Open circular billiards and the Riemann Hypothesis

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The distribution of primes

Prior to Riemann the following results were known:

- Euclid: There are infinitely many primes.
- Euler (1737): Product form of the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}$$

  showing that the sum of reciprocals of primes diverges.

- Legendre (c 1800) conjecture that the primes have density $1/(A \ln x + B)$
- Chebyshev (c 1850) number of primes up to x, $\pi(x)$ satisfies for sufficiently large x

$$0.89 Li(x) < \pi(x) < 1.11 Li(x)$$

  using the Logarithmic Integral

$$Li(x) = \int_0^x \frac{dx}{\ln x}$$

The 1859 paper

Georg Friedrich Bernhard Riemann (1826-1866) published an 8-page paper "On the number of primes less than a given magnitude" in German, revolutionising analytic number theory. New approaches included

- The study of $\zeta(s)$ for complex argument and continuation to $\Re s < 1$,
- The study of the zeros of $\zeta(s)$,
- The use of the Fourier transform,
- The use of Möbius inversion formula,
- Series and integral representations of $\pi(x)$ and related functions,

and of course the Riemann Hypothesis. A translation and discussion of the paper can be found in H. M. Edwards "Riemann's zeta function" 1974 (republished 2001).

Properties of $\zeta(s)$

Using the definition of the Gamma function

$$\frac{\Gamma(s)}{n^s} = \int_0^\infty e^{-nx} x^{s-1} dx$$

and summing over n, Riemann represented the zeta function in the entire complex plane apart from a pole at $s = 1$ as

$$2 \sin \pi s \Gamma(s) \zeta(s) = i \int_\infty^\infty \frac{(-x)^{s-1} dx}{e^x - 1}$$

where the contour surrounds the positive real axis in an anticlockwise direction. He then used properties of the Gamma function to show that the combination

$$\Gamma(s/2) \pi^{-s/2} \zeta(s)$$

is invariant under the transformation s to $1 - s$. Thus there are zeros at negative even integers and possibly with real parts between zero and one. Euler knew exact values for positive even integers; this formula extends these to negative odd integers and zero.

Riemann's result

Riemann introduced a function $f(x)$ counting integers of the form $p^k < x$ with weight $1/k$. He then gave a formula in terms of the zeros of the zeta function with real parts between zero and one ρ:

$$f(x) = Li(x) - \sum_{\rho} Li(x^{\rho}) - \ln 2 - \int_x^{\infty} \frac{dt}{t(t^2-1)\ln t}$$

where the sum is ordered by the magnitude of the imaginary part of ρ. Thus the primes are distributed with an average density of $1/x$ corrected by terms depending on the location of the zeta zeros. In particular:

- The prime number theorem (1896): There are no zeta zeros with real part 1, and hence $\pi(x)$ satisfies

$$\pi(x) \sim Li(x)$$

- The Riemann hypothesis, that all nontrivial zeros have real part 1/2, is equivalent to the statement

$$\pi(x) - Li(x) = O(x^{1/2+\epsilon})$$

Multiplicative number theory

We can replace the "1" in the numerator of $\zeta(s)$ by a multiplicative function, ie satisfying

$$f(mn) = f(m)f(n) \qquad gcd(m,n) = 1$$

and the Euler product and often functional equations will follow. The function needs only to be defined at powers of primes. For example, the Möbius function, used in the Möbius transform:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \qquad \mu(p^k) = \begin{cases} -1 & k = 1 \\ 0 & k > 1 \end{cases}$$

The Euler totient function, giving the number of coprime integers up to n.

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}, \qquad \phi(p^k) = (p-1)p^{k-1}$$

Dirichlet characters (periodic, supported on integers coprime to the period, and completely multiplicative)

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

Subsequent progress

- Numerics: The first few billion zeros lie on the critical line
- 40% of zeros lie on line, and 99% of $\rho = \beta + i\gamma$ satisfy $|\beta - 1/2| \leq 8/\ln|\gamma|$.
- Values of $\zeta(1/2 + t)$: RH implies the Lindelöf hypothesis, that this is $O(t^\epsilon)$; best result so far is $O(t^{1/6+\epsilon})$.
- Connections/restatements using Fourier transforms, Probability, Functional Analysis, spectral theory including of random matrices, other number theoretic functions.
- Study of wider classes of L-functions, including the above questions.

Billiards

A billiard is a dynamical system, arising naturally in mechanics and optics, in which a point particle moves with constant velocity except for specular collisions with the boundary. The behaviour depends on the shape of the boundary:

Integrable Circle, ellipse, rectangle, three triangles.

Pseudo-integrable Polygons with rational angles.

Parabolic Polygons with irrational angles.

Mixed Mushroom, generic curved.

Hyperbolic Stadium, Sinai, Cardioid.

Exotic External field, Riemannian metric, global topology, thin barrier, unbounded domain, alternative reflection laws, moving boundary.

Open Trajectories absorbed at hole(s), subset of boundary, incidence angles, or the interior.

Basic facts/notation for 2D billiards

The most convenient approach is usually the collision map $\Phi(x)$ where x denotes arc length l and angle ψ at the boundary. There is a function $T(x)$ giving the continuous time from x to $\Phi(x)$, from which continuous time properties may be calculated.

The equilibrium measure and mean collision time for a billiard with domain $D \subset R^2$ are

$$d\mu_0 = \frac{\cos\psi dl d\psi}{2|\partial D|} \qquad \int T(x) d\mu_0 = \frac{\pi|D|}{|\partial D|}$$

$\langle\rangle$ will indicate an average with respect to μ_0, so that correlation functions are written $\langle fg \circ \Phi^n\rangle - \langle f\rangle\langle g\rangle$ for functions $f, g : M \to R$.

Some history of circular billiards

An ancient geometry problem, called "Alhazen's problem" (probably due to Ptolemy) consists of finding the angle to aim a billiard particle to collide once with a circular boundary and hit another specified point; there are related internal and external problems and they require solution of a quartic equation.

A billiards game on a circular table, with no pockets, was proposed in 1890 by Charles Dodgson (aka Lewis Carroll), whose mathematics focused on Euclidean geometry and logic. Circular and elliptical billiard tables have occasionally been constructed and patented.

Present work: summary

- Idea: Experiments can measure escape of particles from a cavity; what can this tell us about the dynamics?
- Start with simplest case: long time survival probability of a circular billiard with holes in the boundary.
- Leading order behaviour: survival probability inversely proportional to total hole size in small hole limit.
- Next order, for example subtracting one and two hole probabilities with equal total size: The Riemann Hypothesis.
- Published: Phys. Rev. Lett. **94** 100201 (2005)
- Further work on open billiards: EPL 2007, Physica D 2009, ongoing.

Open dynamics

Define a "hole" as a subset of phase space, at which trajectories escape. Then measure the survival probability P(t) given an initially specified distribution of particles (often μ_0). This has many applications:

- Illuminating structures and Poincare recurrence in the corresponding closed system, for example fractal measures in a Hamiltonian phase space for which the invariant measure is uniform.

- Describing metastability and rare events, for example chemical reaction rates, migration of asteroids.

- Relating dynamics to thermodynamics via the escape rate formalism of Gaspard et al

- Physical escape or scattering problems, eg microlasers, room acoustics

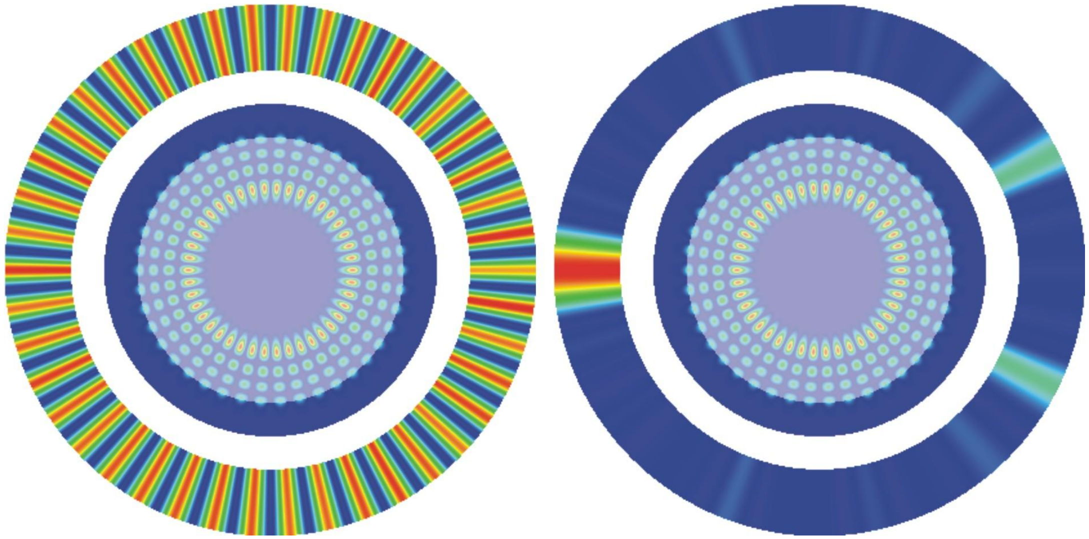
- Nondestructive investigation of internal dynamics by measurements of escaping particles.

Open billiards

Open billiards provide:

- Examples of open dynamical systems covering many cases from integrable to strongly hyperbolic.
- Connections with number theory.
- A description of many physical systems and experiments involving a particle or small(ish) wavelength wave in a cavity.
- Models for statistical mechanics and molecular dynamics.

Application: Microlasers



Microlasers are cavities containing an active (lasing) medium that trap light due to total internal reflection. Thus the "hole" is the entire boundary, but only trajectories sufficiently close to the normal direction can escape. Placing a small scatterer (right) breaks the symmetry and allows strong directivity in conjunction with low losses. Here we have an internal wavelength about 1/6 the radius, ie not too small, yet geometric optics is still useful in determining the optimal position of the scatterer. [CD, Morozov, Sieber, Waalkens, 2008-9; numerous theoretical and experimental papers in the physics literature]

Decay of the survival probability

Most mathematics for open dynamical systems is restricted to the uniform hyperbolic case, and gives an exponential escape rate. More generally the survival probability could have

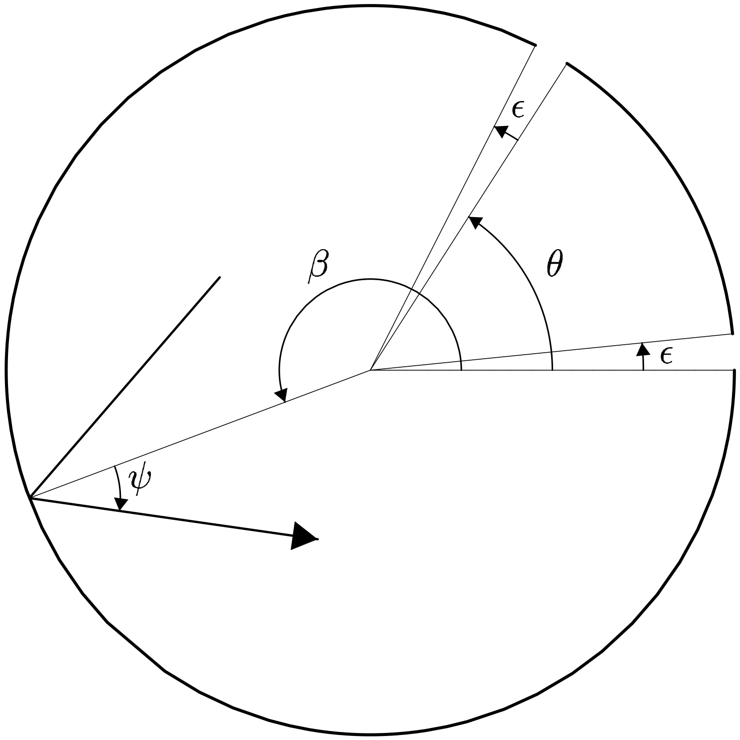
- $P(n) = 0$ for finite n, eg if the hole is large, or some models with square scatterers.
- $P(n)$ decays superexponentially, eg the map $\Phi(x) = \sqrt{x} + x$ on a finite interval containing zero.
- $P(n) \sim 1/n$, eg a marginal family of orbits in a 2D billiard such as the circle or stadium.
- $P(n) \to C$, a constant, eg an elliptical billiard with a small hole at one end; no orbits passing between the foci escape.

A variety of methods is needed.

The open circle

Dynamics $(\varphi, \psi) \to (\varphi + \pi - 2\psi, \psi)$ is just rotation on a circle, periodic for $\psi/\pi = 1/2 - m/n$, dense and uniform for irrational ψ/π. For the open problem, put holes at $\varphi \in [0, \epsilon] \cup [\theta, \theta + \epsilon]$ and find time to escape $t = 2\cos\psi_0 N(\varphi_0, \psi_0)$ where N counts collisions. Starting from the equilibrium measure at time zero, consider the probability $P(t)$ of remaining until time t, specifically

$$P_\infty = \lim_{t\to\infty} tP(t)$$



Long lived orbits are nearly periodic

- The $N+1$ values $\varphi_0, \varphi_1, \ldots \varphi_N$ contain two at a distance less than ϵ if $N+1 > 2\pi/\epsilon$.

- Define the period n to be the smallest positive integer so that $|\varphi_n - \varphi_0| < \epsilon$. Thus $n < 2\pi/\epsilon$.

- In units of n collisions, the orbit precesses slowly enough to be captured by the holes.

- All very long living orbits have small precession and are close to a periodic orbit.

Precisely: Let $t > 4[2\pi/\epsilon]$, then every connected component of the set of (φ, ψ) that survive to time t contains a unique interval of never escaping periodic orbits.

Counting long lived orbits

$$\psi = \psi_{m,n} + \eta, \quad \eta \ll \epsilon$$

Orbit will survive for at least time t, $t/2\cos\psi_{m,n}$ collisions if

$$\varphi_0' = \left(\epsilon + \frac{\eta t}{\cos\psi_{m,n}}, \theta'\right) \cup \left(\theta' + \epsilon + \frac{\eta t}{\cos\psi_{m,n}}, \frac{2\pi}{n}\right)$$

if $\eta > 0$. Adding up these contributions:

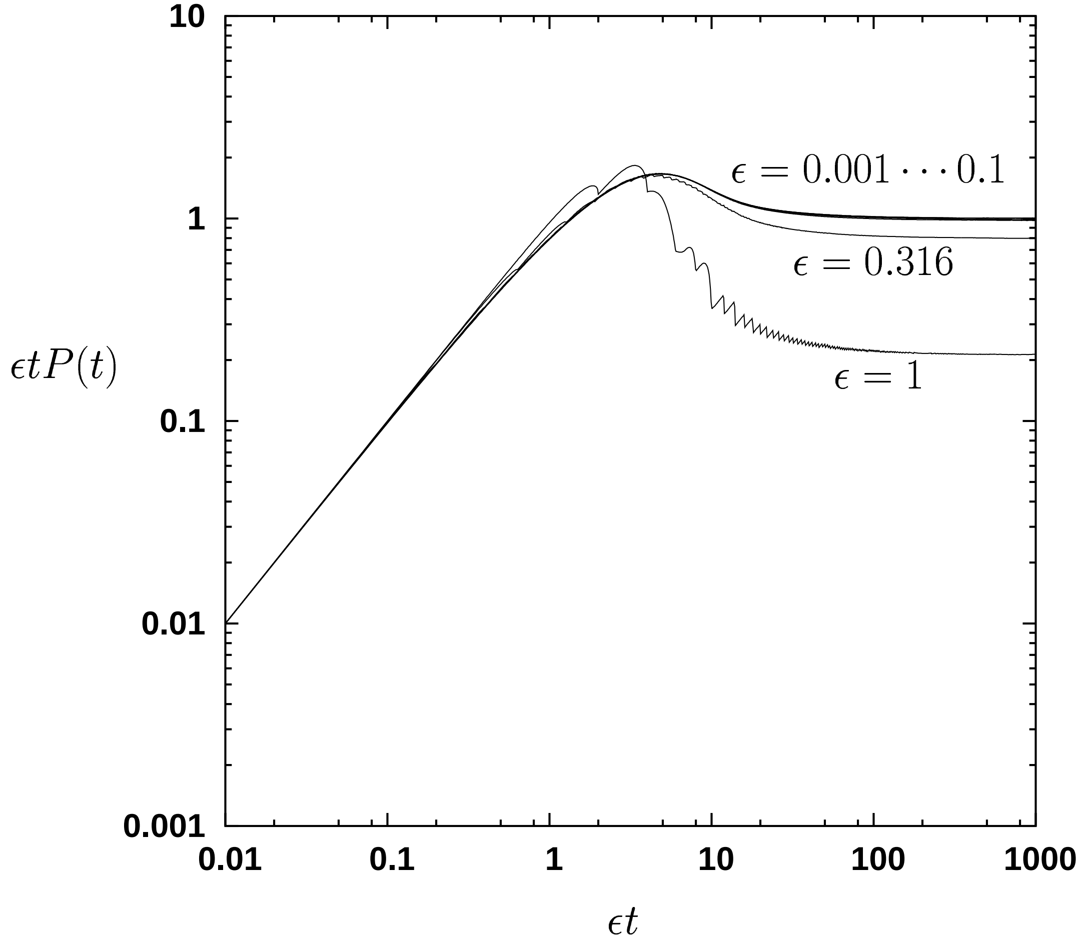
$$tP(t) \sim \frac{1}{4\pi}\sum_{m,n} n[g\left(\frac{2\pi}{n} - \theta' - \epsilon\right) + g(\theta' - \epsilon)]\sin^2\frac{\pi m}{n}$$

where

$$g(x) = \begin{cases} x^2 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

and the sum is over $1 \leq m < n < 2\pi/\epsilon$, $gcd(m,n) = 1$. The symbol $\sim$ means take $t \to \infty$, in which limit upper and lower bounds converge.

Finite time scaling



Summation over m

Use Ramanujan

$$\sum_{m=1}^{n-1} \exp(2\pi i m/n) = \mu(n)$$

where the sum is over $gcd(m,n) = 1$, μ is the Möbius function

$$\mu(n) = \begin{cases} 1 & n = 1 \\ -1 & n \quad \text{prime} \\ \mu(a)\mu(b) & n = ab, \quad gcd(a,b) = 1 \\ 0 & a^2|n, \quad a > 1 \end{cases}$$

and we find

$$P_\infty = \frac{1}{8\pi} \sum_{n=1}^{\infty} n[\phi(n) - \mu(n)][g\left(\frac{2\pi}{n} - \theta' - \epsilon\right) + g(\theta' - \epsilon)]$$

where

$$\phi(n) = n \prod_{p|n} (1 - p^{-1})$$

is the Euler totient function, the number of positive integers $\leq n$ which are coprime to n. Note $\phi(1) = 1$, so the $n = 1$ term vanishes.

Small hole limit

Mellin transforms:

$$\tilde{P}(s) = \int_0^\infty P_\infty \epsilon^{s-1} d\epsilon$$

$$P_\infty = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \epsilon^{-s} \tilde{P}(s) ds$$

leads to

$$\begin{aligned} P_\infty &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{ds\, \epsilon^{-s} (2\pi)^{s+1}}{2s(s+1)(s+2)} \sum_{n=1}^{\infty} \frac{\phi(n) - \mu(n)}{n^{s+1}} \\ &\quad \times \left\{ \left[1 - f\left(\frac{n\theta}{2\pi} \right) \right]^{s+2} + f\left(\frac{n\theta}{2\pi} \right)^{s+2} \right\} \end{aligned}$$

where f denotes fractional part. The small ϵ expansion is obtained by summing residues.

Dirichlet characters

The $\phi(q)$ conjugacy classes modulo q which are coprime to q form a group under multiplication. The group is Abelian, so it has $\phi(q)$ one dimensional representations, called Dirichlet characters $\chi(a)$. They are extended to all conjugacy classes by $\chi(a) = 0$ if $gcd(a, q) > 1$. Thus by definition

$$\chi(ab) = \chi(a)\chi(b)$$

and from representation theory we have the orthogonality relation

$$\frac{1}{\phi(q)} \sum_{\chi} \bar{\chi}(a)\chi(n) = \delta_{a,n}$$

where the bar is complex conjugation and the δ is modulo q. A character is called even (odd) if $\chi(-1) = 1$ (respectively -1).

Rational θ

Write $\theta = 2\pi r/q$. Then we need

$$\sum_{n \equiv a \pmod{q}} \frac{\phi(n) - \mu(n)}{n^{s+1}}$$

First remove common factors, primes indicate division by $b = gcd(a, q)$. This leaves

$$\sum_{n' \equiv a' \pmod{q'}} \frac{\phi(bn') - \mu(bn')}{(bn')^{s+1}}$$

Then insert the orthogonality relation of Dirichlet characters to give

$$\frac{1}{\phi(q')} \sum_{\chi} \bar{\chi}(a') \sum_{n'=1}^{\infty} \chi(n') \frac{\phi(bn') - \mu(bn')}{(bn')^{s+1}}$$

Then expand n' in prime factors and use the multiplicative properties of χ, ϕ and μ to give

$$\frac{1}{b^{s+1}\phi(q')} \sum_{\chi} \frac{\bar{\chi}(a')[\phi(b)L(s,\chi) - \mu(b)]}{L(s+1,\chi) \prod_{p|b}[1 - \chi(p)p^{-s-1}]}$$

Dirichlet L-functions

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left[1 - \frac{\chi(p)}{p^s}\right]^{-1}$$

$$L(s,1) = \zeta(s) \prod_{p|q'} (1 - p^{-s})$$

Poles: The zeta function has a pole at $s = 1$. Nontrivial L-functions have no poles.

Trivial zeroes: Occur at nonpositive integers of the same parity as χ, except that $\zeta(0) = -1/2$.

Critical zeroes: Other zeroes exist only with real part 1/2 according to the (generalised) Riemann hypothesis. This is equivalent to statements about the distribution of prime numbers (in specific conjugacy classes).

Final expression

For two holes separated by angle $\theta = 2\pi r/q$, we find

$$P_\infty = \sum_j \mathrm{Res}_{s=s_j} \tilde{P}(s)\epsilon^{-s}$$

$$\begin{aligned}\tilde{P}(s) &= \frac{(2\pi)^{s+1}}{2s(s+1)(s+2)} \\ &\times \sum_{a=1}^{q} \frac{[1-f(\frac{ap}{q})]^{s+2} + f(\frac{ap}{q})^{s+2}}{b^{s+1}\phi(q')} \\ &\times \sum_{\chi} \frac{\bar{\chi}(a')[\phi(b)L(s,\chi) - \mu(b)]}{L(s+1,\chi)\prod_{p|b}[1-\chi(p)p^{-s-1}]}\end{aligned}$$

with $b = gcd(a,q)$, $a' = a/b$, $q' = q/b$, characters χ modulo q', f is the fractional part, L is a Dirichlet L-function.

Special cases

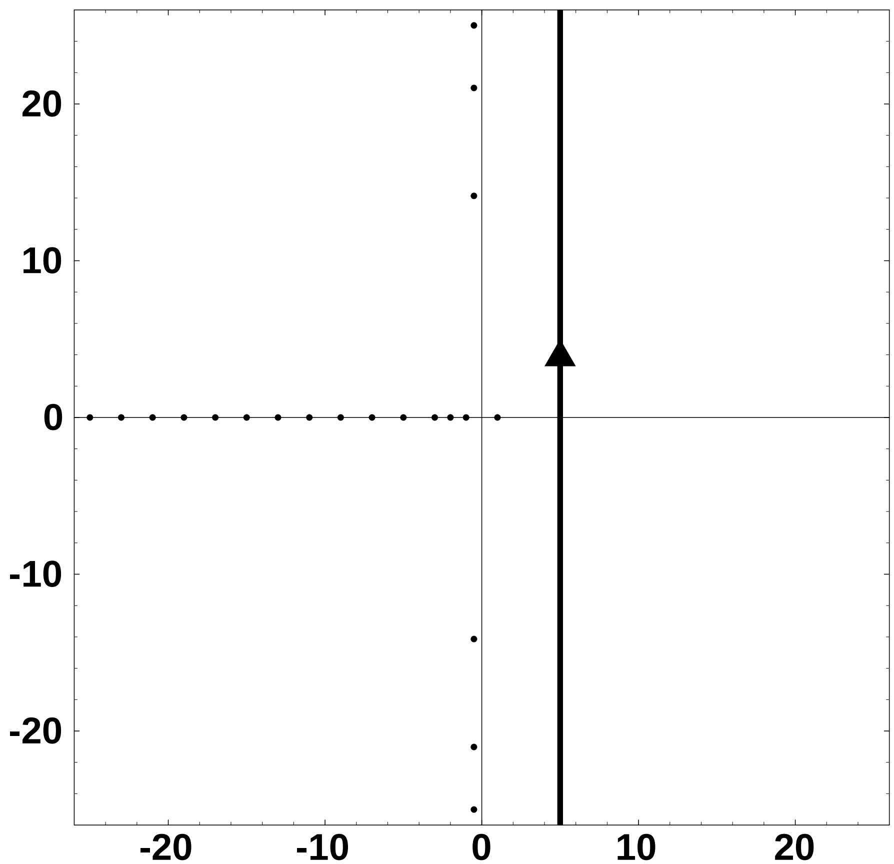
Odd characters cancel, so when $q = 1, 2, 3, 4, 6$ only Riemann zeta functions appear. The one and symmetric two hole cases are

$$\tilde{P}_1(s) = \frac{(2\pi)^{s+1}[\zeta(s) - 1]}{2s(s+1)(s+2)\zeta(s+1)}$$

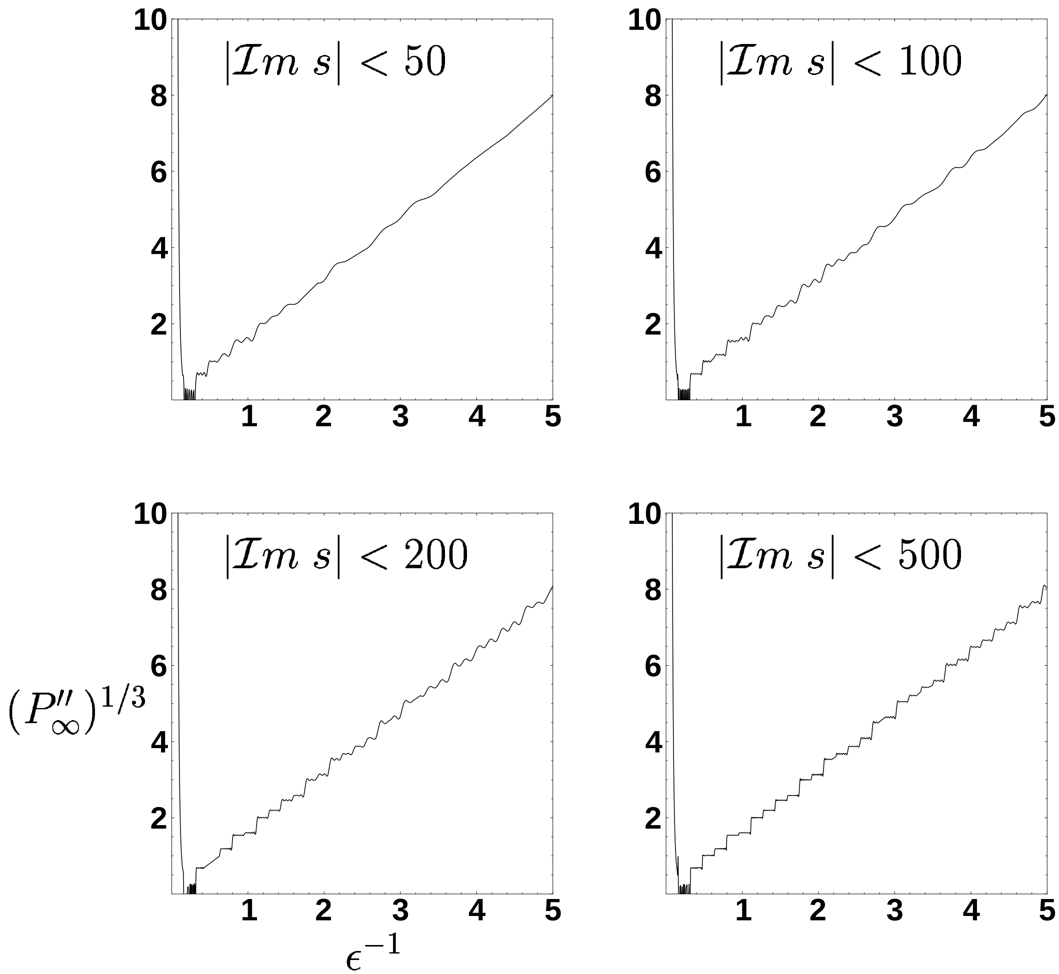
$$\tilde{P}_2(s) = \frac{(\pi)^{s+1}\zeta(s)}{s(s+1)(s+2)\zeta(s+1)}$$

with poles at odd $s \leq 1$ and at $\mathcal{R}e\, s = -1/2$ assuming the Riemann Hypothesis, and for $q = 1$ also $s = -2$.

The contour



Steps as sums over zeta zeros



Riemann reformulated

The Riemann hypothesis is thus

$$\lim_{\epsilon\to 0}\lim_{t\to\infty}\epsilon^{\delta}(tP_1(t)-2/\epsilon)=0$$

for every $\delta > -1/2$ where $P_1(t)$ is the probability of remaining after time t from an initial equilibrium distribution in the one hole problem.

An equivalent formulation is

$$\lim_{\epsilon\to 0}\lim_{t\to\infty}\epsilon^{\delta}t[P_1(t)-2P_2(t)]=0$$

with $P_2(t)$ the symmetric 2-hole probability.

The generalised Riemann hypothesis implies that the statement is also true for two holes with rational θ, but the converse statement is open, as is the case of irrational θ.

Irrational θ in 2-hole

The fractional parts are uniformly distributed, so make a "mean field" approximation, replacing the sum by an integral:

$$\begin{aligned}
\left\langle g\left(\frac{2\pi}{n} - \theta' - \epsilon\right) + g(\theta' - \epsilon)\right\rangle &= \frac{n}{3\pi}\left(\frac{2\pi}{n} - \epsilon\right)^3 \\
\langle \phi(n) \rangle &= \frac{6n}{\pi^2} \\
\langle \mu(n) \rangle &= 0
\end{aligned}$$

so that

$$tP(t) \approx \frac{1}{24\pi^2}\int_0^{2\pi/\epsilon} n^2 \frac{6n}{\pi^2}\left(\frac{2\pi}{n} - \epsilon\right)^3 dn = \frac{1}{\epsilon}$$

which is the same as all tested values of rational $\theta \neq 0$. Taking an irrational θ as a limit of rational θ suggests that $\tilde{P}(s)$ cannot be continued past a line of singularities at $\mathcal{R}e\, s = -1/2$.

Comparison with other open billiards

- Circle:

$$\begin{aligned} P_\infty &= \lim_{t\to\infty} tP(t) \\ &= \frac{1}{8\pi}\sum_{n=1}^{\infty} n[\phi(n)-\mu(n)]\left[g\left(\frac{2\pi}{n}-\theta'-\epsilon\right)+g(\theta'-\epsilon)\right] \\ &= \frac{2}{\epsilon}+o(\epsilon^{1/2-\delta}) \qquad \text{1 hole; assumes RH} \end{aligned}$$

- Diamond (dispersing billiard with corners):

$$\begin{aligned} \gamma &= -\lim_{t\to\infty}\frac{1}{t}\ln P(t) = \frac{\epsilon}{\langle T\rangle}+O(\epsilon^2) \\ \gamma_{AB} &= \gamma_A+\gamma_B-\frac{1}{\langle T\rangle}\left\{\sum_{j=-\infty}^{\infty}\langle u_A^0 u_B^j\rangle+\sum_{n=3}^{\infty}[Q_{nAB}(0)-Q_{nA}(0)-Q_{nB}(0)]\right\} \end{aligned}$$

- Stadium (defocusing billiard with intermittency):

$$P_\infty = \frac{(3\ln 3+4)\left[(a+h_1)^2+(a-h_2)^2\right]}{(16a+8\pi r)}$$

Other remarks

The combination ϵt appears in $P(t)$ for

- All billiards, sufficiently small t.
- The circle, small ϵ and all t (numerically).
- The diamond, small ϵ and large t (at least).
- The stadium, not for large t.

Periodic orbits play an important role for

- The diamond, correlations increase when the hole covers short periodic orbits
- The circle, is completely determined by periodic orbits at long times
- The stadium, is dominated by its marginal family of periodic orbits plus the neighbourhood.
- Fixed squares: No periodic orbits, no long time survival probability!

The future

- The ellipse, a more generic integrable model.
- ϵt scaling and dynamical effects.
- Other dynamical behaviour, eg mixed systems.
- Higher dimensions.
- Exotic billiards.
- Quantum connections.
- Applications: microlasers, room acoustics, ...

Thank you for your attention!