General relativity solution sheet 2

1. The first equation reads
$$A_1 = B_{11}C_1 + B_{21}C_2 + B_{31}C_3$$
$$A_2 = B_{12}C_1 + B_{22}C_2 + B_{32}C_3$$
$$A_3 = B_{13}C_1 + B_{23}C_2 + B_{33}C_3$$
The second equation is invalid because the terms have different free indices, for example the first term has two free indices i and j, while the second term has free indices j and k. The third equation is
$$A_{11} + A_{22} + A_{33} = 6$$
The fourth equation reads
$$A_1 + B_{11}A_1 + B_{12}A_2 + B_{13}A_3 + C_{111} + C_{212} + C_{313} = 0$$
$$A_2 + B_{21}A_1 + B_{22}A_2 + B_{23}A_3 + C_{121} + C_{222} + C_{323} = 0$$
$$A_3 + B_{31}A_1 + B_{32}A_2 + B_{33}A_3 + C_{131} + C_{232} + C_{333} = 0$$
The fifth equation is invalid because an index cannot appear more than twice in a term.

2. $$A_{ij}S_{ji} = -A_{ji}S_{ji} = -A_{ji}S_{ij} = -A_{ij}S_{ji}$$
using the antisymmetry in the first equality, the symmetry in the second equality and exchanging the dummy indices in the third equality. Finally, any quantity equal to the negative of itself is zero.

3. Gauss' law (derived in lectures from spherical symmetry and the divergence theorem) states that the gravitational field is given by $|\mathbf{g}| = GM/r^2$ where M is the mass interior to the given radius, here zero. The observer cannot do local measurements to infer the existence of the hollow sphere, but can observe its effect on distant objects. In particular, it will lead to time dilation of the observer, hence a blue shift of incoming light. Examples of such a situation might include the Oort cloud of comets in the far outer solar system, and dark matter inferred from galaxy rotation curves and cosmological measurements.

4. (a) Use Gauss' law as in the previous question to obtain
$$\mathbf{g} = \begin{cases} -\frac{4}{3}\pi\rho G\mathbf{r} & r < R \\ -\frac{4}{3}\pi\rho G R^3 \frac{\mathbf{r}}{r^3} & r > R \end{cases}$$
where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = \sqrt{x^2 + y^2 + z^2}$. Differentiating, we find
$$\frac{\partial g_i}{\partial x_j} = \begin{cases} -\frac{4}{3}\pi\rho G\delta_{ij} & r < R \\ \frac{4}{3}\pi\rho G R^3 \frac{3x_i x_j - r^2\delta_{ij}}{r^5} & r > R \end{cases}$$
where δ_{ij} is one when $i = j$ and zero otherwise.
(b) We note that the answer in (a) is scale invariant, ie we get the same answer if we divide both R and r by two. Thus the mass distribution given by an infinite number of spheres $j = 0, 1, 2, \ldots$ of fixed density ρ, centred at $x_j = 2^{-j}$ and of radius $R_j = 2^{-j-2}$ leads to an infinite gravity gradient at the origin. This singularity is relatively weak (logarithmic with distance from the origin), so it is probably not of physical importance. It is possible to create similar "curvature singularities" in general relativity, again of minor physical interest. Much stronger singularities are created in black holes, but we can never observe them ("cosmic censorship conjecture").

5. In two dimensions, the effect of a rotation by a an angle θ followed by a rotation by an angle ϕ is a rotation by an angle $\theta+\phi$, irrespective of the order, hence abelian.

   In three dimensions, rotations do not commute, for example let X be a rotation by an angle $\pi/2$ about the x axis, and Y be a rotation by an angle $\pi/2$ about the y axis,

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

   it is easy to check that $XY \neq YX$.

6. (a) We have

$$\dot{x} = \dot{r}\sin\theta\cos\phi + \dot{\theta}r\cos\theta\cos\phi - \dot{\phi}r\sin\theta\sin\phi$$
$$\dot{y} = \dot{r}\sin\theta\sin\phi + \dot{\theta}r\cos\theta\sin\phi + \dot{\phi}r\sin\theta\cos\phi$$
$$\dot{z} = \dot{r}\cos\theta - \dot{\theta}r\sin\theta$$

   so the Lagrangian is

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) + \frac{mM}{r}$$

   (setting $G=1$ as in lecture notes). We use $p_i = \partial L/\partial \dot{q}^i$ to find

$$p_r = m\dot{r}$$
$$p_\theta = mr^2\dot{\theta}$$
$$p_\phi = mr^2\sin^2\theta\dot{\phi}$$

   with obvious inversion to find $\dot{q}^i$ in terms of p_i.

$$H = \sum_i p_i\dot{q}^i - L = \frac{1}{2m}(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}) - \frac{mM}{r}$$

   (b) Point A is at $(\sin\alpha, -\cos\alpha)$ while point B is at $(\sin\alpha+\sin\beta, -\cos\alpha-\cos\beta)$. The Lagrangian is given by kinetic minus potential energy,

$$L = \frac{m}{2}[2\dot{\alpha}^2 + \dot{\beta}^2 + 2\dot{\alpha}\dot{\beta}\cos(\alpha-\beta)] + mg(2\cos\alpha + \cos\beta)$$

   As above, we find

$$p_\alpha = 2m\dot{\alpha} + m\dot{\beta}\cos(\alpha-\beta)$$
$$p_\beta = m\dot{\beta} + \dot{\alpha}\cos(\alpha-\beta)$$

   Inverting, we find

$$\dot{\alpha} = \frac{p_\alpha - p_\beta\cos(\alpha-\beta)}{m[2-\cos^2(\alpha-\beta)]}$$
$$\dot{\beta} = \frac{2p_\beta - p_\alpha\cos(\alpha-\beta)}{m[2-\cos^2(\alpha-\beta)]}$$

   then after some algebra,

$$H = \frac{p_\alpha^2 + 2p_\beta^2 - 2p_\alpha p_\beta\cos(\alpha-\beta)}{2m(2-\cos^2(\alpha-\beta))} - mg(2\cos\alpha + \cos\beta)$$

   Using a metric and variational form of geodesics, we will discover a method of circumventing some of this algebra later in the course.