General relativity solution sheet 7

1. Given the fact that the covariant derivative of a vector
$$A^{\alpha}_{;\beta} = A^{\alpha}_{,\beta} + \Gamma^{\alpha}_{\mu\beta} A^{\mu}$$
transforms as a tensor, we find
$$A^{\alpha'}_{,\beta'} + \Gamma^{\alpha'}_{\mu'\beta'} A^{\mu'} = \Lambda^{\alpha'}_{\gamma} \Lambda^{\delta}_{\beta'} (A^{\gamma}_{,\delta} + \Gamma^{\gamma}_{\epsilon\delta} A^{\epsilon})$$
Then on the LHS we write
$$A^{\mu'} = \Lambda^{\mu'}_{\epsilon} A^{\epsilon}$$
which is the vector transformation law, and
$$A^{\alpha'}_{,\beta'} = (\Lambda^{\alpha'}_{\gamma} A^{\gamma})_{,\delta} \Lambda^{\delta}_{\beta'}$$
which is the chain rule for partial derivatives. The derivatives of $\vec{A}$ cancel, leaving
$$A^{\gamma} \Lambda^{\alpha'}_{\gamma,\delta} \Lambda^{\delta}_{\beta'} + \Gamma^{\alpha'}_{\mu'\beta'} \Lambda^{\mu'}_{\epsilon} A^{\epsilon} = \Lambda^{\alpha'}_{\gamma} \Lambda^{\delta}_{\beta'} \Gamma^{\gamma}_{\epsilon\delta} A^{\epsilon}$$
Relabeling γ for ϵ in the first term, and noting that the equation holds for all $\vec{A}$ we have
$$\Gamma^{\alpha'}_{\mu'\beta'} \Lambda^{\mu'}_{\epsilon} = \Lambda^{\alpha'}_{\gamma} \Lambda^{\delta}_{\beta'} \Gamma^{\gamma}_{\epsilon\delta} - \Lambda^{\alpha'}_{\epsilon,\delta} \Lambda^{\delta}_{\beta'}$$
Finally multiplying both sides by $\Lambda^{\epsilon}_{\nu'}$ (which is the matrix inverse of the $\Lambda^{\mu'}_{\epsilon}$) we obtain the result:
$$\Gamma^{\alpha'}_{\nu'\beta'} = \Lambda^{\alpha'}_{\gamma} \Lambda^{\delta}_{\beta'} \Lambda^{\epsilon}_{\nu'} \Gamma^{\gamma}_{\epsilon\delta} - \Lambda^{\delta}_{\beta'} \Lambda^{\epsilon}_{\nu'} \Lambda^{\alpha'}_{\epsilon,\delta}$$
There are a couple of equivalent forms of this expression, but it should include the usual tensor transformation law, together with a term containing a derivative of Λ, with indices matching correctly.

2. We know that $g_{\mu\nu,\alpha\beta}$ is symmetric on both pairs of indices, but there are no other relationships. In n dimensions, a pair of symmetric indices has $\sum_{i=1}^{n} i = n(n+1)/2$ independent components, so both pairs have $n^2(n+1)^2/4$.

   On the other hand, $\Lambda^{\alpha}_{\beta',\gamma'\delta'}$ is equal to
$$\frac{\partial^3 x^{\alpha}}{\partial x^{\beta'} \partial x^{\gamma'} \partial x^{\delta'}}$$
which is symmetric on the lower three indices; the upper index is free and takes all values. In n dimensions, this will lead to
$$n \sum_{j=1}^{n} \sum_{i=1}^{j} i = n \sum_{j=1}^{n} \frac{j(j+1)}{2} = n \frac{n(n+1)(n+2)}{6}$$
Substituting values of n:

n	$g_{\mu\nu,\alpha\beta}$	$\Lambda^{\alpha}_{\beta',\gamma'\delta'}$	curvature
1	1	1	0
2	9	8	1
3	36	30	6
4	100	80	20

3. (a) Area is

$$\int_0^{\psi}\int_0^{2\pi}\sqrt{|g|}d\phi d\theta = \int_0^{\psi}\int_0^{2\pi}\sin\theta d\phi d\theta = 2\pi(1-\cos\psi)$$

where ψ is the fixed value of θ (also below). (b) Use a parametrisation $\lambda = \phi$ for the path given by $\theta = \psi$ (constant). The parallel transport equation for a vector $\vec{V}$ is

$$u^{\alpha}V^{\beta}{}_{;\alpha} = 0$$

where $u^{\alpha} = dx^{\alpha}/d\lambda = (0,1)$ is the tangent vector. The connection coefficients (see the solution 5.4) are $\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \cot\theta$, $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$. In full, the parallel transport equation reads

$$V^{\theta}{}_{,\phi} - \sin\psi\cos\psi V^{\phi} = 0.$$

$$V^{\phi}{}_{,\phi} + \cot\psi V^{\theta} = 0$$

which gives

$$V^{\theta}{}_{,\phi\phi} + \cos^2\psi V^{\theta} = 0$$

with solution

$$V^{\theta} = A\sin(\phi\cos\psi) + B\cos(\phi\cos\psi)$$

$$V^{\phi} = A\mathrm{cosec}\psi\cos(\phi\cos\psi) - B\mathrm{cosec}\psi\sin(\phi\cos\psi)$$

After one time round the path, ϕ increases by 2π, so we deduce a rotation of $2\pi\cos\psi$, which after a change of orientation, is equivalent to $2\pi(1-\cos\psi)$ from part (a).

Why should these answers be related? It turns out that curvature is additive in two dimensions, and that the amount of rotation of a parallel transported vector is given by the amount of curvature enclosed within the curve. Of course a sphere has constant curvature, so the total curvature is proportional to the area.

4. The line element is given by the solution to 6.1(a):

$$ds^2 = (a+b\cos\phi)^2 d\theta^2 + b^2 d\phi^2$$

(a) In a similar manner to the sphere (covered in lectures), the value of the Hamiltonian $g^{\mu\nu}p_{\mu}p_{\nu}/2$ is conserved; in terms of the tangent vector (4-velocity in GR) $\vec{u}$ it is

$$\frac{1}{2}g_{\mu\nu}u^{\mu}u^{\nu} = \frac{1}{2}[(a+b\cos\phi)^2(u^{\theta})^2 + b^2(u^{\phi})^2]$$

of course we can ignore the half. We can also think of this conservation as normalisation of the tangent vector.

The other conservation law follows from the fact that the metric does not depend on θ, hence

$$p_{\theta} = (a+b\cos\phi)^2 u^{\theta}$$

is conserved.

(b) We have $g_{\theta\theta,\phi} = -2b\sin\phi(a + b\cos\phi)$ which leads to

$$\Gamma^{\theta}_{\theta\phi} = \Gamma^{\theta}_{\phi\theta} = \frac{-b\sin\phi}{a + b\cos\phi}$$

$$\Gamma^{\phi}_{\theta\theta} = \sin\phi(\frac{a}{b} + \cos\phi)$$

so the geodesic equation reads

$$\frac{d^2\theta}{d\lambda^2} - \frac{2b\sin\phi}{a + b\cos\phi}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0$$

$$\frac{d^2\phi}{d\lambda^2} + \sin\phi(\frac{a}{b} + \cos\phi)\left(\frac{d\theta}{d\lambda}\right)^2 = 0$$

(c) Use the general formula

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\mu\delta}\Gamma^{\mu}_{\beta\gamma}$$

from which the only contributions come from

$$R^{\theta}{}_{\phi\theta\phi} = -\Gamma^{\theta}_{\phi\theta,\phi} - \Gamma^{\theta}_{\theta\phi}\Gamma^{\theta}_{\phi\theta}$$

which reduces to

$$R^{\theta}{}_{\phi\theta\phi} = \frac{b\cos\phi}{a + b\cos\phi}$$

Note that this is physically plausible: It is positive on the outside of the torus, and negative on the inside; it is small for large a since in that limit the torus resembles a cylinder; it is singular at the centre when $a = b$.

(d) Since the metric is diagonal, raising and lowering indices is obtained by multiplying and dividing by a single metric component. We have

$$R_{\theta\phi\theta\phi} = g_{\theta\theta}R^{\theta}{}_{\phi\theta\phi}$$

Now the Ricci tensor is obtained by contracting the first and third indices,

$$R_{\phi\phi} = g^{\mu\nu}R_{\mu\phi\nu\phi} = g^{\theta\theta}R_{\theta\phi\theta\phi} = R^{\theta}{}_{\phi\theta\phi}$$

$$R_{\theta\phi} = R_{\phi\theta} = 0$$

$$R_{\theta\theta} = g^{\mu\nu}R_{\mu\theta\nu\theta} = g^{\phi\phi}R_{\phi\theta\phi\theta} = g^{\phi\phi}R_{\theta\phi\theta\phi} = g^{\phi\phi}g_{\theta\theta}R^{\theta}{}_{\phi\theta\phi}$$

where we have made extensive use of the fact that the Riemann tensor is antisymmetric with repect to exchange of indices $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$. Note that we can summarise these results in the equation

$$R_{\mu\nu} = g_{\mu\nu}g^{\phi\phi}R^{\theta}{}_{\phi\theta\phi}$$

The curvature scalar is obtained by contracting on the remaining two indices,

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = 2g^{\phi\phi}R^{\theta}{}_{\phi\theta\phi}$$

Finally the Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

since $R_{\mu\nu}$ is proportional to $g_{\mu\nu}$. Einstein's field equations are not very useful in two dimensions!

5. (a) Use the comma-to-semicolon rule:

$$4\pi J^\mu = F^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{|g|}} (\sqrt{|g|} F^{\mu\nu})_{,\nu}$$

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

In the first simplication, we need the fact that F is antisymmetric, which follows from the second equation.
(b) Direct substitution leads to

$$A^{\nu,\mu}{}_{\nu} - A^{\mu,\nu}{}_{\nu} = 4\pi J^\mu$$

which is equivalent to

$$A^{\nu}{}_{,\nu}{}^{\mu} - A^{\mu,\nu}{}_{\nu} = 4\pi J^\mu$$

however when we convert these to covariant derivatives they no longer commute: the difference is

$$A^{\nu;\mu}{}_{\nu} - A^{\nu}{}_{;\nu}{}^{\mu} = R^{\nu}{}_{\sigma\nu}{}^{\mu} A^\sigma = R_\sigma{}^\mu A^\sigma$$

Substitution of the results of part (a) gives

$$A^{\nu;\mu}{}_{\nu} - A^{\mu;\nu}{}_{\nu} = 4\pi J^\mu$$

which is indeed the correct generalisation.

6.

$$G^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} G^\gamma{}_\gamma = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R^\gamma{}_\gamma - \frac{1}{2} g^{\alpha\beta} \left(R^\gamma{}_\gamma - \frac{1}{2} g^\gamma{}_\gamma R^\delta{}_\delta \right)$$

Now $g^\gamma{}_\gamma$ is just summing the diagonal of the unit matrix, so it gives 4, the dimension of the space. Thus the RHS collapses to simply $R^{\alpha\beta}$ as required. To make this work in dimension d, put q instead of $1/2$ in the above calculation and show that everything cancels if $q = 2/d$. The derivation of the Einstein tensor from the Bianchi identities in the lectures does not involve the dimension of the space, so it is not the trace reverse of R in dimensions other than four. There are other pathologies with gravity in other dimensions, for example for $d > 4$ there are no stable orbits, even in the nonrelativistic limit. In $d = 3$ the Einstein tensor determines the curvature, so vacuum solutions are flat. In $d = 2$ the Einstein tensor vanishes identically. There are benefits to studying these, however, for example low dimensional gravity is easier to quantize.