Chapter 6

First Order PDEs

6.1 Characteristics

6.1.1 The Simplest Case

Suppose $u(x,t)$ satisfies the PDE

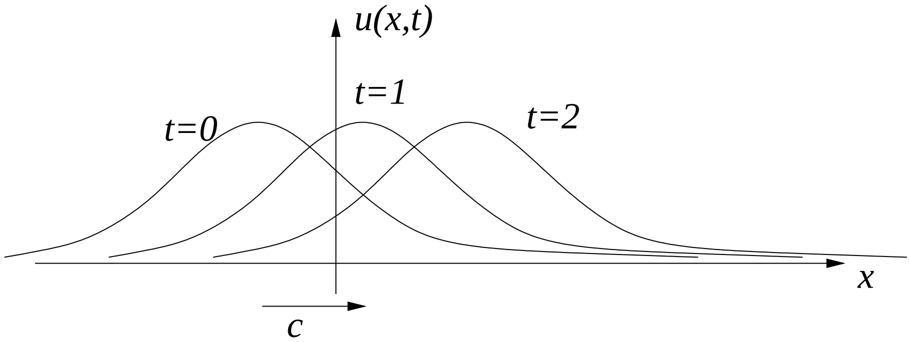
$$au_t + bu_x = 0$$

where b, c are constant.

If $a = 0$, the PDE is trivial (it says that $u_x = 0$ and so $u = f(t)$. If $a \neq 0$, it reduces to

$$u_t + cu_x = 0 \text{ where } c = b/a. \tag{6.1}$$

We know from §5.4 that the solution is $f(x - ct)$. This represents a wave travelling in the x direction with speed c, and with constant shape.



We now take a new and more general point of view.

The equation (6.1) can be written

$$u_t + \frac{dx}{dt} u_x = 0, \qquad \text{or} \qquad dt\, u_t + dx\, u_x = 0 \tag{6.2}$$

where $dx/dt = c$. Then integrating gives $x = \xi + ct$ where ξ is an integration constant such that $x = \xi$ when $t = 0$.

From the chain rule,

$$\frac{\partial u}{\partial s} = \frac{\partial t}{\partial s} u_t + \frac{\partial x}{\partial s} u_x = 0 \tag{6.3}$$

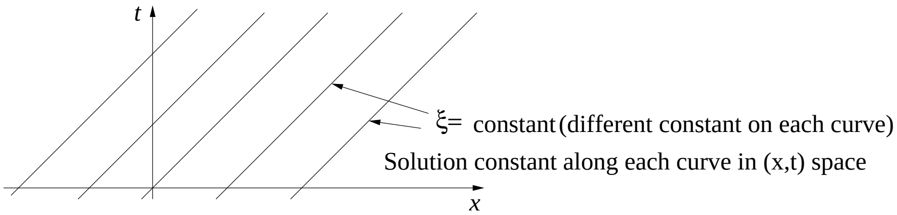
where s is a parameter lin. indep. of ξ. We think of (ξ, s) being a change of variables from (x, t) and so $u(\xi, s) \equiv u(x, t)$. Integrating (6.3) gives

$$u(\xi, s) = g(\xi)$$

where g is an arbitrary function. If we choose $s = 0$ on $t = 0$, then an I.C. of $u(x, 0) = f(x)$ is transformed to $u(\xi, 0) = f(\xi)$ and so must have $g \equiv f$. I.e. solution is $u(\xi, s) = f(\xi)$ or, in terms of (x, t),

$$u(x, t) = f(\xi) = f(x - ct).$$

Solution is constant and equal to its initial value along curves $\xi = x - ct =$ constant. Such curves are called **Characteristic curves**.



Each curve is a straight line passing through $t = 0$ at $x = \xi$ with slope $1/c$.

This agrees with §5.4.

6.1.2 A more complicated example

Consider a 1st order **inhomogeneous** linear PDE with **non-constant** coefficients:

$$u_t + x u_x = \sin t$$

with I.C. $u(x, 0) = f(x)$.

As before, introduce (ξ, s) s.t. $x = \xi$, $s = 0$ on $t = 0$. By chain rule

$$\frac{\partial u}{\partial s} = \frac{\partial t}{\partial s} u_t + \frac{\partial x}{\partial s} u_x = \sin t$$

Matching terms, we have

$$\frac{\partial t}{\partial s} = 1, \qquad \Rightarrow t = s, \qquad \text{(since } s = 0 \text{ on } t = 0\text{)}$$

$$\frac{\partial x}{\partial s} = x, \qquad \Rightarrow x = \xi e^s, \qquad \text{(since } x = \xi \text{ on } s = 0\text{)}$$

Also,

$$\frac{\partial u}{\partial s} = \sin t = \sin s, \qquad \text{with} \quad u(\xi, 0) = f(\xi)$$

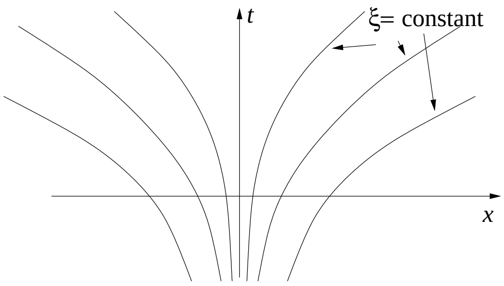
Integrate up, using I.C. to get

$$u(\xi, s) = -\cos s + 1 + f(\xi)$$

Inverting variables: $s = t$ and $\xi = x\mathrm{e}^{-s} = x\mathrm{e}^{-t}$. So solution is, in terms of (x, t)

$$u(x, t) = 1 - \cos t + f(x\mathrm{e}^{-t})$$

In this example, characteristics are not straight lines; given by $\xi = x\mathrm{e}^{-t} =$ constant. Or $t = \log(x/\xi)$ for different constant values of ξ.



6.1.3 First Order Quasilinear PDEs

We consider the PDE

$$\boxed{u_t + g(u)u_x = 0} \tag{6.4}$$

where g is a given function of one variable. The equation is called *quasilinear*, because it is linear in u_t and u_x, but may be nonlinear in u. Also, have I.C.

$$u(x, 0) = f(x)$$

As before, let $x = \xi$ and $s = 0$ on $t = 0$. Then we have

$$\frac{\partial t}{\partial s} = 1, \qquad \frac{\partial x}{\partial s} = g(u), \qquad \frac{\partial u}{\partial s} = 0$$

The first one gives $t = s$, the last one gives $u(\xi, s) = f(\xi)$. Then

$$\frac{\partial x}{\partial s} = g(f(\xi))$$

which we can integrate up to give

$$x = sg(f(\xi)) + \xi$$

so that $x = \xi$ on $s = 0$ as required. Using $s = t$, we have

$$\boxed{x = tg(f(\xi)) + \xi} \tag{6.5}$$

as the equation defining characteristic curves. These are **straight** lines in the (x, t) plane with slope $1/g(f(\xi))$. Solution is constant along characteristics, with

$$u(x, t) = f(\xi) = f(x - tg(f(\xi))).$$

I.e. solution determined by data at $t = 0$.

Note: Since $u = f(\xi)$, can eliminate ξ:

$$u(x, t) = f(x - tg(u))$$

which is an *implicit* equation defining $u(x,t)$.

Remark: The method of characteristics works straightforwardly for quasilinear equations of the form

$$a(u,x,t)u_t + b(u,x,t)u_x = c(u,x,t)$$

in terms of ODEs for the variables u, x and t, which may in general be coupled. The ICs may be defined along any curve not tangent to a characteristic curve. In this case $s = 0$ on the IC curve and ξ is an arbitrarily chosen parameter along the curve. More details (also for second order problems) in the Methods course.

6.2 Examples of Nonlinear Wave Problems

6.2.1 Examples with Shock Waves

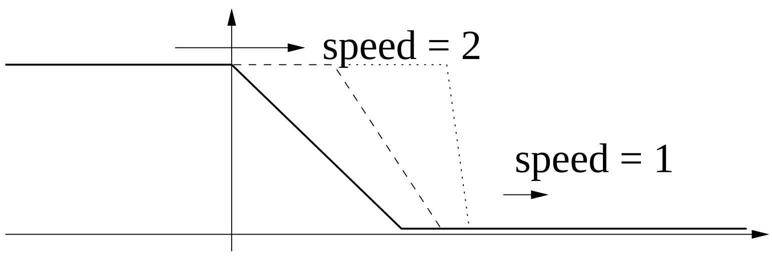
Example 1. Suppose

$$g(u) = 1 + u, \qquad \text{so} \qquad u_t + (1+u)u_x = 0. \tag{6.6}$$

with

$$u(x,0) = f(x) = \begin{cases} 1 & \text{for } x \leq 0 \\ 1-x & \text{for } 0 < x < 1 \\ 0 & \text{for } x \geq 1 \end{cases}$$

A simple physical interpretation of the solution goes as follows. The wave speed $c = 1 + u = 1+$initial height. So for $x < 0$, $c = 1 + 1 = 2$ and travels faster than for $x > 1$, where $c = 1 + 0 = 1$.

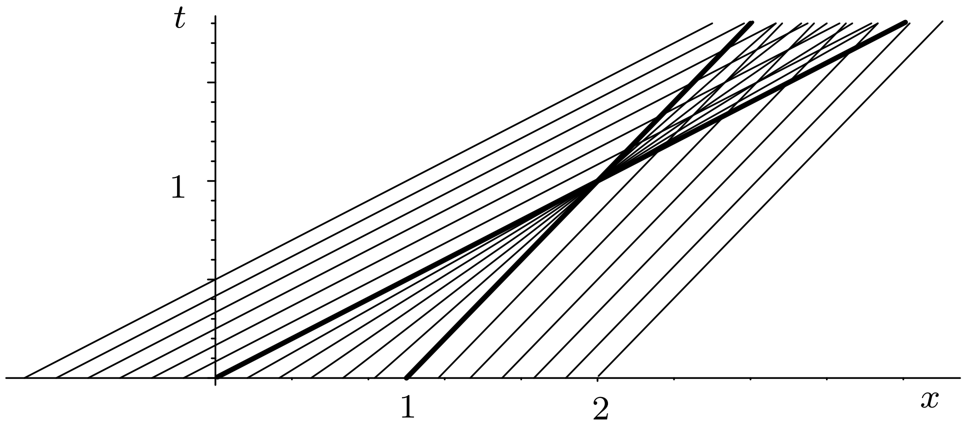


In the I.C. we can interchange x with ξ.

So

- For $\xi < 0$, $g(f(\xi)) = 1 + u(\xi,0) = 1 + 1 = 2$. So slope of char. is $1/2$ and equation of char. is $x = 2t + \xi$ (from (6.5)).
- For $\xi \geq 1$, $g(f(\xi)) = 1 + u(\xi,0) = 1 + 0 = 1$. So slope of char is 1 and equation of char. is $x = t + \xi$
- For $0 < \xi < 1$, $g(f(\xi)) = 1 + 1 - \xi = 2 - \xi$. So slope of char is $1/(2-\xi)$ and equation of char is $x = (2-\xi)t + \xi$.

Sketch of chars:



- The two bounding curves are (A): $x = 2t$ and (B): $x = t + 1$. They cross when (x, t) are the same, i.e. at $1 + t = 2t$ or $t = 1$, $x = 2$.
- Interior curves are given by $x = (2 - \xi)t + \xi$ for $0 < \xi < 1$. Easy to see they all pass through $x = 2$, $t = 1$.

Solution: $u = f(\xi)$ so

$$u(x,t) = \begin{cases} 1 & \text{for } \xi \leq 0 \\ 0 & \text{for } \xi \geq 1 \\ 1 - \xi & \text{for } 0 < \xi < 1 \end{cases}$$

or, in terms of (x, t),

$$u(x,t) = \begin{cases} 1 & \text{for } x \leq 2t \\ 0 & \text{for } x \geq 1 + t \\ \left(\dfrac{1 + t - x}{1 - t}\right) & \text{for } 2t < x < 1 + t \end{cases}$$

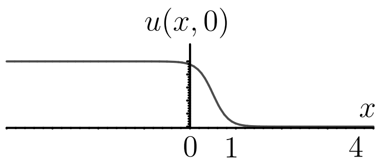
(last equation comes from $x = (2 - \xi)t + \xi$ implies $\xi(1 - t) = x - 2t$ implies $\xi = (x - 2t)/(1 - t)$ and so $1 - \xi = 1 - ((x - 2t)/(1 - t)))$.

Can see that the solution **blows up** at $t = 1$. This is where the characteristics cross one another. Always the case: *when characteristics cross the solution breaks down*. Indicative of **shock waves**.

Physically, in this problem, it is where the linear ramp becomes vertical; infinite gradient implies derivatives don't exist.

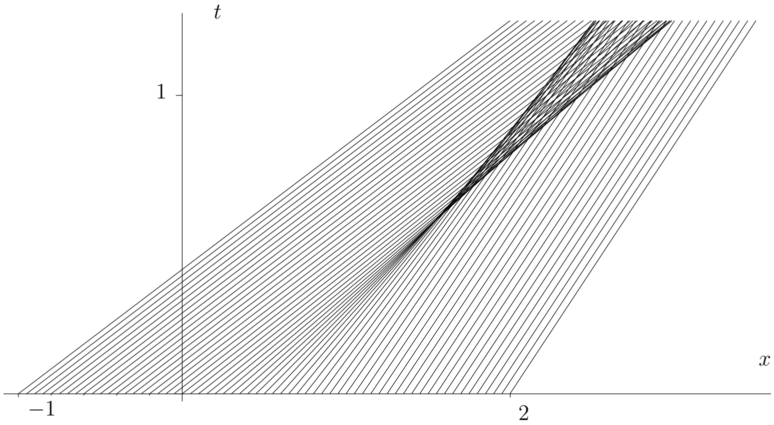
Very interesting... but not here.

Example 2. The same PDE but a smooth initial condition: $\frac{1}{2}[1 - \tanh 3(x - \frac{1}{2})]$. Here is its graph:



It is a smoothed-out version of Example 1.

Characteristics:

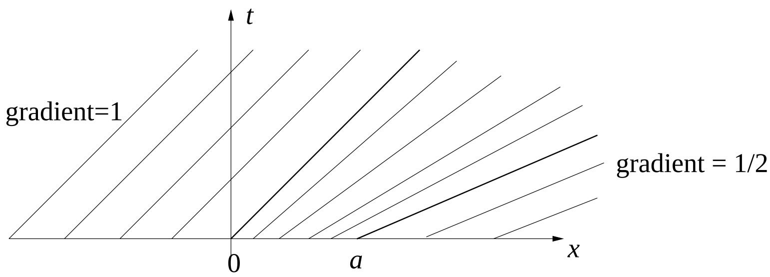


6.2.2 Example with an Expansion Fan

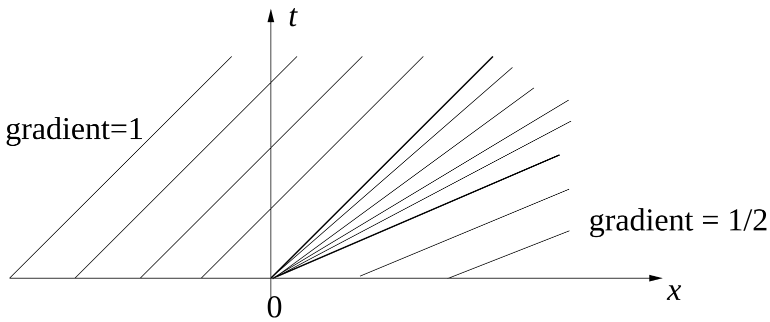
Example 3: The same PDE $u_t + (1+u)u_x = 0$, but this time with an initial condition which increases with x:

$$u(x,0) = f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ x/a & \text{for } 0 < x < a \\ 1 & \text{for } x \geq a \end{cases} \qquad \text{where } a > 0$$

The characteristics spread out, and there are no shocks in this problem.



In the limit as $a \to 0$ the solution is called an *expansion fan*.



6.3 Traffic Flow

This theory was invented in Manchester in 1955 by Sir James Lighthill and G. B. Whitham.

6.3.1 The Traffic Flow Equation

Let x be distance along a road (not necessarily straight).

Traffic density $\rho(x,t)$ on a road is defined as the number of cars (or other vehicles) per unit distance at the point x and time t.

Then the number of cars at time t in the region $a < x < b$ is $\int_a^b \rho(x,t)\,dx$.

ρ is really a subtle kind of average.

Conservation Law: No cars can be created or destroyed and so can use the conservation law in §2.3.1 ('stuff' is now cars).

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \phi}{\partial x}$$

where $\phi(x,t)$ is **flux**. In this context, flux is *the rate at which cars are crossing the fixed point* x. I.e. it is (density of cars) $\times$ (speed of cars).

$$\phi(x,t) = \rho(x,t)u(x,t)$$

where $u(x,t)$ is the **Traffic speed**. So we have

$$0 = \rho_t + (\rho u)_x = \rho_t + \rho_x u + \rho u_x \tag{6.7}$$

This **one** equation involves **two** dependent variables ρ and u. Need another equation linking ρ and u to close the model.

Instead of $u = u(x,t)$, propose $u = u(\rho) = u(\rho(x,t))$.

I.e. ones speed is not dependent where you are on the road, or what time it is, only on the density of the traffic surrounding you. So

$$\phi = \rho u = \rho u(\rho) \equiv f(\rho), \qquad \text{say.}$$

Now we get

$$0 = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} f(\rho) = \rho_t + f'(\rho)\rho_x \tag{6.8}$$

This is a quasi-linear PDE of the type already investigated in §6.1.3.

6.3.2 The Quadratic Model

Assumptions:

1. When $\rho = 0$, $u = u_{max}$, the maximum speed a car will travel at on an empty road (the speed limit ?)
2. $u = 0$ when $\rho = \rho_c$ where $\rho_c = 1/\text{spacing between cars in a jam}$.

Consider the simplest model, in which u is a linear function of ρ:

$$u = u_{max}\left(1 - \frac{\rho}{\rho_c}\right) \quad \text{for} \quad \rho \le \rho_c \tag{6.9}$$

Now $f(\rho) = \rho u = u_{max}\rho(1 - \rho/\rho_c)$ and

Plug into the conservation eqn (6.8) to get

$$\boxed{\rho_t + c(\rho_c - 2\rho)\rho_x = 0} \tag{6.10}$$

where

$$c = \frac{u_{max}}{\rho_c}. \tag{6.11}$$

This is the type of PDE considered in §6.2, but with a negative coefficient for the quadratic term $\rho\rho_x$.

Wave speed = $1/(\text{slope of characteristics}) = c(\rho_c - 2\rho)$. Can be +ve or -ve.

Speed of traffic is $c[\rho_c - \rho] > 0$ from (6.9) and (6.11).

So the *wave speed* $<$ *traffic speed.*

This means that changes in density travel more slowly than cars. So when you drive, you go faster than the changes in density; that's why you have to slow down to avoid thickening of traffic. This is the other way round from water waves, where as you float with the wave, breakers come up behind you. The reason is the nonlinear term $\rho\rho_x$ in (6.10) has a minus sign, where the nonlinear term in (6.6) has a plus sign.

It is easy to verify that if $\rho(x, 0)$ is an increasing function, you get shock formation, so when traffic starts to get thicker (as when a motorway lane closes), discontinuities tend to develop. Anyone who has driven on a busy motorway knows that traffic jams can form suddenly and for no obvious reason. They are shock waves formed by the steepening of initially smooth changes of density.

Remark:

These quasi-linear equations also closely connected to other observable phenomena, such as glacier flows and sedimentation in river deltas.

