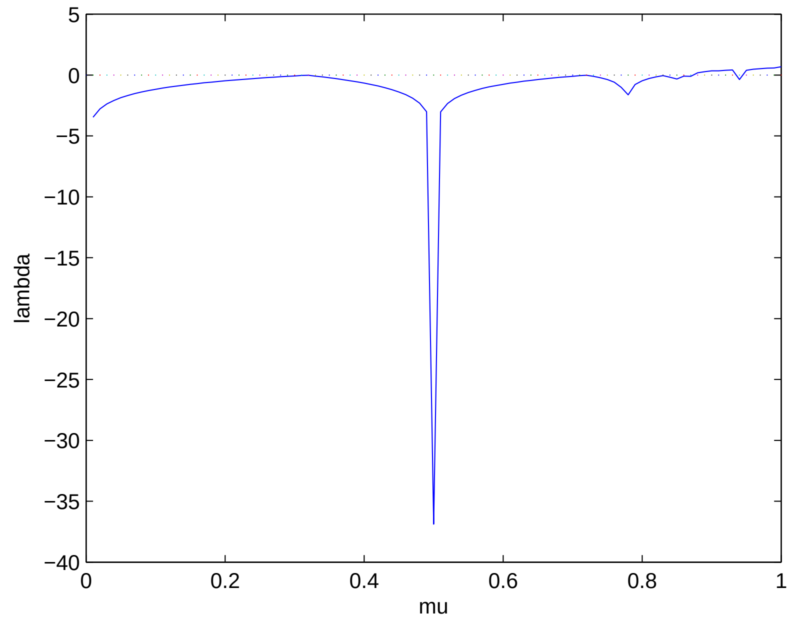

Figure 3.4: The Lyapunov exponents for the sine map.

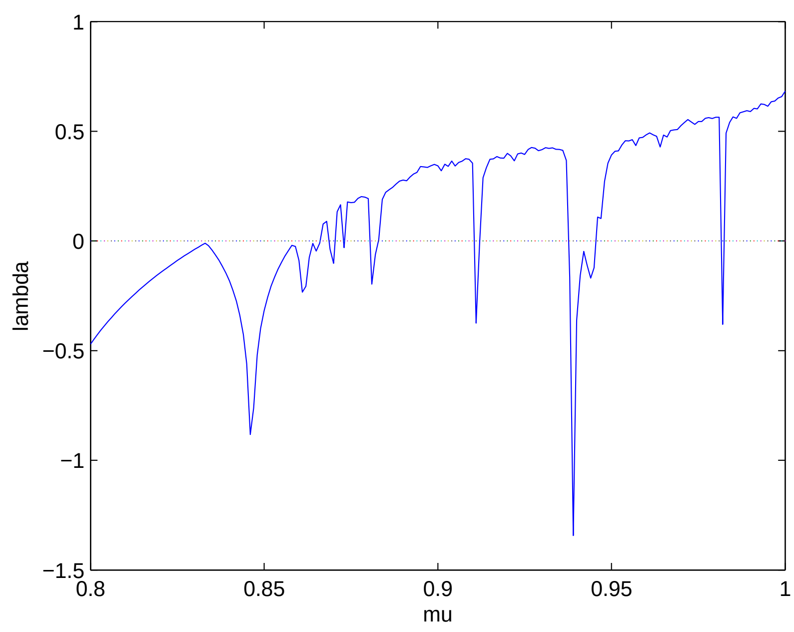

Figure 3.5: A close up of the chaotic region.

the sine map is qualitatively identical to the logistic map, and the superficial similarity has resulted in a much deeper connection. It would be natural to consider splitting the interval at the critical point $x = \frac{1}{2}$, and considering the symbolic dynamics that result. A good summary of symbolic dynamics for unimodal maps, and the properties that can be determined, can be found in [Hao, 1991].

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