General relativity solution sheet 6

1. (a) We have

$$dx = -\sin\theta(a + b\cos\phi)d\theta - \cos\theta b\sin\phi d\phi$$

$$dy = \cos\theta(a + b\cos\phi)d\theta - \sin\theta b\sin\phi d\phi$$

$$dz = b\cos\phi d\phi$$

so

$$ds^2 = dx^2 + dy^2 + dz^2 = (a + b\cos\phi)^2 d\theta^2 + b^2 d\phi^2$$

(b) We have

$$dx = \cos\theta dr - r\sin\theta d\theta$$

$$dy = \sin\theta dr + r\cos\theta d\theta$$

$$dz = adr$$

so

$$ds^2 = dx^2 + dy^2 + dz^2 = (1 + a^2)dr^2 + r^2 d\theta^2$$

2. We have

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 = \left(\frac{d\theta}{d\psi}\right)^2 d\psi^2 + \sin^2\theta d\phi^2$$

which is conformally flat if

$$\frac{d\theta}{d\psi} = \sin\theta$$

$$\psi = \int \frac{d\theta}{\sin\theta} = \ln\tan\frac{\theta}{2}$$

or

$$\theta = 2\arctan e^{\psi}$$

A general metric in n dimensions has $n(n+1)/2$ independent components (ie undetermined functions). This can be reduced by a suitable choice of the n functions $x^{\alpha'}(x^{\beta})$, leaving $n(n-1)/2$. On the other hand, a conformally flat matric has only a single undetermined function, the scalar field. Thus it seems plausible (and in fact is the case) that all two dimensional manifolds are conformally flat, but not in higher dimensions.

3. Substituting the four eigenvectors (ie $z = 1, i, -1, -i$) we find that this matrix has eigenvalues $(3, -1, -1, -1)$ respectively, so a suitable change of basis leads to a diagonal matrix with these entries. Scaling the first coordinate by a factor of $\sqrt{3}$ leads to Minkowski form.

4. We use

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}(g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta})$$

with $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{\phi\phi} = r^2\sin^2\theta$ and off-diagonal elements zero (see for example problem 4.3). The inverse matrix is, trivially, $g^{rr} = 1$, $g^{\theta\theta} = 1/r^2$ and $g^{\phi\phi} = 1/(r^2\sin^2\theta)$.

Thus the only nonvanishing derivatives are $g_{\theta\theta,r} = 2r$, $g_{\phi\phi,r} = 2rsin^2\theta$ and $g_{\phi\phi,\theta} = 2r^2 \sin\theta\cos\theta$. We have from these

$$\Gamma^{\theta}_{\theta r} = \Gamma^{\theta}_{r\theta} = \frac{1}{2} g^{\theta\theta} g_{\theta\theta,r} = 1/r$$

$$\Gamma^{r}_{\theta\theta} = -\frac{1}{2} g^{rr} g_{\theta\theta,r} = -r$$

$$\Gamma^{\phi}_{\phi r} = \Gamma^{\phi}_{r\phi} = \frac{1}{2} g^{\phi\phi} g_{\phi\phi,r} = 1/r$$

$$\Gamma^{r}_{\phi\phi} = -\frac{1}{2} g^{rr} g_{\phi\phi,r} = -r\sin^2\theta$$

$$\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \frac{1}{2} g^{\phi\phi} g_{\phi\phi,\theta} = \cot\theta$$

$$\Gamma^{\theta}_{\phi\phi} = -\frac{1}{2} g^{\theta\theta} g_{\phi\phi,\theta} = -\sin\theta\cos\theta$$

Alternatively, we could have obtained the same results by differentiating the basis vectors, or the variational method.

5. (a) We have

$$u^{\beta} v^{\alpha}{}_{;\beta} - v^{\beta} u^{\alpha}{}_{;\beta} = u^{[\beta} v^{\alpha]}{}_{,\beta} - v^{\beta} u^{\alpha}{}_{,\beta} + u^{\beta}\Gamma^{\alpha}_{\gamma\beta} v^{\gamma} - v^{\beta}\Gamma^{\alpha}_{\gamma\beta} u^{\gamma}$$

but the Γ terms are identical (exchanging β and γ and using the symmetry of Γ), and cancel.

(b) We have

$$\tilde{\nabla}_{[\alpha}\rho_{\beta\ldots]} = \partial_{[\alpha}\rho_{\beta\ldots]} - \rho_{\mu[\ldots}\Gamma^{\mu}_{\beta\alpha]} - \ldots$$

In this case all the Γ terms vanish because Γ is symmetric on its lower indices, and we are antisymmetrising them.

(c) First let us write the given formula explicitly for the metric,

$$(\ln|g|)_{,\alpha} = g^{\beta\gamma} g_{\gamma\beta,\alpha}$$

Now we have

$$F^{\alpha\beta\ldots}{}_{;\alpha} = F^{\alpha\beta\ldots}{}_{,\alpha} + \Gamma^{\alpha}_{\mu\alpha} F^{\mu\beta\ldots} + \Gamma^{\beta}_{\mu\alpha} F^{\alpha\mu\ldots} + \ldots$$

All but the first Γ term vanishes because Γ is symmetric on μ and α, but F is antisymmetric. The trace of Γ itself simplifies,

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2} g^{\alpha\nu}(g_{\nu\mu,\alpha} + g_{\nu\alpha,\mu} - g_{\mu\alpha,\nu}) = \frac{1}{2} g^{\alpha\nu} g_{\nu\alpha,\mu}$$

since the other two terms cancel — there is symmetry on α and ν. The remaining term is simply

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2}(\ln|g|)_{,\mu} = \frac{(\sqrt{|g|})_{,\mu}}{\sqrt{|g|}}$$

from the given relation, which leads directly to the desired result. This simplification for divergences suggests we should calculate with vectors and tensors multiplied by $\sqrt{|g|}$; such quantities are called vector or tensor densities.