Mathematical Methods 3 — 2006

Sheet A — Revision

This worksheet is revision of topics that you should be familiar with from previous courses. This include partial differentiation, PDE's, solving ODE's, the use of separation of variables, and complex integration. Some of these questions are quite tough, but should not involve anything mathematically new to you.

1. Find all the first and second partial derivatives of the following functions

$$\text{(i) } u(x,y) = \frac{x}{x^2+y^2}, \qquad \text{(ii) } u(r,\theta) = \cos 3\theta\, \mathrm{e}^{-r^2/2}, \qquad \text{(iii) } u(x,y,z) = f(ax - by + cz)$$

   (a, b, c are constants and f is an arbitrary function.)

2. Use partial differentiation to eliminate the functions f and g to obtain a partial differential equation for the following functions (e.g. $u(x,y) = f(x) + g(y)$ satisfies $u_{xy} = 0$)

   (a) $u(x,y) = f(ax)$, $\quad a$ constant
   (b) $u(x,t) = f(x^2/\kappa t)$, $\quad \kappa$ constant
   (c) $u(x,t) = f(x - ct) + g(x + ct)$, $\quad c$ constant
   (d) $u(x,y) = f(x + iy) + g(x - iy)$

3. Show that $u = f(xy) + xg(y/x)$ is a general solution of $x^2 u_{xx} - y^2 u_{yy} = 0$.

4. Use the substitution $v = u_y$ to find the general solution, $u(x,y)$ of $u_{xy} + u_y = 0$. Your answer should involve two arbitrary functions $g(x)$ and $h(y)$, say.

5. The heat equation (an example of a *parabolic equation*) in one spatial dimension is given by $u_t = \kappa u_{xx}$. Assume a solution of the form $u(x,t) = Af(\kappa t)\mathrm{e}^{-[xf(\kappa t)]^2}$ and show that this satisfies the heat equation provided the function $f(v)$ satisfies the differential equation $f'(v) = -2[f(v)]^3$.

   Hence deduce that a solution of the heat equation may be expressed in the form

$$u(x,t) = \frac{A}{2\sqrt{(\kappa t + c)}}\mathrm{e}^{-x^2/4(\kappa t + c)}$$

   for a constant, $c = 1/[4f^2(0)]$. Under what conditions would such a solution be valid ?

6. *Tricky.* The motion of one-dimensional waves propagating on the surface of a shallow fluid having a slowly varying depth $h(x)$ may be approximated by the following ODE

$$\frac{d}{dx}\left(h(x)\frac{d\eta}{dx}\right) + \frac{\sigma^2}{g}\eta = 0$$

   where $\eta(x)$ represents the height of the surface of the fluid, σ is the angular frequency of the wave motion and g is gravitational acceleration.

Transform this equation by making the substituion $\eta(x) = \psi(x)f(h(x))$ and eliminating the term proportional to $\psi'(x)$ that results to obtain the ODE

$$\frac{d^2\psi}{dx^2} + \kappa(x)\psi = 0$$

where

$$\kappa(h(x)) = \frac{\sigma^2}{gh(x)} - \frac{1}{2}\left(\frac{h''(x)}{h(x)}\right) + \frac{1}{4}\left(\frac{h'(x)}{h(x)}\right)^2$$

[HINT: As an intermediate step you should find that you need to solve the equation $2hf'(h) + f(h) = 0$ for $f(h)$.]

7. Laplace's equation in two dimensions is given by

$$u_{xx} + u_{yy} = 0.$$

Use separation of variables (i.e. assume a solution of the form $u(x, y) = f(x)g(y)$) to show that the solution of Laplace's equation in the semi-infinite rectangle $0 \le x < \infty$, $0 \le y \le 1$ satisfying

$$\begin{aligned} u(x,0) = u(x,1) &= 0 \qquad \text{for } 0 < x < \infty \\ u_x(0,y) &= 1 \qquad \text{for } 0 < y < 1 \\ \text{and } u(x,y) &\to 0 \qquad \text{as } x \to \infty \text{ for } 0 < y < 1 \end{aligned}$$

is given by

$$u(x, y) = -\sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi^2} \sin((2n-1)\pi y)\mathrm{e}^{-(2n-1)\pi x}.$$

8. Use the separation of variables method to find the solution of the heat equation $u_{xx} = u_t$ in $0 < x < \pi$, $t > 0$ with boundary conditions on the ends $x = 0$ and $x = \pi$ given by $u(0, t) = 0$, $u_x(\pi, t) = 0$ for $t > 0$ and an initial condition at time $t = 0$ given by $u(x, 0) = 1$ for $0 < x < \pi$. Your solution should be expressed in terms of an infinite sum and you should confirm that $u(x, t) \to 0$ as $t \to \infty$.

9. These questions are designed as a revision of complex function theory (contour integration). Show that

   (a) $\displaystyle\int_0^\infty \frac{\sin ka}{k} dk = \begin{cases} \frac{1}{2}\pi, & a > 0 \\ -\frac{1}{2}\pi, & a < 0 \end{cases}$

   (b) $\displaystyle\int_0^\infty \frac{\cos kx}{k^2 + a^2} dk = \frac{\pi}{2a}\mathrm{e}^{-a|x|}$

10. *Hard.* The flexural oscillations of an elastic beam of finite length occupying $x \in [0, a]$ are governed by the PDE

$$D\frac{\partial^4 W}{\partial x^4} + m\frac{\partial^2 W}{\partial t^2} = 0, \qquad 0 < x < a$$

for the displacement $W(x, t)$ where m is the mass per unit length of the beam and D is the "flexural rigidity". The conditions that the left-hand end at $x = 0$ is clamped and the end at $x = a$ is free are given by

$$W(0,t) = \frac{\partial}{\partial x}W(0,t) = \frac{\partial^2}{\partial x^2}W(a,t) = \frac{\partial^3}{\partial x^3}W(a,t) = 0.$$

(a) Assume a time harmonic dependence of $W(x,t) = w(x)\mathrm{e}^{i\sigma t}$ where σ is the angular frequency may be factored and show that $w(x)$ now satisfies the ODE

$$\frac{d^4 w}{dx^4} - \mu^4 w = 0, \qquad 0 < x < a \tag{1}$$

with $\mu^4 = m\sigma^2/D$ and

$$w(0) = \frac{d}{dx}w(0) = \frac{d^2}{dx^2}w(a) = \frac{d^3}{dx^3}w(a) = 0. \tag{2}$$

(b) Find the four linearly independent general solutions of (1) using the standard method of assuming a solution of the form $w = A\mathrm{e}^{rx}$ and solving the "characteristic equation" for r.

(c) Hence solve (1) subject to the four boundary conditions (2) to show that the modes of oscillation are given by

$$w(x) = (\sinh \mu x - \sin \mu x) + \frac{(\sinh \mu a + \sin \mu a)}{(\cosh \mu a + \cos \mu a)}(\cosh \mu x - \cos \mu x)$$

(modulo an arbitrary multiplicative constant) where μ must satisfy

$$\cosh \mu a \cos \mu a = -1. \tag{3}$$

(d) Using graphical considerations, approximate the location of a sequence of values of $\mu = \mu_n$ from (3). These values determine the possible frequencies $\sigma = \sigma_n$ that the beam may undergo free oscillations from the relation $\sigma_n^2 = D\mu_n^4/m$

(e) Experiment (what fun !!).

The equations in this question describe the oscillations of a ruler held fixed against a table at one end and "pinged" at the other end (provided the ruler has a fairly rectangular cross section).

The task is to determine experimentally the "flexural rigidity", D of the ruler. Fix a length of ruler protruding over the table. You also need to find the weight of the ruler and work out the mass per unit length, m. Ping the ruler and try and estimate the frequency, f (in Hertz – oscillations per second) at which it makes oscillations. To get σ, which is the *angular frequency*, use $\sigma = 2\pi f$. The ping you gave the ruler will have excited the first mode or *fundamental mode* of oscillation. That is, $\mu = \mu_1 \approx \pi/2a$ from part (d). So now we have $D = m\sigma^2/\mu_1^4 \approx 16ma^4\sigma^2/\pi^4$ and you should be able to calculate D.

This may seem a fairly trivial and pointless exercise, but determining the flexural rigidity and modes of oscillation of steel girders, aircraft panels and ice sheets (for example) are extremely important in engineering design.