General relativity solution sheet 4

1. Use $\vec{a}\cdot\vec{b} = g_{\mu\nu}a^\mu b^\nu$ where $g_{\mu\nu}$ is a diagonal matrix with entries $(1, -1, -1, -1)$. We have

$$g_{\mu\nu}a^\mu(\alpha b^\nu + \beta c^\nu) = \alpha g_{\mu\nu}a^\mu b^\nu + \beta g_{\mu\nu}a^\nu c^\nu$$

using the usual distributive law for multiplication of real numbers. The second part is

$$g_{\mu\nu}a^\mu b^\nu = g_{\nu\mu}a^\nu b^\mu = g_{\nu\mu}b^\mu a^\nu = g_{\mu\nu}b^\mu a^\nu$$

relabelling indices, commuting multiplication of real numbers and noting that $g_{\mu\nu}$ is a symmetric matrix, ie $g_{\mu\nu} = g_{\nu\mu}$. The third part is

$$\partial_t(g_{\mu\nu}a^\mu b^\nu) = g_{\mu\nu}(\partial_t a^\mu)b^\nu + g_{\mu\nu}a^\mu(\partial_t b^\nu)$$

using the product rule for real functions, and noting that $g_{\mu\nu}$ is constant (in special relativity).

2. Let us write the 4-momenta as initial photon: $\vec{p}_1$, initial electron $\vec{p}_2$, final photon: $\vec{p}_3$, final electron $\vec{p}_4$. We can write three of these immediately (setting $c = 1$ for the moment).

$$\vec{p}_1 = (E, E, 0, 0)$$

$$\vec{p}_2 = (m, 0, 0, 0)$$

$$\vec{p}_3 = (E', E'\cos\theta, E'\sin\theta, 0)$$

assuming (without loss of generality) that the photon initially moves in the x-direction, and scatters into the x, y plane. We write down 4-momentum conservation as suggested by the hint,

$$\vec{p}_4 = \vec{p}_1 + \vec{p}_2 - \vec{p}_3$$

and square both sides, noting that $\vec{p}\cdot\vec{p}$ is the mass squared.

$$\vec{p}_4^2 = \vec{p}_1^2 + \vec{p}_2^2 + \vec{p}_3^2 + 2\vec{p}_1\cdot\vec{p}_2 - 2\vec{p}_1\cdot\vec{p}_3 - 2\vec{p}_2\cdot\vec{p}_3$$

$$m^2 = 0 + m^2 + 0 + 2mE - 2(EE' - EE'\cos\theta) - 2mE'$$

$$E' = \frac{mE}{m + E - E\cos\theta}$$

Finally, using $\lambda = hc/E$ and reintroducing the necessary factors of c, we get

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)$$

3. Recall that the relativistic scalar product (and hence whether a 4-vector is spacelike or timelike) does not depend on the choice of reference frame. Since $\vec{u}$ is timelike, there is an observer with 4-velocity $\vec{u}/\sqrt{\vec{u}\cdot\vec{u}}$ with respect to which $\vec{u} = (\sqrt{\vec{u}\cdot\vec{u}}, 0, 0, 0)$. A 4-vector orthogonal to this is of the form $\vec{v} = (0, x, y, z)$ which is clearly spacelike. The converse is false: $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$ in any reference frame are clearly two orthogonal spacelike vectors.

4. The relevant transformation laws are $\vec{e}_{\alpha'} = \Lambda^\beta_{\alpha'}\vec{e}_\beta$ and $\tilde{\omega}^{\alpha'} = \Lambda^{\alpha'}_\beta\tilde{\omega}^\beta$. From the first, we have

$$\Lambda^\beta_{\alpha'} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

noting the convention that the upper index corresponds to the rows of the matrix. The second transformation law has the primed and unprimed indices in the opposite positions. Since the operation of both Λ ("Lorentz") transformations leads back to the first basis, corresponding to the identity transformation, these two matrices are inverses. Thus we find

$$\Lambda^{\alpha'}_{\beta} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and hence $\tilde{\omega}^{1'} = \tilde{\omega}^1 - \tilde{\omega}^2$ and $\tilde{\omega}^{2'} = \tilde{\omega}^2$.

5.

$$\tilde{\omega}^\mu(\vec{x}) = \tilde{\omega}^\mu(x^\nu \vec{e}_\nu) = x^\nu \tilde{\omega}^\mu(\vec{e}_\nu) = x^\mu$$

using linearity of $\tilde{\omega}^\mu$ at the second step, and $\tilde{\omega}^\mu(\vec{e}_\nu) = \delta^\mu_\nu$ at the third step.

$$g(\vec{x}, \vec{y}) = \tilde{x}(\vec{y}) = x_\mu \tilde{\omega}^\mu(\vec{y}) = x_\mu y^\mu$$

since $y^\mu = \tilde{\omega}^\mu(\vec{y})$ from above. We have $x_\mu = \tilde{x}(\vec{e}_\mu)$ (analagous to the first part). Therefore

$$x_\mu = \tilde{x}(\vec{e}_\mu) = g(\vec{x}, \vec{e}_\mu) = g(x^\nu \vec{e}_\nu, \vec{e}_\mu) = g_{\nu\mu} x^\nu$$

since g is linear with respect to its first argument. For any vector $\vec{x}$ we have

$$\tilde{e}_\mu(\vec{x}) = g(\vec{e}_\mu, \vec{x}) = g(\vec{e}_\mu, x^\nu \vec{e}_\nu) = g_{\mu\nu} x^\nu = g_{\mu\nu} \tilde{\omega}^\nu(\vec{x})$$

and therefore

$$\tilde{e}_\mu = g_{\mu\nu} \tilde{\omega}^\nu$$

6. For any one-forms $\tilde{\rho}$ and $\tilde{\sigma}$ and vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ we have

$$\begin{aligned} T(\tilde{\rho}, \tilde{\sigma}, \vec{a}, \vec{b}, \vec{c}) &= T(\rho_\alpha \tilde{\omega}^\alpha, \sigma_\beta \tilde{\omega}^\beta, a^\gamma \vec{e}_\gamma, b^\delta \vec{e}_\delta, c^\epsilon \vec{e}_\epsilon) \\ &= T^{\alpha\beta}{}_{\gamma\delta\epsilon} \rho_\alpha \sigma_\beta a^\gamma b^\delta c^\epsilon \\ &= T^{\alpha\beta}{}_{\gamma\delta\epsilon} \vec{e}_\alpha(\tilde{\rho}) \vec{e}_\beta(\tilde{\sigma}) \tilde{\omega}^\gamma(\vec{a}) \tilde{\omega}^\delta(\vec{b}) \tilde{\omega}^\epsilon(\vec{c}) \\ &= T^{\alpha\beta}{}_{\gamma\delta\epsilon} \vec{e}_\alpha \otimes \vec{e}_\beta \otimes \tilde{\omega}^\gamma \otimes \tilde{\omega}^\delta \otimes \tilde{\omega}^\epsilon (\tilde{\rho}, \tilde{\sigma}, \vec{a}, \vec{b}, \vec{c}) \end{aligned}$$

The coefficients $T^{\alpha\beta}{}_{\gamma\delta\epsilon}$ are determined by the second equality:

$$T^{\alpha\beta}{}_{\gamma\delta\epsilon} = T(\tilde{\omega}^\alpha, \tilde{\omega}^\beta, \vec{e}_\gamma, \vec{e}_\delta, \vec{e}_\epsilon)$$

7. The gradient of a scalar field is a one-form: $\nabla F = F_{,i} \omega^i$. We can convert this into a vector with the help of the metric, $(\nabla F)^i = g^{ij} F_{,j}$. The curl is a further antisymmetric derivative, by analogy with the ordinary cross product: $(\nabla \times \mathbf{V})^i = \epsilon^{ij}{}_k V^k{}_{,j}$. Thus we have

$$(\nabla \times \nabla F)^i = \epsilon^{ij}{}_k g^{kl} F_{,jl} = \epsilon^{ijl} F_{,jl}$$

noting that the metric does not depend on position. This is zero, because ϵ^{ijl} is antisymmetric on (j, l), while $F_{,jl}$ is symmetric (since partial derivatives commute). The argument is the same as in lectures: interchange indices and use the symmetry properties to return them to their former places. This gives the original expression with a minus sign, thus it must vanish.