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Stochastic Stabilization of Chaos and the Cosmic Microwave Background

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The cosmic microwave background (CMB) is a contemporary echo of the Big Bang. The recently announced WMAP 1-year sky maps [1] provide exceptionally accurate data for the CMB, making it possible to probe the physics of the early Universe down to an unprecedented level of detail. Fluctuations in the CMB have a distribution that is close to gaussian (i.e. normal) [2]. There has been considerable interest in identifying physical mechanisms that might lead to deviations from the gaussian distribution [2,3]. One class of cosmological models that have been much studied are those in which the Universe has constant negative curvature; all photon trajectories are then exponentially unstable and the gaussian distribution of the CMB fluctuations has been related to general properties of quantum wavefunctions in chaotic systems [4,5,6]. Inhomogeneities in the distribution of matter imply a non-constant curvature. Here we show that, surprisingly, random perturbations in the curvature can stabilize photon trajectories. We argue that this leads to quantifiable non-gaussian fluctuations in the CMB, as well as having other potentially important cosmological consequences.

Keywords: Cosmic Microwave Background; Stochastic Stabilization; Cosmological Chaos

1. Introduction

Just after its birth, the Universe was very hot and dense; the protons and electrons formed a gas of ionized matter coupled to radiation through the constant scattering of photons. As it expanded and cooled there came a point when the radiation decoupled from the matter - this happened approximately 380,000 years after the Big Bang. That radiation now forms the Cosmic Microwave Background (CMB) [7]. The CMB is isotropic down to one part in 10^5. At this scale there are temperature variations, or fluctuations, which carry the imprint of structures in the early Universe [7,8].

One interesting and much explored possibility is that the Universe has negative spatial curvature [9,10,11,12,13]. For this to be consistent with observed properties of the CMB, the absolute value of the curvature must be small: the Universe is flat to within about 1%. If the curvature is taken to be constant (and negative) then all of the geodesics are unstable: neighbouring geodesics diverge from each other exponentially quickly. Generally, cosmological models with negative spatial curvature are regarded as open, or non-compact. However, in the case of non-positively curved three space the local metric structure does not determine the global topology uniquely; in particular, it is possible for this to be compact [5,14]. The geodesics on compact surfaces of constant negative curvature are strongly chaotic: as well as being unstable the dynamics is rapidly mixing. It has been suggested that the latter property accounts for the remarkable homogeneity observed in the Universe [9]. Quantum wavefunctions in systems where the classical mechanics is strongly chaotic may be modelled by gaussian random functions (i.e. by a superposition of random waves)[6], explaining the fact that fluctuations in the CMB have a value distribution that is close to gaussian [4,5]. These wavefunctions have, in various systems, also been observed to exhibit scars along short periodic orbits [15], and this has been linked with non-isotropic structures in the CMB and in the distribution of galaxies [4].

It is obvious that the Universe is not exactly homogeneous and isotropic. Matter is not smoothly distributed, but is organized into galaxies, galaxy clusters and even superclusters of galaxy clusters. (This complex hierarchy is, according to inflation theory [16], a result of the gravitational amplification of quantum fluctuations in the very early Universe [17,18,19,20,21]). Consequently, the curvature is not in reality constant; it fluctuates. We show here that, surprisingly, these fluctuations can stabilize geodesics, even if they are random; that is, if they are seen by a photon, as it moves, to be *stochastic*.

2. Stochastic stabilization of photons

In order to establish the principle of stochastic stabilization, and to describe some implications for cosmology, the calculations we report contain the essential ingredients for the effect, but ignore many additional, comparatively weaker phenomena present in the early Universe. With this in mind, we begin with an expanding isotropic homogeneous (Friedmann) cosmology perturbed by density fluctuations with nonrelativistic velocities, neglecting perturbations of vector and tensor character such as gravitational waves, and pressure fluctuations, for example caused by relativistic neutrinos. In the coordinate system called the 'conformal Newtonian' or 'longitudinal' gauge [22], with the spatial variables in Robertson-Walker form, such a spacetime is described by the metric

$$ds^2 = R^2(\tau)\left\{(1+2\Phi)d\tau^2 - \frac{1-2\Phi}{[1+\frac{K}{4}(x^2+y^2+z^2)]^2}(dx^2+dy^2+dz^2)\right\}. \quad (1)$$

Here, $R(\tau)$ is the scale factor or spatial curvature radius of the Universe, $\tau = \int R^{-1}dt$ is conformal time, $\Phi \ll 1$ is the Newtonian gravitational potential, $K=-1$

is the sign of the spatial curvature corresponding to a hyperbolic geometry, (x, y, z) are comoving coordinates, expanding at the same rate as the Universe, and units are chosen in which Newton's constant G and the speed of light c are equal to unity.

The paths of photons through the spacetime are null geodesics. Since the equations for these are conformally invariant [23], the factor $R^2(\tau)$ may be neglected as it does not appear in the final results. The separation of two close photon trajectories is determined by the geodesic deviation equation, which, when written in terms of conformal time with respect to a basis aligned to the instantaneous direction of motion, leads to an equation for the separation in the two orthogonal directions:

$$\frac{d^2}{d\tau^2}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = -\begin{pmatrix}K + 2\Phi_{\xi\xi} + \Phi_{\zeta\zeta} & 2\Phi_{\xi\eta}\\ 2\Phi_{\xi\eta} & K + 2\Phi_{\eta\eta} + \Phi_{\zeta\zeta}\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \tag{2}$$

where constant exponential separation arising from the spatial curvature K, whose value we take to be -1, is modified by the second derivatives of the Newtonian potential, namely tidal forces.

These tidal forces are, in principle, entirely characterised by a complete knowledge of the fluctuations in the matter density. As it moves, a photon will see them as a time-varying force. We shall be interested in the case when this force fluctuates rapidly (because the speed of light is large) and randomly (i.e. stochastically) with zero mean.

In order to illustrate the qualitative behaviour of the solutions of this class of equations, it is sufficient to consider a single component, satisfying

$$\frac{d^2u}{d\tau^2} = -\left(Af(\tau) - 1\right)u. \tag{3}$$

Here A is a control parameter and $f(\tau)$ is a stochastic forcing function, which we take to have zero mean. This has a mechanical analogy: it also describes an inverted pendulum with a vertically moving pivot in the limit of small oscillations; see Figure 1. In this case u denotes the angular displacement from the vertical and the forcing term describes the height of the pivot. The gravitational force destabilizing the pendulum corresponds to the smooth hyperbolic geometry of the unperturbed cosmology, while the motion of the pivot corresponds to the metric perturbations induced by fluctuations in the matter density.

Although equation (3) may appear rather simple, its solutions exhibit a rich variety of qualitatively different behaviours in the long-time limit. If $A = 0$ the solutions grow like $\exp(\tau)$. When $A \neq 0$ we are interested in when the frequency ω characterising the fluctuations in $f(\tau)$ is large. We therefore write $u(\tau) =< u(\tau) > +u_f(\tau)$, where $< \cdots >$ denotes a local time average over scales large compared to ω^{-1}. Hence $< u >$ varies slowly, and u_f is small is varies rapidly. Averaging over the rapid fluctuations [24] gives, as $\omega \to \infty$, that

$$\frac{d^2}{d\tau^2}\langle u\rangle = \left(1 - A^2\langle v^2\rangle\right)\langle u\rangle, \tag{4}$$

where $v(\tau) = \int f(\tau)d\tau$ and the constant of integration is chosen so that v has zero mean. It follows that if $A^2\langle v^2\rangle < 1$ then u grows exponentially quickly as $\tau \to \infty$,

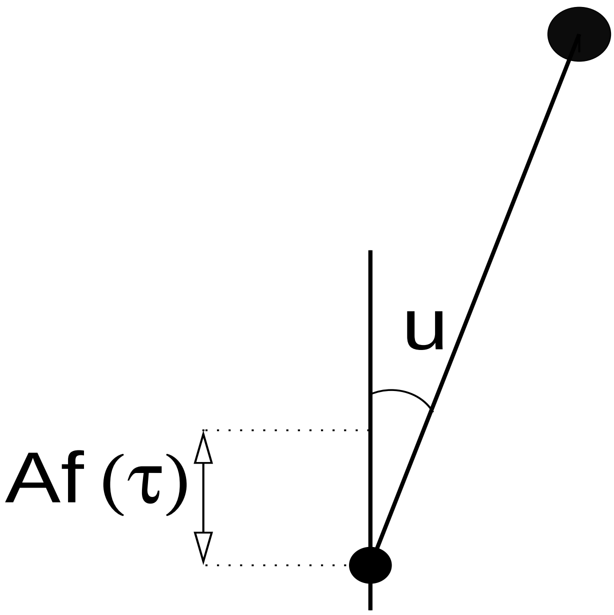

Fig. 1. The pendulum referred to in the text. The pivot moves along a vertical line, its height given at time τ by $Af(\tau)$. The arm of the pendulum makes an angle u, which is assumed to be small, with the upward vertical. When $A = 0$, $u = 0$ is an unstable equilibrium.

but if $A^2 \left\langle v^2 \right\rangle > 1$ then u is bounded and oscillatory (i.e. stabilized). Numerical simulations are shown in next section.

3. Numerical results

The photon stabilization, and the accuracy of (4), is illustrated by numerical simulations, the results of which are represented in Figures 2, 3 and 4. In these simulations we took $f(\tau) = \sum_n \sin(\omega_n \tau + \varphi_n)$, with 100 frequencies ω_n chosen at random from [120, 600] and phases φ_n chosen at random from $(-\pi, \pi]$. Thus $\left\langle v^2 \right\rangle = \frac{1}{2} \sum_n \frac{1}{\omega^2} \approx \frac{1}{1440}$ and so for $A < \sqrt{1440} \approx 38$ the expectation is that u grows like $\exp\left(\tau\sqrt{1 - A^2/1440}\right)$, while for $A > \sqrt{1440}$ it is expected to oscillate with frequency $\sqrt{(A^2/1440 - 1)}$. In order to compare with astrophysical time-scales, note that $\tau = 50$ corresponds to a red-shift ratio of $50^2 = 2500$ in a matter dominated Universe $\left(R \sim \sqrt{t}\right)$ when the spatial curvature scale is of the same order as the space-time curvature. Observations indicate that the Universe is closer to spatial flatness than this, so $\tau = 50$ corresponds to a time longer than that since the Big Bang.

Returning to (2), one can split the variables ξ and η into slow and fast components as for u. Performing a local time average we find, in the high-frequency limit,

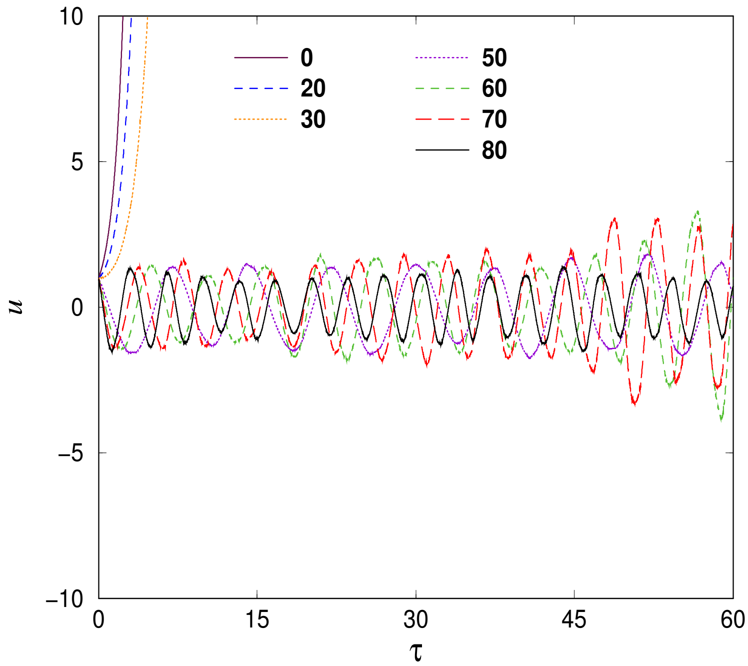

Fig. 2. Solutions of (3) with the forcing $f(\tau)$ given by a combination of 100 sinusoidal functions with angular frequencies chosen randomly with respect to the uniform distribution on $[120, 600]$, and phases chosen randomly from the uniform distribution on $(-\pi, \pi]$. When $A = 0$, the solution increases exponentially, corresponding to the fact that the photon trajectories are unstable. This is also the case when $A = 20$ and $A = 30$. When $A = 50, 60, 70$ and 80 the solution is oscillatory, indicating stability.

that

$$\frac{d^2}{d\tau^2}\begin{pmatrix}\langle\xi\rangle\\ \langle\eta\rangle\end{pmatrix} = \left(|K|\mathbf{I} - A^2\left\langle\mathbf{v}^2\right\rangle\right)\begin{pmatrix}\langle\xi\rangle\\ \langle\eta\rangle\end{pmatrix}, \tag{5}$$

where $\mathbf{v}$ is the time integral of the perturbation matrix in (2) (i.e. the matrix appearing in (2) with K=0) and $\mathbf{I}$ is the identity matrix. The stability of the trajectory thus depends on the eigenvalues of the matrix $\left(|K|\mathbf{I} - A^2\left\langle\mathbf{v}^2\right\rangle\right)$; specifically, if $A\sqrt{\left\langle(v_{11} \pm v_{12})^2\right\rangle} < 1$ it is unstable, and if $\left(|K|\mathbf{I} - A^2\left\langle\mathbf{v}^2\right\rangle\right)$; specifically, if $A\sqrt{\left\langle(v_{11} \pm v_{12})^2\right\rangle} > 1$ it is stabilized by the stochastic fluctuations. Note that v_{11} and v_{12} are of the order of ω^{-1}, and so the larger ω is the larger A has to be to give stabilization.

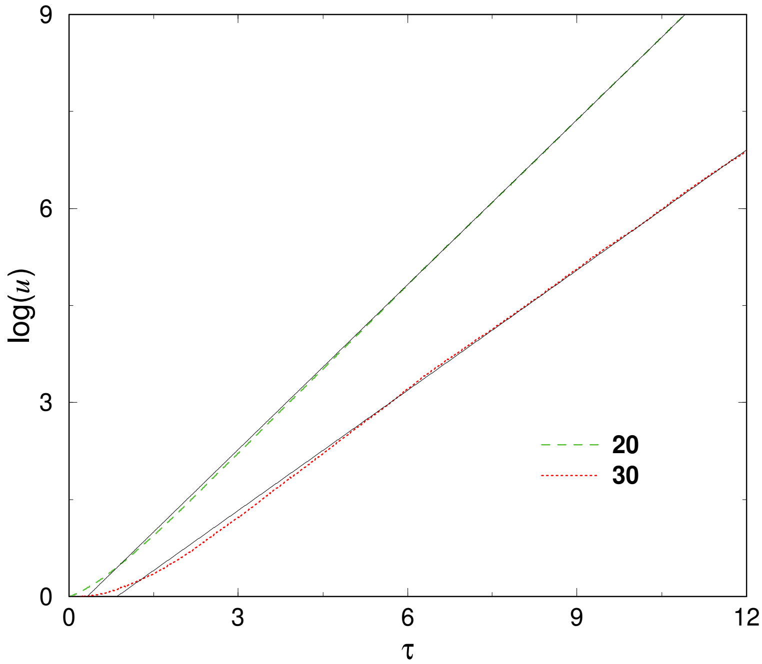

Fig. 3. Plots of $\log u$ vs. τ for the data corresponding to $A = 20$ and $A = 30$ in Figure 2. The continuous black lines are best-fitting straight lines when $\tau > 6$. For $A = 20$ the gradient is about 0.85 (the theoretical value is $0.8498\ldots$). For $A = 30$ the gradient is about 0.62 (the theoretical value is $0.6124\ldots$).

4. Conclusions

Stabilization has several potentially important consequences. First, rather than being fully chaotic, the geodesic dynamics will have both chaotic trajectories and stable islands. It is well established in the context of quantum chaos that stable islands lead to a quantifiable non-gaussian component in the value distribution of wavefunctions, whose precise form depends on the size and position of the islands [25]. Second, the motion of matter in the Universe will cause the curvature to change with time. Since the matter dynamics is non-relativistic and only weakly coupled to the photon dynamics, the curvature can be considered to be a system parameter, rather than a dynamical variable. As it varies, orbits undergo bifurcations. These bifurcations give rise to a separate non-gaussian component in the value distribution of quantum wavefunctions, quantified in the moments by universal scaling exponents in the short-wavelength limit that have been calculated using the theory of singularity-dominated strong fluctuations [26]. We are thus led to predict related non-gaussian components in the CMB. Third, stabilization typically amplifies scarring; in particular, orbits undergoing bifurcation give rise to what have been called *superscars* [26]. Thus stabilization is likely to enhance considerably the influence of scars on the

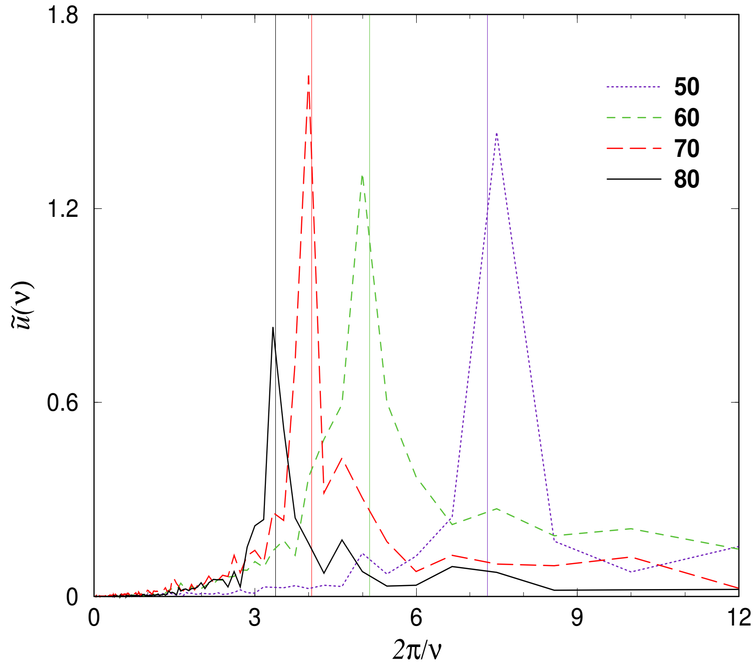

Fig. 4. Plots of the fourier transform of the data for $u(\tau)$ shown in Figure 2, as a function of period $2\pi/\nu$ (i.e. ν is the frequency conjugate to τ) when $A = 50, 60, 70$ and 80. In each case the vertical dotted line marks the theoretical oscillation period, $2\pi/\sqrt{A^2/1440 - 1}$.

CMB and on the generation of structure in the distribution of galaxies [4]. This may explain the previously puzzling observation that scars in constant-curvature models are too weak to support the structures they have been conjectured to generate [4]. Fourth, stable islands are likely to have a major influence on the proposed relationship between geodesics and galactic structure. These islands generically exhibit a fractal hierarchy of structures and this may relate to the hierarchical structures seen in the distribution of galaxies and clusters of galaxies. Stable orbits also typically inhibit the rate of mixing, and so may play a role in the suggested links between mixing and homogenization [9].

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