Mathematical Methods 3 — 2006

Sheet I — Green's functions for ODE's

In this worksheet we construct Green's functions and find general solutions in integral form to inhomogeneous second order ODE's with both unmixed and periodic boundary conditions and initial conditions

1. Consider the equation of motion for a particle of mass $m > 0$ on the end of a damped spring with spring constant $\sigma > 0$ and damping $k > 0$ under external forcing $h(t)$:

$$m\frac{d^2y}{dt^2} + k\frac{dy}{dt} + \sigma y = h(t)$$

Rewrite this equation in self-adjoint form.

2. Find the Green's function $g(x, \xi)$, $0 < x, \xi < b$, for the operator

$$L = -\frac{d^2}{dx^2} + k^2$$

on $0 < x < b$ subject to the homogeneous unmixed boundary conditions

$$g(0, \xi) = 0, \qquad g'(b, \xi) = 0, \qquad 0 < \xi < b$$

where the prime indicates differentation w.r.t. x.

3. By defining and constructing a suitable Green's function, find an integral representation for the solution of

$$\begin{aligned} \frac{d^2y}{dx^2} + 9y &= f(x), \qquad 0 < x < 1, \\ y(0) &= y(1), \\ y'(0) &= y'(1). \end{aligned}$$

4. Consider the following problem for the function $y(x)$:

$$\begin{aligned} \frac{d^2y}{dx^2} + y &= f(x), \qquad 0 < x < \pi/2 \\ y(0) &= c_0, \\ y(\pi/2) - y'(\pi/2) &= c_1. \end{aligned}$$

Find a suitable Green's function $g(x, \xi)$ for this problem and hence obtain an integral representation of the solution $y(x)$.

5. Consider the problem of a mass $m > 0$ falling under gravity, G, subject to air resistance kv where $k > 0$, $v = dy/dt$ is the velocity, so that the distance fallen $y(t)$ may be described by the system

$$\frac{d^2y}{dt^2} + \lambda\frac{dy}{dt} = G$$

where $\lambda = k/m$. Initially, $y(0) = H$, $y'(0) = 0$.

(a) Express the ODE in self-adjoint form, writing $(L_t y)(t) = f(t)$ where

$$L_t \equiv \frac{d}{dt}\left(p(t)\frac{d}{dt}\right) + q(t)$$

and $p(t)$, $q(t)$ and $f(t)$ should all be determined in terms of given quantities.

(b) Define a Green's function $g(t, \tau)$ for this problem, and hence show that

$$g(t, \tau) = \frac{(\mathrm{e}^{-\lambda t} - \mathrm{e}^{-\lambda \tau})}{\lambda} H(\tau - t)$$

(c) Use Green's formula given in the lecture notes for ODE's with initial conditions, to obtain the solution for $y(t)$. Hence, confirm that the ultimate free fall velocity that the mass can achieve is G/λ.

6. Consider the problem for a function $y(r)$ given by

$$\frac{d}{dr}\left(\frac{1}{r}\frac{dy}{dr}\right) - \frac{3}{r^3}y = f(r), \qquad 0 < r < 1$$

with boundary conditions $y(0) = 0$ and $y(1) = c_1$.

(a) Define a Green's function by

$$Lg(r, \rho) = \delta(r - \rho), \qquad 0 < r, \rho < 1$$

where $L = \dfrac{d}{dr}\left(\dfrac{1}{r}\dfrac{d}{dr}\right) - \dfrac{3}{r^3}$ with $g(0, \rho) = g(1, \rho) = 0$. Show that

$$g(r, \rho) = \begin{cases} \frac{1}{4} r^3(\rho^3 - \rho^{-1}), & 0 < r < \rho \\ \frac{1}{4} \rho^3(r^3 - r^{-1}), & \rho < r < 1 \end{cases}$$

[HINT: To solve the homogeneous ODE, use $g(r, \rho) = Ar^\lambda$ as a trial solution.]

(b) Using $g(r, \rho)$ show that

$$y(\rho) = c_1 \rho^3 + \int_0^1 g(r, \rho) f(r) dr.$$

(c) Confirm that $y(0) = 0$ and that $y(1) = c_1$.