General relativity problem sheet 7

1. $*$ Find the transformation law for connection coefficients, which is not the usual tensor transformation law.

2. Calculate the number of independent components of $g_{\mu\nu,\alpha\beta}$ and $\Lambda^{\alpha}_{\beta',\gamma',\delta'}$ in $n = 1, 2, 3$ dimensions (4 dimensions was given in lectures): the difference is the number of independent components of the Riemann curvature tensor.

3. (a) Calculate the area enclosed by a curve of constant θ on a sphere of unit radius. Hint: use the metric.
$*$ (b) Show that the net effect of parallel transporting a vector once around such a curve is rotation by a angle equal to this area.

4. For the torus in question 6.1(a) find
(a) Two conserved quantities in terms of $u^{\alpha} = dx^{\alpha}/ds$ that permit solution of the geodesic equation.
(b) The connection coefficients and hence the geodesic equation in its original form.
(c) The Riemann curvature component $R^{\theta}{}_{\phi\theta\phi}$.
(d) In terms of the component found in (c), and the metric components $g_{\theta\theta}$ and $g^{\phi\phi}$ compute all lowered components of the Ricci and Einstein tensors.

5. Maxwell's equations in SR can be written in the form

$$F^{\mu\nu}{}_{,\nu} = 4\pi J^{\mu}$$

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

where F is the electromagnetic field tensor (containing electric and magnetic fields), $\vec{J}$ is the four-current (containing charge density and flux) and $\tilde{A}$ is the four-potential (containing electric potential and magnetic vector potential).
(a) Write down the most likely GR forms of these equations, and simplify them using the results of problem 6.5.
$*$ (b) Substitute the second SR equation into the first, and show that there are two inequivalent GR generalisations of the resulting equation, obtained by commuting derivatives. Write down the difference in terms of the Ricci tensor, and find which of the two possibilities corresponds to substitution of the (unsimplified) equations found in part (a).

6. The equation $G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R^{\gamma}{}_{\gamma}$ is often expressed as saying that G is the "trace reverse" of R. Show that R is also the trace reverse of G. How would you define "trace reverse" in $d \neq 4$ dimensions? Would the Einstein tensor so defined be automatically divergence free?