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Magnetized plasmas in the early universe

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The fluid equations governing a magnetized plasma in a spatially flat Friedmann-Robertson-Walker metric are formulated using the 3+1 formalism of Thorne and MacDonald. They are the generalization of our previous work for a zero external field. These equations are first solved in the ultrarelativistic limit where the electron-positron plasma is treated. The equations turn out to be conformally invariant, so that the results closely resemble those of flat spacetime. The nonrelativistic limit is then studied in both the matter- and radiation-dominated eras. Here it is found that the various magnetoplasma modes redshift at different rates, and are governed by the rate of expansion of the Universe as well as whether the dynamics are dominated by matter or radiation.

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I. INTRODUCTION

Plasma physics and cosmology are two well-established fields in theoretical physics. In particular, analysis of the linear modes of plasmas and the Lemaitre-Friedmann-Robertson-Walker model of our Universe are thoroughly understood, yet a blending of these two disparate theories into a model for the behavior of plasmas in the early Universe has received little attention in the past. The linearized theory of plasmas in the early Universe is an ideal area of study to pursue before tackling more complicated nonlinear effects.

There will be several periods of interest to us in cosmological evolution where we can study the behavior of plasmas. From approximately $t = 10^{-3}$ to $t = 1$ seconds when the temperature $T \gtrsim 10^{10}$ K, there predominantly existed an electron-positron plasma at ultrarelativistic temperatures. This was followed by primordial element formation, after which a plasma consisting mainly of electrons and protons (hydrogen ions), with traces of helium and other light elements existed. The plasma was believed to be in thermal equilibrium with photons, and the energy density of the photons also exceeded that of matter [1], so that this period has become known as the radiation-dominated era. As the temperature decreased with expansion, at a few thousand degrees or $t \simeq 10^{13}$ seconds, recombination of the ionized hydrogen atoms occurred so that matter became decoupled from the photons. Also around this time the mass-energy density of matter came to exceed that of the radiation so that the Universe entered what has become known as the matter-dominated era. We will consider the pertinent plasmas at each of these stages of evolution.

We will also be concerned with the existence of primordial magnetic fields applied to the plasmas. The literature on the origins and evolution of these fields is vast. Cheng, Schramm, and Truran [2] have recently investigated the effects of these fields on big bang nucleosynthesis and temperature, and have obtained limits on their size. Other good references on cosmical magnetic fields are Zeldovich *et al.* [3] and Parker [4]. In particular, nonlinear dynamo theories and the evolution of these fields are considered here. Some nonlinear plasma effects have also been investigated by Tajima and Taniuti [5], who consider the nonlinear interaction of photons and phonons in electron-positron plasmas. They hint at the importance of these effects to cosmology.

At present we restrict ourselves to the linearized theories in a Friedmann Universe. The first to study plasmas in this regime appear to be Holcomb and Tajima [6,7]. Their first paper [6] obtains the equation of motion for free photons, longitudinal and transverse oscillations as well as Alfvén waves for a plasma at ultrarelativistic temperatures in a radiation-dominated Friedmann Universe using a fluid approach. The second paper [7] discusses plasmas at nonrelativistic temperatures in the matter-dominated Universe. Here more attention is paid to plasmas in a constant external magnetic field.

Dettmann, Frankel, and Kowalenko [8] also have studied the early Universe unmagnetized plasmas in the ultrarelativistic (UR) (i.e., $T \gg m_e$) and nonrelativistic (NR) (i.e., $T \ll m_e$) limits in both the pre- and post-recombination eras. For UR plasmas, they recover the results of [6], which show that the various modes of oscillation redshift at the same rate. In particular, this is shown to arise from the conformal flatness of the metric. Note that all bulk motions of the plasma are considered small, which will also be the case in this paper. They also extend this treatment to a kinetic theory approach. In contrast to [7] however, they find that for post-recombination unmagnetized plasmas the modes of oscillation have different time dependences to that of photons, and the frequencies of each redshift at different rates. They also show that similar effects result in prerecombination NR plasmas.

We extend the work in [8] to that of plasmas in a constant external magnetic field, obtaining results in both the UR and NR limits for both matter- and radiation-dominated Universes. We remark at this point in the paper that the presence of a constant external magnetic field would in principle perturb the Robertson-Walker metric that we use, thus making the Universe anisotropic.

However, we can still employ this metric in this paper, as we can either assume a small enough magnetic field, compared to the limits from the cosmic microwave background radiation [9], or choose to solve our equations for magnetic fields which are coherent on scales smaller than the horizon.

Section II briefly reviews the 3+1 formalism of Thorne and MacDonald which is used to display the general relativistic equations in a more familiar form reminiscent of conventional plasma physics. We also introduce the metric and its conformal flatness property.

Section III treats the electron-positron plasma at UR temperatures, and solves completely the longitudinal and transverse modes of oscillation.

Section IV treats an electron plasma on a background of positive ions in the NR limit. The longitudinal modes of oscillation are solved completely in both the pre- and post-recombination eras. The transverse equations turn out to be highly coupled, and require a detailed numerical study which we defer to future work. We find that the results in [7] are both incorrect and incomplete.

II. FORMALISM

The plasmas we wish to investigate in this paper are governed by general relativistic equations, which come in a four-dimensional covariant form. It is very convenient to reduce these equations to the familiar 3+1 notation employed in plasma physics by splitting the electromagnetic field tensor $F^{\mu\nu}$ into electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, respectively, as well as splitting the stress energy tensor. Then we may be able to use our intuition gained from NR plasma physics, in particular being able to use many similar procedures in the method of solution of the equations.

The 3+1 split of spacetime equations has been investigated by numerous authors. We will use the formalism due to Thorne and MacDonald [10]. As this has been set out in great detail in [10], as well as having been extensively reviewed in [6] and [8], we will be very brief here in presenting the required equations. Note that we will use unrationalized units with $c = G = k_b = 1$, where k_b is Boltzmann's constant.

In our four-dimensional spacetime we introduce a family of space-filling three-dimensional spacelike hypersurfaces and an arbitrary time parameter η. This global time parameter labels the hypersurfaces and increases smoothly as one moves forward in time from hypersurface to hypersurface. Orthogonal to these hypersurfaces will exist a congruence of timelike curves, which we regard as the world lines of a family of fiducial observers (FIDO's) with respect to which the various physical quantities under consideration are measured.

We will be using the spatially flat Robertson-Walker metric, often written as

$$ds^2 = -dt^2 + R^2(t)(dx^2 + dy^2 + dz^2) \ . \quad (2.1)$$

Here x, y, and z are comoving coordinates and t is proper time of the FIDOs which are at fixed x, y, and z. We make the coordinate transformation

$$\eta = \int \frac{1}{R} dt \ , \quad (2.2)$$

$$ds^2 = R^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2) \ , \quad (2.3)$$

to exhibit the conformal flatness property of the metric. Although both t and η are global time parameters, we find η more convenient to work with, as will become clear when the plasma equations are exhibited.

FIDO proper time t is related to η by the lapse function α:

$$\alpha = \frac{dt}{d\eta} = \frac{1}{\sqrt{-\eta^{;\mu}\eta_{;\mu}}} = R \ , \quad (2.4)$$

$$u^\mu = -\alpha\eta^{;\mu} \ . \quad (2.5)$$

Here u^μ is the FIDO four-velocity, from which we can define the metric of the spatial hypersurfaces:

$$\gamma^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \ . \quad (2.6)$$

We may use $\gamma^{\mu\nu}$ and u^μ to project four-tensors into their equivalent FIDO observed scalars and spatial three-vectors. The spatial three-vectors are orthogonal to u^μ. We will use the FIDO measured quantities ρ_e, j^μ, E^μ, B^μ, ϵ, S^μ, and $W^{\mu\nu}$, which have been defined in Eqs. (2.4)–(2.10) of [8].

We will choose the orthonormal basis to work in for our spatial tensors. This is physically the most appropriate choice, as the components of tensors take on the numerical values as measured by a FIDO, and $\overleftrightarrow{\gamma}$ becomes a Euclidean metric. Vectors and tensors measured with respect to an orthonormal basis will be denoted by carets; however, all derivatives will be explicitly with respect to the coordinates $(\eta, \mathbf{x})$ and denoted by $(', \boldsymbol{\partial})$, respectively.

The 3+1 form of Maxwell equations, charge conservation, Lorentz force equations, and energy and momentum conservation for continuous media have been derived in [10]. They have subsequently been adapted to our problem in [6] and [8]. We simply remark here that Eqs. (2.30)–(2.37) in [8] are our constitutive equations for the magnetized plasma. Equations (2.38)–(2.42) in [8] define the more convenient barred variables ($\bar{\mathbf{E}}$, $\bar{\mathbf{B}}$, and so on) in terms of the orthonormal variables ($\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$).

III. ULTRARELATIVISTIC LIMIT

A. Formulation

We wish to study a magnetized electron-positron plasma in the early Universe at UR temperatures of approximately $T \gtrsim 10^{10}$ K. We will use a hydrodynamic treatment, and thus generalize our previous zero field results [8].

Since all the frequencies scale the same way with respect to R, it turns out to be fairly straightforward to apply classical kinetic theory to the UR limit, since there are no explicit time dependences. This has been done for a zero external magnetic field $\mathbf{B}_0 = 0$ by [8]. We do not attempt this treatment here for $\mathbf{B}_0 \neq 0$, but rather use the hydrodynamic treatment throughout this paper. This is in keeping with the spirit of the approach used in the NR limit, where analytic results cannot be obtained using kinetic theory, and involved numerical work is the only way to make progress.

We should also briefly mention the possibility of formulating this problem quantum field theoretically to take into account particle creation and/or annihilation and other quantum processes correctly. This type of approach is quite involved and beyond the scope of the present paper. Naturally it would yield a more inclusive description, but as discussed at some length in [8], the semiclassical approach is appropriate to study the leading-order properties of the plasma. An important exception is the case of very strong magnetic fields ($eB_0/m^2 \gtrsim 1$), in which there are large quantum effects [11]. We confine ourselves to weak magnetic fields in this paper.

The hydrodynamic treatment involves treating the plasma as a two-component perfect fluid – no viscosity or heat conduction. The interaction of the fluid with electromagnetic fields is governed by Eqs. (2.36),(2.37) of [8], which we refer to as the energy and momentum equations. The quantities ϵ, $\hat{\mathbf{S}}$, and $\overset{\leftrightarrow}{\hat{\mathbf{W}}}$ in these equations are given by

$$\epsilon = \Gamma^2(\rho + p\hat{v}^2) \ , \tag{3.1}$$

$$\hat{\mathbf{S}} = \Gamma^2(\rho + p)\hat{\mathbf{v}} \ , \tag{3.2}$$

$$\overset{\leftrightarrow}{\hat{\mathbf{W}}} = \Gamma^2(\rho + p)\hat{\mathbf{v}} \otimes \hat{\mathbf{v}} + p\,\overset{\leftrightarrow}{\mathbf{1}} \ , \tag{3.3}$$

where now $\hat{\mathbf{v}}$ and the boost factor Γ correspond to the bulk motion of the plasma, ρ is the total (rest plus internal) energy density in the rest frame, and p is the pressure in the rest frame. Assuming equal densities of particles and antiparticles, that is taking only leading order terms thus neglecting chemical potential, we may express these quantities in terms of temperature using

$$\rho = \frac{2S+1}{2\pi^2}\Gamma(4)\left\{ \begin{array}{c} \tau(4) \\ \zeta(4) \end{array} \right\} T^4 \ , \tag{3.4}$$

$$p = \frac{1}{3}\rho \ , \tag{3.5}$$

where S is the spin of the particles. $\tau(z)$ is defined by

$$\tau(z) = (1 - 2^{1-z})\zeta(z) \ , \tag{3.6}$$

where $\zeta(z)$ is the Riemann zeta function. Here τ refers to a fermion plasma, and we have also introduced the ζ function result in (3.4) to allow our formalism to concurrently apply to a boson–antiboson plasma [12–15].

We also require the particle number density, which in contrast with the NR case is not proportional to ρ:

$$n = \frac{2S+1}{2\pi^2}\Gamma(3)\left\{ \begin{array}{c} \tau(3) \\ \zeta(3) \end{array} \right\} T^3 \ . \tag{3.7}$$

We may remove explicit time dependence from our equations by defining

$$\bar{T} = RT \ ,$$
$$\bar{n} = R^3 n \ ,$$

which displays the correct time scaling of these variables in a radiation-dominated Friedmann universe.

Assuming the bulk motion of the fluid is NR, i.e., $|\hat{\mathbf{v}}| \ll 1$, we linearize our equations:

$$\bar{T} = \bar{T}_0 + \bar{T}_1 \ ,$$
$$\bar{n} = \bar{n}_0 + \bar{n}_1 \ .$$

Then the equation of particle conservation (2.34) in [8], which becomes

$$\bar{n}_1' + \bar{n}_0 \boldsymbol{\partial} \cdot \hat{\mathbf{v}} = 0, \tag{3.8}$$

can be combined with the energy equation (2.36) in [8], to produce the adiabatic equation

$$\frac{\bar{T}_1'}{\bar{T}_0} = \frac{1}{3}\frac{\bar{n}_1'}{\bar{n}_0} \ . \tag{3.9}$$

Extending the work of [8], we now assume a constant external magnetic field $\bar{\mathbf{B}}_0$, so that

$$\bar{\mathbf{B}} = \bar{\mathbf{B}}_0 + \bar{\mathbf{B}}_1 \ , \tag{3.10}$$

and the momentum equation (2.37) in [8] gives

$$\hat{\mathbf{v}}' = \frac{\tau(3)}{4\tau(4)\bar{T}_0} q(\bar{\mathbf{E}} + \hat{\mathbf{v}} \times \bar{\mathbf{B}}_0) - \frac{\boldsymbol{\partial}\bar{T}_1}{\bar{T}_0} \tag{3.11}$$

for fermions.

The above are the hydrodynamic equations of each component of the fluid which we require. Following techniques outlined in various plasma physics texts, for example, Refs. [16]–[19], we may connect them with the Maxwell equations to yield the various longitudinal and transverse modes of oscillation, which we now discuss.

B. Longitudinal (electrostatic) oscillations

Longitudinal modes consist of $\bar{\mathbf{k}} \parallel \bar{\mathbf{E}}$ oscillations. We only require Poisson's equation

$$\boldsymbol{\partial} \cdot \bar{\mathbf{E}} = 4\pi\bar{\rho}_e \tag{3.12}$$

from the Maxwell set. The source quantities required from the Maxwell equations are

$$\bar{\rho}_e = \sum_s q_s \bar{n}_{s1} \ , \tag{3.13}$$

$$\bar{\jmath} = \sum_s q_s \bar{n}_{s0} \hat{\mathbf{v}} \ , \tag{3.14}$$

where the sum on s is over electron and positron species. Since there is no explicit time dependence in the equations, we use Fourier transform techniques, assuming harmonic space and time dependence $e^{i(\bar{\mathbf{k}}\cdot\mathbf{x} - \bar{\omega}\eta)}$ for the linearized quantities.

There are two possible types of longitudinal modes possible, namely $\bar{\mathbf{E}} \parallel \bar{\mathbf{B}}_0$ and $\bar{\mathbf{E}} \perp \bar{\mathbf{B}}_0$. For the first case, if we set

$$\bar{\mathbf{E}} = \bar{E}\hat{\mathbf{z}} \ , \ \bar{\mathbf{k}} = \bar{k}\hat{\mathbf{z}} \ , \ \bar{\mathbf{B}}_0 = \bar{B}_0\hat{\mathbf{z}} \ , \tag{3.15}$$

we rederive the longitudinal mode found in [8] by combining Eqs. (3.8)–(3.13). This wave propagates along the z direction, and is given by the dispersion relation

$$\bar{\omega}^2 = \bar{\omega}_p^2 + \frac{\bar{k}^2}{3} \ , \tag{3.16}$$

where

$$\bar{\omega}_p^2 = 4\pi e^2 \frac{2S+1}{2\pi^2} \frac{\tau(3)^2}{\tau(4)} \bar{T}_0^2 \tag{3.17}$$

is the plasma frequency of an UR fermion plasma. We achieve the same dispersion relation as the $\bar{B}_0 = 0$ case, since oscillations along $\bar{\mathbf{B}}_0$ are not affected by the magnetic field.

In the $\bar{\mathbf{B}}_0 \neq 0$ case, the particles also execute Larmor gyrations in the x and y directions with frequency

$$\bar{\omega} = \bar{\omega}_c \ , \tag{3.18}$$

where

$$\bar{\omega}_c = \frac{1}{4} \frac{\tau(3)}{\tau(4)} \frac{e\bar{B}_0}{\bar{T}_0} \tag{3.19}$$

is the cyclotron frequency for an UR fermion plasma.

For the $\bar{\mathbf{E}} \perp \bar{\mathbf{B}}_0$ mode, if we set $\bar{\mathbf{B}}_0 = \bar{B}_0 \hat{\mathbf{z}}$ we obtain from (3.11) two coupled velocity equations

$$-i\bar{\omega}\hat{v}_x = \frac{1}{4} \frac{\tau(3)}{\tau(4)} \frac{q}{\bar{T}_0} \left(\bar{E}_x + \hat{v}_y \bar{B}_0\right) - i\bar{k}_x \frac{1}{3} \frac{\bar{n}_1}{\bar{n}_0} \ , \tag{3.20}$$

$$-i\bar{\omega}\hat{v}_y = \frac{1}{4} \frac{\tau(3)}{\tau(4)} \frac{q}{\bar{T}_0} \left(\bar{E}_y - \hat{v}_x \bar{B}_0\right) - i\bar{k}_y \frac{1}{3} \frac{\bar{n}_1}{\bar{n}_0} \ . \tag{3.21}$$

For simplicity, assume $\bar{\mathbf{E}} = \bar{E}\hat{\mathbf{x}}$, $\bar{\mathbf{k}} = \bar{k}\hat{\mathbf{x}}$. We may then combine the above two equations along with the adiabatic equation (3.9) and substitute into Poisson's equation to obtain the dispersion relation

$$\bar{\omega}^2 = \bar{\omega}_p^2 + \bar{\omega}_c^2 + \frac{1}{3}\bar{k}^2 \ . \tag{3.22}$$

Here it is evident the plasma oscillations and cyclotron oscillations combine their effects, in analogy to the flat spacetime plasma results.

C. Transverse (electromagnetic) oscillations

There is a greater variety of transverse modes possible, depending on the orientation of $\bar{\mathbf{E}}$, $\bar{\mathbf{B}}_0$, and $\bar{\mathbf{k}}$. The simplest case is the ordinary wave, where $\bar{\mathbf{E}} \parallel \bar{\mathbf{B}}_0$ and $\bar{\mathbf{E}} \perp \bar{\mathbf{k}}$. In this case, Poisson's equation implies there are no charge density fluctuations, so we combine the other Maxwell equations to obtain

$$\bar{\mathbf{E}}'' = -\boldsymbol{\partial} \times \boldsymbol{\partial} \times \bar{\mathbf{E}} - 4\pi \bar{\jmath}' \ . \tag{3.23}$$

This equation, combined with the z component of the momentum equation (3.11) leads to the same mode as found in [8], which is logical, since $\bar{\mathbf{E}} \parallel \bar{\mathbf{B}}_0$, so that the external magnetic field term of the momentum equation $\hat{\mathbf{v}} \times \bar{\mathbf{B}}_0$ makes no contribution. The dispersion relation for the electromagnetic fields is consequently

$$\bar{\omega}^2 = \bar{\omega}_p^2 + \bar{k}^2 \ . \tag{3.24}$$

The above equation also gives the harmonic dependence of the z component of the velocity, which oscillates with the same frequency as the electromagnetic fields. We may also solve for the x and y components of velocity, though, of course, the electromagnetic fields will not exist in the x-y plane. Combining (3.8) and (3.9) with (3.11), we immediately obtain the frequency of oscillation of electrons and positrons in the x-y plane:

$$\bar{\omega}^2 = \bar{\omega}_c^2 + \frac{\bar{k}^2}{3} \ . \tag{3.25}$$

This particle motion perpendicular to the electric field is peculiar to a two-component plasma. As will be demonstrated later in the NR limit, a one-component plasma will not exhibit such effects.

A considerably more complex case is the extraordinary mode $\bar{\mathbf{k}} \perp \bar{\mathbf{B}}_0$, $\bar{\mathbf{E}} \perp \bar{\mathbf{B}}_0$. In complete generality, let us write our fields as

$$\bar{\mathbf{E}} = \bar{E}_x \hat{\mathbf{x}} + \bar{E}_y \hat{\mathbf{y}} \ ,$$
$$\bar{\mathbf{k}} = \bar{k}_x \hat{\mathbf{x}} + \bar{k}_y \hat{\mathbf{y}} \ ,$$
$$\bar{\mathbf{B}}_0 = \bar{B}_0 \hat{\mathbf{z}} \ . \tag{3.26}$$

Particle conservation (3.8) and the adiabatic equation (3.9) yield the relation

$$\frac{\bar{T}_1}{\bar{T}_0} = \frac{1}{3\bar{\omega}} (\bar{k}_x \hat{v}_x + \bar{k}_y \hat{v}_y) \ . \tag{3.27}$$

This may be substituted into the momentum equation (3.11) and solved for $\hat{\mathbf{v}}$ to give

$$\hat{v}_x = i\frac{\bar{\omega}_{cs}}{\bar{\omega}\bar{B}_0} \left[\left(1 - \frac{\bar{k}_y^2}{3\bar{\omega}^2}\right) \bar{E}_x + \left(\frac{\bar{k}_x \bar{k}_y}{3\bar{\omega}^2} + i\frac{\bar{\omega}_{cs}}{\bar{\omega}}\right) \bar{E}_y \right] \left(1 - \frac{\bar{\omega}_{cs}^2 + \bar{k}^2/3}{\bar{\omega}^2}\right)^{-1} \ , \tag{3.28}$$

$$\hat{v}_y = i\frac{\bar{\omega}_{cs}}{\bar{\omega}\bar{B}_0} \left[\left(1 - \frac{\bar{k}_x^2}{3\bar{\omega}^2}\right) \bar{E}_y + \left(\frac{\bar{k}_x \bar{k}_y}{3\bar{\omega}^2} - i\frac{\bar{\omega}_{cs}}{\bar{\omega}}\right) \bar{E}_x \right] \left(1 - \frac{\bar{\omega}_{cs}^2 + \bar{k}^2/3}{\bar{\omega}^2}\right)^{-1} \ . \tag{3.29}$$

Note that here $\bar{\omega}_{cs}$ refers to the frequency of either positrons or electrons ($\bar{\omega}_c$ of electrons $= -\bar{\omega}_c$ of positrons), so that these particles' respective velocities will be in opposite directions. This will lead to some simplification when the velocities are substituted into Maxwell equations. Note also that the momentum equation gives no z component of velocity in the extraordinary wave case.

We substitute this result along with (3.14) into (3.23) to obtain a set of homogeneous simultaneous equations for

$\bar{E}_x$ and $\bar{E}_y$, which only yields a solution if the determinant of the system is zero. This condition yields a rather complicated dispersion relation:

$$\bar{k}_x^2\bar{k}_y^2\left[1-\frac{\bar{\omega}_p^2/3}{\bar{\omega}^2-\bar{\omega}_c^2-\bar{k}^2/3}\right]^2=\left[\bar{\omega}^2-\bar{k}_y^2-\bar{\omega}_p^2\left(\frac{\bar{\omega}^2-\bar{k}_y^2/3}{\bar{\omega}^2-\bar{\omega}_c^2-\bar{k}^2/3}\right)\right]\left[\bar{\omega}^2-\bar{k}_x^2-\bar{\omega}_p^2\left(\frac{\bar{\omega}^2-\bar{k}_x^2/3}{\bar{\omega}^2-\bar{\omega}_c^2-\bar{k}^2/3}\right)\right] \ . \tag{3.30}$$

Note that the extraordinary mode has rotational symmetry in the x-y plane, so that we may in fact set either $\bar{k}_x$ or $\bar{k}_y$ equal to zero to facilitate an immediate solution of (3.30). The roots of the first set of square brackets on the right hand side of (3.30) give the longitudinal modes as exhibited in (3.22). The roots of the second set of square brackets on the right hand side of (3.30) give the dispersion relationship for the transverse modes, which are obtained now from

$$(\bar{\omega}^2-\bar{k}^2)=\bar{\omega}_p^2\left(\bar{\omega}^2-\frac{\bar{k}^2}{3}\right)\left(\bar{\omega}^2-\bar{\omega}_c^2-\frac{\bar{k}^2}{3}\right)^{-1} \ , \tag{3.31}$$

which reduces to the ordinary mode if we set $\bar{\omega}_c=0$ as expected. The solution of this equation is

$$\bar{\omega}^2=\frac{1}{2}\left(\bar{\omega}_p^2+\bar{\omega}_c^2+\frac{4}{3}\bar{k}^2\pm\sqrt{\bar{\omega}_p^4+\bar{\omega}_c^4+\frac{4}{9}\bar{k}^4+2\bar{\omega}_p^2\bar{\omega}_c^2+\frac{4}{3}\bar{\omega}_p^2\bar{k}^2-\frac{4}{3}\bar{\omega}_c^2\bar{k}^2}\right) \ . \tag{3.32}$$

In solving (3.31) we have multiplied up to produce a quadratic equation in $\bar{\omega}^2$. In so doing we have introduced a spurious solution as so often happens in such cases. By setting $\bar{\omega}_c=0$ in (3.31) and (3.32), respectively, we readily deduce that the positive square root in (3.32) is the correct solution for the transverse modes of the plasma.

We may also deduce from the equations for $\bar{E}_x$ and $\bar{E}_y$ that the waves are linearly polarized, in contrast to the NR flat spacetime case where they are elliptically polarized. The reason for this is that in the NR case, the plasma is composed of electrons and ions which have different masses, whereas here we have electron and positron effects competing, which lead to cancellations resulting in only linear polarization.

There is one more case to consider, namely, when $\bar{\mathbf{k}}\parallel\bar{\mathbf{B}}_0$ and $\bar{\mathbf{E}}\perp\bar{\mathbf{B}}_0$. In the NR case, this mode consists of left- and right-circularly polarized waves. As in the ordinary wave case, the momentum equation (3.11) yields solutions for all the components of velocity of the particles. Hence if we set

$$\begin{aligned}\bar{\mathbf{E}}&=\bar{E}_x\hat{\mathbf{x}}+\bar{E}_y\hat{\mathbf{y}} \ ,\\ \bar{\mathbf{k}}&=\bar{k}\hat{\mathbf{z}} \ ,\\ \bar{\mathbf{B}}_0&=\bar{B}_0\hat{\mathbf{z}} \ ,\end{aligned} \tag{3.33}$$

we find that in the z direction

$$\hat{v}_z=\frac{\bar{k}}{\bar{\omega}}\frac{\bar{T}_1}{\bar{T}_0} \ , \tag{3.34}$$

which gives the frequency of particle oscillations in the z direction:

$$\bar{\omega}^2=\frac{1}{3}\bar{k}^2 \ . \tag{3.35}$$

Note that this equation only describes the oscillations of the particles along the z axis, and does not represent the transverse electromagnetic wave.

The x and y components of the particle velocities may be used to describe the propagation of a transverse wave. The momentum equation has solutions in the x-y plane:

$$\hat{v}_x=\frac{\bar{\omega}_{cs}}{\bar{\omega}\bar{B}_0}\left(i\bar{E}_x-\frac{\bar{\omega}_{cs}}{\bar{\omega}}\bar{E}_y\right)\left(1-\frac{\bar{\omega}_{cs}^2}{\bar{\omega}^2}\right)^{-1} \ , \tag{3.36}$$

$$\hat{v}_y=\frac{\bar{\omega}_{cs}}{\bar{\omega}\bar{B}_0}\left(i\bar{E}_y+\frac{\bar{\omega}_{cs}}{\bar{\omega}}\bar{E}_x\right)\left(1-\frac{\bar{\omega}_{cs}^2}{\bar{\omega}^2}\right)^{-1} \ . \tag{3.37}$$

This time, when we substitute into (3.23), we find the equations for $\bar{E}_x$ and $\bar{E}_y$ decouple, and each yields the dispersion relation

$$\frac{\bar{k}^2}{\bar{\omega}^2}=1-\left(\frac{\bar{\omega}_p^2/\bar{\omega}^2}{1-\bar{\omega}_c^2/\bar{\omega}^2}\right) \ , \tag{3.38}$$

which once again differs from the flat spacetime NR results. Here, as in the extraordinary wave case, the waves are linearly polarized, exhibiting no left- or right-circular polarization, in contrast with that observed in the flat spacetime NR case.

When we solve the dispersion relation (3.38), we find

$$\bar{\omega}^2=\frac{1}{2}\left(\bar{\omega}_p^2+\bar{\omega}_c^2+\bar{k}^2\pm\sqrt{\bar{\omega}_p^4+\bar{\omega}_c^4+\bar{k}^4+2\bar{\omega}_p^2\bar{\omega}_c^2+2\bar{\omega}_p^2\bar{k}^2-2\bar{\omega}_c^2\bar{k}^2}\right) \ , \tag{3.39}$$

where once again a spurious mode is introduced. An identical argument to that used in the extraordinary wave case shows us that only the positive square root gives the correct dispersion relationship for these transverse modes.

To summarize this section, we have obtained all the important modes for an UR particle–antiparticle plasma which may have existed in the first second of the Universe. The results were fairly easy to obtain due to the fact that all of the equations scale the same way, and as can be seen, all of the modes redshift identically to a free photon.

IV. NONRELATIVISTIC LIMIT

A. Formulation

We now turn to a NR treatment of magnetized plasmas in the early Universe. The period in cosmological history we are now considering is the pre- and post-recombination eras. At NR temperatures we need no longer consider electron-positron plasmas. Now that element formation has occurred, we will treat the plasma as a one-component electron gas on a background of positive ions (protons or helium nuclei).

There are two physically different eras we must consider in this regime. In the pre-recombination era thermal photons dominated the temperature of the fluid and hence the geometry:

$$T \sim R^{-1} , \quad R = \left(\frac{t}{t_i}\right)^{\frac{1}{2}} = \frac{\eta}{2t_i} . \tag{4.1}$$

Here t_i is a fiducial time constant which for simplicity we often take to be of the same order as t, so that R is of order unity. This will facilitate analyzing the asymptotics of our solutions, as the comoving coordinates x, y, and z correspond closely to physical (proper) distances and our "conformalized" variables are of the same order as the corresponding physically measured ones.

In the post-recombination era, the thermal photons had decoupled from matter so that the geometry was matter dominated:

$$T \sim R^{-2} , \quad R = \left(\frac{t}{t_i}\right)^{\frac{2}{3}} = \frac{\eta^2}{9t_i^2} . \tag{4.2}$$

We will now find that our equations have explicit time dependences from factors of R occurring, and unlike the UR results, the various frequencies redshift at different rates:

$$\text{photon } \omega \sim R^{-1} , \tag{4.3}$$

$$\text{plasma } \omega_p = \sqrt{\frac{4\pi n e^2}{m}} \sim R^{-\frac{3}{2}} , \tag{4.4}$$

$$\text{cyclotron } \omega_c = \frac{eB}{m} \sim R^{-2} . \tag{4.5}$$

This is contrary to the results found by Holcomb [7], as are all of our specific solutions that now ensue.

To set up the NR equations, we begin again with the fluid quantities (3.1)–(3.3). This time we use the NR equations of state for an ideal gas:

$$\rho = n(m + \tfrac{3}{2}T) , \tag{4.6}$$

$$p = nT , \tag{4.7}$$

$$m \gg T . \tag{4.8}$$

Substituting (3.1)–(3.3) into the energy equation (2.36) in [8], and combining this with the continuity equation (3.8), we once again derive an adiabatic equation

$$R^2 T = \kappa (R^3 n)^{\frac{2}{3}} , \tag{4.9}$$

where κ is an arbitrary constant. Some concern might be raised as to the adiabaticity of pre-recombination plasmas. Here the temperature is dominated by the photons which continually heat up the plasma. Our linear theory though, considers small oscillations which occur over a much shorter time scale than these thermal equilibrium processes; hence, adiabaticity is a good approximation.

We may linearize the adiabatic equation and introduce the time-independent number density $\bar{n} = R^3 n$ once more to obtain

$$\frac{\bar{n}_1}{\bar{n}_0} = \frac{3}{2}\frac{T_1}{T_0} . \tag{4.10}$$

If we linearize the force equation, assume an $e^{i\bar{\mathbf{k}}\cdot\mathbf{x}}$ dependence in the linearized variables, and eliminate T_1 using the adiabatic equation (4.9), we obtain

$$\bar{\mathbf{v}}' + i\bar{\mathbf{k}}R\frac{\bar{n}_1}{\bar{n}_0}\frac{5}{3}\frac{T_0}{m} = \frac{q}{m}\left(\bar{\mathbf{E}} + \frac{\bar{\mathbf{v}}}{R}\times\bar{\mathbf{B}}_0\right) , \tag{4.11}$$

where we have introduced $\bar{\mathbf{v}} = R\hat{\mathbf{v}}$, and have included the external magnetic field so that $\bar{\mathbf{B}} = \bar{\mathbf{B}}_0 + \bar{\mathbf{B}}_1$. We also require the continuity equation

$$i\bar{\mathbf{k}}\cdot\bar{\mathbf{v}}R^{-1} = -\frac{\bar{n}_1'}{\bar{n}_0} . \tag{4.12}$$

Equations (4.11) and (4.12) are our fluid equations for both electron and ion species. The species can be connected by the Maxwell equations, but as already stated, we will consider the ions as a positive background, and include only the dynamical effects of electrons in the Maxwell equations.

B. Longitudinal oscillations

For longitudinal oscillations, we assume $\bar{\mathbf{k}} \parallel \bar{\mathbf{E}}$, and require only Poisson's equation from the Maxwell set:

$$i\bar{\mathbf{k}}\cdot\bar{\mathbf{E}} = 4\pi q\bar{n}_1 . \tag{4.13}$$

There are two cases to consider: when $\bar{\mathbf{E}} \parallel \bar{\mathbf{B}}_0$ or when $\bar{\mathbf{E}} \perp \bar{\mathbf{B}}_0$. In a similar fashion to the UR limit, the first case gives precisely the same results as found in [8], as the external magnetic field has no contribution along the direction of $\bar{\mathbf{k}}$. For the second case, without loss of generality we let

$$\bar{\mathbf{E}} = \bar{E}\hat{\mathbf{x}} , \quad \bar{\mathbf{k}} = \bar{k}\hat{\mathbf{x}} , \quad \bar{\mathbf{B}}_0 = \bar{B}_0\hat{\mathbf{z}} , \tag{4.14}$$

and substitute (4.12) into (4.13) to obtain

$$\bar{v}_x = \frac{R}{4\pi e\bar{n}_0}\bar{E}' , \tag{4.15}$$

along with two nontrivial components of the force equation:

$$\bar{v}_x' + i\bar{k}R\frac{\bar{n}_1}{\bar{n}_0}\frac{5}{3}\frac{T_0}{m} = -\frac{e}{m}\left(\bar{E} + \frac{\bar{v}_y}{R}\bar{B}_0\right) \ , \tag{4.16}$$

$$\bar{v}_y' = \frac{e}{m}\frac{\bar{v}_x}{R}\bar{B}_0 \ . \tag{4.17}$$

Equations (4.15)–(4.17) constitute the relevant set for a description of longitudinal modes. They may be combined to produce a third order differential equation for $\bar{E}$. We consider the pre- and post-recombination cases separately.

1. *Pre-recombination*

In this case $\bar{T}_0 = RT_0$ is the pertinent time-independent quantity. The third order equation obtained looks complicated, but may be factored conveniently:

$$\frac{d}{d\eta}\eta^2\left[\bar{E}'' + \frac{1}{\eta}\bar{E}' + \frac{2t_i}{\eta}\left(\bar{\omega}_p^2 + \frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2 + \frac{2\bar{\omega}_c^2 t_i}{\eta}\right)\bar{E}\right] = 0 \ . \tag{4.18}$$

The expression in square brackets yields an homogeneous second-order equation which is easily solved. The third solution is equivalent to the inhomogeneous solution of the second-order equation:

$$\bar{E}'' + \frac{1}{\eta}\bar{E}' + \frac{2t_i}{\eta}\left(\bar{\omega}_p^2 + \frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2 + \frac{2\bar{\omega}_c^2 t_i}{\eta}\right)\bar{E} = \frac{c}{\eta^2} \ , \tag{4.19}$$

where c is an arbitrary constant. We may set $c = 0$ and ignore this third solution. This can be seen in a simple way by comparing the analogous flat spacetime NR result found in all elementary plasma texts. In the flat spacetime case, the general solution for charge density is

$$n_1 = a_1 e^{i\omega t} + a_2 e^{-i\omega t} + \frac{c}{\omega^2} \ , \tag{4.20}$$

where

$$\omega^2 = \omega_p^2 + \omega_c^2 + \frac{5}{3}\frac{T_0}{m}k^2 \ . \tag{4.21}$$

Here a_1 and a_2 are arbitrary constants, and c/ω^2 forms the analogous inhomogeneous solution. But $c = 0$ is necessary, as our initial assumption constituted a constant background density n_0 with a perturbed oscillating density n_1, which contained no constant background itself. Thus $c \neq 0$ violates this assumption of charge neutrality. This argument follows through for (4.19), with the exception that the nonoscillating solution of n_1 is now also time dependent.

Taking $c = 0$, (4.19) may be solved (see [20]) to give

$$\bar{E} = Z_{4i\bar{\omega}_c t_i}\left[\sqrt{8t_i\left(\bar{\omega}_p^2 + \frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2\right)}\eta^{1/2}\right] \ , \tag{4.22}$$

where Z is any Bessel function. Any two linearly independent Bessel functions of the correct order are appropriate solutions. We choose the Hankel functions $H^{(1)}$ and $H^{(2)}$, because they resemble most the flat spacetime $e^{\pm i\omega t}$ solutions.

We may assume the frequencies involved are much greater than the reciprocal age of the Universe, that is

$$\bar{\omega}_p\eta \gg 1 \ , \ \bar{\omega}_p t_i \gg 1 \ , \ \bar{\omega}_c t_i \gg 1 \ . \tag{4.23}$$

Then we can proceed to derive asymptotic expansions for (4.22) in various limits.

The appropriate asymptotic expansions for the above conditions are found in Watson [21], pages 262–268, and are also used in [8]. These expansions of $Z_\nu(z)$ are valid for large ν/i and z. We will merely quote the results here.

We find the first two terms in the expansion of $\bar{E}$. Comparing successive terms in the expansion, we derive a condition for convergence,

$$4t_i^{3/4}t^{1/4}\left[\bar{\omega}_p^2 + \bar{\omega}_c^2\left(\frac{t}{t_i}\right)^{-1/2} + \frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2\right]^{1/2} \gg 1 \ , \tag{4.24}$$

which in terms of FIDO-related quantities is essentially just saying

$$\omega t \gg 1 \ , \tag{4.25}$$

where

$$\omega^2 = \omega_p^2 + \omega_c^2 + \frac{5}{3}\frac{T_0}{m}k^2 \ . \tag{4.26}$$

Here ω is the same as its flat spacetime counterpart. Since we are using a fluid approximation (i.e., long wavelengths), we must assume

$$\omega_p^2 \gg \frac{5}{3}\frac{T_0}{m}k^2 \ , \tag{4.27}$$

hence the condition (4.25) suggests that our expansions are valid for either $\omega_p \gg \omega_c$ or $\omega_p \ll \omega_c$, as long as condition (4.23) is met. To be more specific, let us give an estimate of the magnitudes of our frequencies. Assuming $n_0 \sim 10^9$ particles per cubic meter, which is the generally accepted value around recombination, and $t_i \sim 10^{13}$ seconds, we find

$$\omega_p t_i \sim 10^{19} \ , \ \omega_c t_i \sim 10^{20} B_0 \ , \tag{4.28}$$

where B_0 is the magnetic field which is measured in Gauss. Thus the theoretically possible values of B_0 lie well within our required range, so that we may assume either $\omega_p \gg \omega_c$ or $\omega_p \ll \omega_c$.

Neglecting some irrelevant constant factors, we now exhibit the first two terms for the electric field:

$$\bar{E} \sim a^{-1/4}\exp\left(\pm i\left\{4a^{1/2} - 4t_i\bar{\omega}_c \operatorname{arcsinh}\left[\frac{\sqrt{2t_i}\bar{\omega}_c}{\sqrt{\left(\bar{\omega}_p^2 + \frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2\right)\eta}}\right] - b + \cdots\right\}\right) \ , \tag{4.29}$$

where

$$a = \bar{\omega}_c^2 t_i^2 + \frac{\eta t_i}{2}\left(\bar{\omega}_p^2 + \frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2\right) ,$$

$$b = \frac{1}{32}a^{-1/2}\left(1 - \frac{5\bar{\omega}_c^2 t_i^2}{3a}\right) . \quad (4.30)$$

We may obtain the locally measured frequency ω from this result by differentiating the argument of the exponential with respect to t:

$$\omega = \left(\frac{t}{t_i}\right)^{-3/4}\left[\bar{\omega}_0^2 + \bar{\omega}_c^2\left(\frac{t}{t_i}\right)^{-1/2}\right]^{1/2}\left\{1 - \frac{1}{128}\frac{\bar{\omega}_0^2}{t^2}\left(\frac{t}{t_i}\right)^{3/2}\frac{\bar{\omega}_0^2 - 4\bar{\omega}_c^2\left(\frac{t}{t_i}\right)^{-1/2}}{\left[\bar{\omega}_0^2 + \bar{\omega}_c^2\left(\frac{t}{t_i}\right)^{-1/2}\right]^3} + \cdots\right\}$$

$$= \left(\omega_0^2 + \omega_c^2\right)^{1/2}\left[1 - \frac{1}{128}\frac{\omega_0^2}{t^2}\frac{\omega_0^2 - 4\omega_c^2}{\left(\omega_0^2 + \omega_c^2\right)^3} + \cdots\right] , \quad (4.31)$$

where for ease of presentation we have defined

$$\bar{\omega}_0^2 = \bar{\omega}_p^2 + \tfrac{5}{3}\tfrac{\bar{T}_0}{m}\bar{k}^2 \ , \ \bar{\omega}_0^2 = R^3\omega_0^2 \ . \quad (4.32)$$

Thus the frequencies resemble their flat spacetime counterparts with time-dependent correction terms. Note however that the local frequencies ω_0 and ω_c decrease in time as the plasma becomes less dense. We may recover the flat spacetime results by taking the long time limit in the above equation.

There is another limit we may consider for our solution (4.22). This is the case $\bar{\omega}_p \gg \bar{\omega}_c$, where $\bar{\omega}_c t_i$ is not necessarily large, i.e., the $\bar{B}_0 \to 0$ limit. This differs from the above expansion in that although there we were allowed the condition $\bar{\omega}_p \gg \bar{\omega}_c$, we still required $\bar{\omega}_c t_i \gg 1$, since the expansion was for Bessel functions of large order.

The relevant expansion in this case is given in Watson [21] page 198, which is for $H_\nu(z)$ where $z \gg 1$, but ν is unrestricted. Comparing successive terms in the expansion, a condition for convergence is

$$\sqrt{\eta t_i}\bar{\omega}_p \gg \bar{\omega}_c^2 t_i^2 \quad \text{for} \ \ \bar{\omega}_c t_i > 1$$

$$\sqrt{\eta t_i}\bar{\omega}_p \gg 1 \quad \text{for} \ \bar{\omega}_c t_i < 1 \ . \quad (4.33)$$

To leading order we simply recover the solutions of [8], so $\bar{\omega}_c$ only displays an influence from the first correction term downwards. To first order we find

$$\bar{E} \sim \eta^{-1/4}\exp\left(\pm i\left\{\sqrt{8t_i\eta\left(\bar{\omega}_p^2 + \frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2\right)} - \frac{8\bar{\omega}_c^2 t_i^2 + \frac{1}{8}}{\sqrt{8t_i\eta\left(\bar{\omega}_p^2 + \frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2\right)}}\right\}\right) , \quad (4.34)$$

$$\omega = \bar{\omega}_0\left(\frac{t}{t_i}\right)^{-3/4}\left[1 + \frac{1}{2}\frac{\bar{\omega}_c^2}{\bar{\omega}_0^2}\left(\frac{t}{t_i}\right)^{-1/2} + \frac{1}{128}\frac{1}{\bar{\omega}_0^2 t^2}\left(\frac{t}{t_i}\right)^{3/2} + \cdots\right]$$

$$= \omega_0\left(1 + \frac{1}{2}\frac{\omega_c^2}{\omega_0^2} + \frac{1}{128}\frac{1}{\omega_0^2 t^2} + \cdots\right) . \quad (4.35)$$

The FIDO-measured frequency can be seen to contain the first few terms of the binomial expansion of $\sqrt{1 + \omega_c^2/\omega_0^2}$, plus additional time-dependent corrections. Again the flat spacetime result is obtained in the long time limit.

2. Post recombination

In the post-recombination case, temperature varies as R^{-2}, i.e., $\bar{T}_0 = R^2 T_0$ is the time-independent quantity. Once again, using (4.15)–(4.17) we obtain a third-order equation in $\bar{E}$, which may conveniently be factored:

$$\frac{d}{d\eta}\eta^4\left\{\bar{E}'' + \frac{2}{\eta}\bar{E}' + \left[\frac{9t_i^2}{\eta^2}\bar{\omega}_p^2 + \frac{81t_i^4}{\eta^4}\left(\bar{\omega}_c^2 + \frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2\right)\right]\bar{E}\right\} = 0 \ . \quad (4.36)$$

As in the pre-recombination case, we have two homogeneous solutions to a second-order equation contained in the curly brackets, and a third inhomogeneous solution, which again we may discard from arguments of charge neutrality.

The solution is

$$\bar{E} = \eta^{-1/2} Z_{\sqrt{\frac{1}{4} - q\bar{\omega}_p^2 t_i^2}} \left(9t_i^2 \sqrt{\bar{\omega}_c^2 + \frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2}\, \eta^{-1} \right) . \tag{4.37}$$

As before, we choose the two linearly independent Hankel function solutions, and proceed to obtain appropriate asymptotic expansions for the conditions (4.23). Comparing successive terms in the series, we obtain the condition

$$3t_i \left[\bar{\omega}_p^2 - \frac{1}{36t^2}\left(\frac{t}{t_i}\right)^2 + \left(\bar{\omega}_c^2 + \frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2\right)\left(\frac{t}{t_i}\right)^{-2/3} \right]^{1/2} \gg 1 , \tag{4.38}$$

which is equivalent to (4.25) and (4.26). Thus once again the relative magnitudes of ω_p and ω_c are unimportant; rather what is important is (4.23).

The expansion for (4.37) to first order turns out to be

$$\bar{E} \sim \eta^{-1/2} a^{-1/4} \exp\left(\pm i \left\{ a^{1/2} - \sqrt{9\bar{\omega}_p^2 t_i^2 - \frac{1}{4}}\, \text{arcsinh}\left[\frac{\eta}{3t_i} \left(\frac{\bar{\omega}_p^2 - \frac{1}{36t_i^2}}{\bar{\omega}_c^2 + \frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2} \right)^{1/2} \right] - b + \cdots \right\} \right) , \tag{4.39}$$

where

$$a = 9t_i^2 \left[\bar{\omega}_p^2 - \frac{1}{36t_i^2} + \frac{9t_i^2}{\eta^2}\left(\bar{\omega}_c^2 + \frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2 \right) \right] , \tag{4.40}$$

$$b = \frac{1}{8} a^{-1/2} \left[1 - \frac{15}{a}\left(\bar{\omega}_p^2 t_i^2 - \frac{1}{36} \right) \right] . \tag{4.41}$$

The fluid approximation dictates that, for post recombination,

$$\bar{\omega}_p^2 \gg \frac{\bar{T}_0}{m}\bar{k}^2 \frac{t_i^2}{\eta^2} . \tag{4.42}$$

We find the locally measured frequency to be

$$\omega = \left(\frac{t}{t_i}\right)^{-1} \left[\bar{\omega}_t^2 + \bar{\omega}_{ck}^2 \left(\frac{t}{t_i}\right)^{-2/3} \right]^{1/2} \left[1 - \frac{1}{72}\left(\frac{t}{t_i}\right)^{4/3} \frac{\bar{\omega}_{ck}^2}{t^2} \frac{\bar{\omega}_{ck}^2 \left(\frac{t}{t_i}\right)^{-2/3} - 4\bar{\omega}_t^2}{\left(\bar{\omega}_{ck}^2\left(\frac{t}{t_i}\right)^{-2/3} + \bar{\omega}_t^2\right)^3} + \cdots \right]$$

$$= \left(\omega_t^2 + \omega_{ck}^2\right)^{1/2} \left[1 - \frac{1}{72}\frac{\omega_{ck}^2}{t^2} \frac{\omega_{ck}^2 - 4\omega_t^2}{(\omega_{ck}^2 + \omega_t^2)^3} + \cdots \right] , \tag{4.43}$$

where for ease of presentation we have defined

$$\begin{aligned} \bar{\omega}_t^2 &= \bar{\omega}_p^2 - \tfrac{1}{36t^2}\left(\tfrac{t}{t_i}\right)^2 , & \bar{\omega}_t^2 &= R^3\omega_t^2 \\ \bar{\omega}_{ck}^2 &= \bar{\omega}_c^2 + \tfrac{5}{3}\tfrac{\bar{T}_0}{m}\bar{k}^2 , & \bar{\omega}_{ck}^2 &= R^4\omega_{ck}^2 . \end{aligned} \tag{4.44}$$

We may wish to take the $\bar{B}_0 \to 0$ limit in a post-recombination plasma as well. By examining the argument of the Bessel function in (4.37) we see that we will merely recover the results of [8] if we assume $\frac{\bar{T}_0}{m}\bar{k}^2 > 1$. To obtain a full small-argument expansion for $\bar{E}$, we have to take a very long wavelength limit. To obtain an estimate for λ we substitute the correct numerical values for the various quantities in (4.37) to give

$$9t_i^2 \sqrt{\frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2}\,\eta^{-1} \sim \frac{10^{20}}{\lambda} , \tag{4.45}$$

where λ the wavelength is measured in meters. So for this to be a small expansion parameter, we would require $\lambda \gg 10^{20}$ meters. At recombination the radius of the visible Universe was $\sim 3 \times 10^{21}$ m; hence, we are now dealing with oscillations of wavelength approaching this order.

We now require the small z expansions of $H_\nu^{(1)}$ and $H_\nu^{(2)}$. To leading order we simply recover the plasma oscillations

$$\omega = \bar{\omega}_p \left(\frac{t}{t_i}\right)^{-1} = \omega_p . \tag{4.46}$$

Including the next order, for general ν and $z \ll 1$ from [21] and [20], we find

$$H_\nu^{(1),(2)}(z) \sim \mp \frac{i}{\pi}\Gamma(\nu)\left(\frac{1}{2}z\right)^{-\nu}\left(1 + \frac{z^2}{4(\nu-1)} + \cdots\right) , \tag{4.47}$$

which gives for (4.37)

$$\bar{E} \sim \eta^{-1/2} \exp\left(\pm i \left\{ 3\bar{\omega}_p t_i \ln\left(\frac{9t_i^2}{2\eta} \sqrt{\bar{\omega}_c^2 + \frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2} \right) + \frac{27t_i^4}{4\eta^2}\frac{1}{\bar{\omega}_p t_i}\left(\bar{\omega}_c^2 + \frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2 \right) + \cdots \right\} \right) , \tag{4.48}$$

$$\omega = \bar{\omega}_p \left(\frac{t}{t_i}\right)^{-1} \left[1 + \frac{1}{2}\left(\frac{t}{t_i}\right)^{-2/3} \frac{\bar{\omega}_c^2 + \frac{5}{3}\frac{\bar{T}_0}{m}\bar{k}^2}{\bar{\omega}_p^2} + \cdots \right]$$
$$= \omega_p \left(1 + \frac{1}{2}\frac{\omega_c^2 + \frac{5}{3}\frac{T_0}{m}k^2}{\omega_p^2} + \ldots \right) . \tag{4.49}$$

Thus to this order, the flat spacetime, frequencies as given in Eq. (4.26) have been recovered.

Comparing the various results of our longitudinal modes calculations we see that we have found the amplitude of oscillations $\bar{E}$ to decay in time in most intricate fashions. This is in marked contrast to that of a free photon, or to flat spacetime plasma physics, where all amplitudes are constant. We have also found time-dependent correction terms to the locally measured frequencies which could not be guessed. Thus the curved spacetime, within which we work, significantly affects the time-dependent behavior of plasmas.

C. Transverse oscillations

Transverse waves turn out to present a far more formidable problem mathematically. One case which is solvable analytically is the ordinary wave mode $\bar{\mathbf{E}} \parallel \bar{\mathbf{B}}_0$, $\bar{\mathbf{k}} \perp \bar{\mathbf{B}}_0$.

Poisson's equation implies $\bar{n}_1 = 0$. We let

$$\bar{\mathbf{k}} = \bar{k}\hat{\mathbf{x}} \;,\; \bar{\mathbf{E}} = \bar{E}\hat{\mathbf{z}} \;,\; \bar{\mathbf{B}}_0 = \bar{B}_0\hat{\mathbf{z}} \;. \tag{4.50}$$

Then the force equation (4.11) yields

$$\bar{v}_x' = -\frac{e}{mR}\bar{v}_y\bar{B}_0 \;, \tag{4.51}$$

$$\bar{v}_y' = \frac{e}{mR}\bar{v}_x\bar{B}_0 \;, \tag{4.52}$$

$$\bar{v}_z' = -\frac{e}{m}\bar{E} \;. \tag{4.53}$$

With $\bar{\mathbf{B}} = \bar{\mathbf{B}}_0 + \bar{\mathbf{B}}_1$, we obtain, from the electromagnetic equations of Maxwell's set,

$$\bar{\mathbf{B}}_1' = -i\bar{\mathbf{k}} \times \bar{\mathbf{E}} \;, \tag{4.54}$$

$$\bar{\mathbf{E}}' = i\bar{\mathbf{k}} \times \bar{\mathbf{B}}_1 + 4\pi e\bar{n}_0\frac{\bar{\mathbf{v}}}{R} \;, \tag{4.55}$$

the results

$$\bar{B}_{1y}' = -i\bar{k}\bar{E} \;, \tag{4.56}$$

$$\bar{E}' = i\bar{k}\bar{B}_{1y} + 4\pi e\bar{n}_0\frac{\bar{v}_z}{R} \;, \tag{4.57}$$

and also $\bar{B}_{1x}' = \bar{B}_{1z}' = 0$. Hence from the definition of $\bar{\mathbf{B}}_0$, we assume no constant perturbed field so that $\bar{B}_{1x} = \bar{B}_{1z} = 0$. Thus we find $\bar{v}_x = \bar{v}_y = 0$, and if we let

$$\bar{v}_z \equiv \bar{v} \;,\; \bar{B}_{1y} \equiv -\bar{B}_1 \;, \tag{4.58}$$

we obtain

$$\bar{v}' = -\frac{e}{m}\bar{E} \;, \tag{4.59}$$

$$\bar{E}' = -i\bar{k}\bar{B}_1 + 4\pi e\bar{n}_0\frac{\bar{v}}{R} \;, \tag{4.60}$$

$$\bar{B}' = -i\bar{k}\bar{E} \;. \tag{4.61}$$

This set of equations is identical to the transverse set found in [8]; thus, $\bar{B}_0$ has no effect on the ordinary transverse mode of a plasma. The full solution and asymptotic evaluation of this mode is presented in [8]. Note that due to the fact that we are considering a one-component plasma, the particles have no velocity in the x-y plane, in contrast with the UR limit, where it turned out to be permissible due to the two components producing current cancellations in Maxwell equations.

The other transverse modes that are possible yield complicated systems of coupled differential equations. We will merely derive the equations and not concentrate on the solutions here. We begin with the momentum and continuity equations (4.11) and (3.8) along with an equation obtained by combining (4.54) and (4.55):

$$\bar{\mathbf{E}}'' = \bar{\mathbf{k}}(\bar{\mathbf{k}}.\bar{\mathbf{E}}) - \bar{k}^2\bar{\mathbf{E}} + 4\pi e\frac{\bar{n}_0}{R}\left(\bar{\mathbf{v}}' - \bar{\mathbf{v}}\frac{R'}{R} \right) . \tag{4.62}$$

From these equations we may find two transverse modes, and corresponding to each of these modes a pre- and post-recombination case, with $\bar{T}_0 = RT_0$ and $\bar{T}_0 = R^2T_0$, respectively. We list the resulting sets of equations, which cannot be simplified further by any techniques known to the authors.

1. Extraordinary waves

This mode consists of $\bar{\mathbf{k}} \perp \bar{\mathbf{B}}_0$, $\bar{\mathbf{E}} \perp \bar{\mathbf{B}}_0$ oscillations. We let

$$\bar{\mathbf{E}} = \bar{E}_x\hat{\mathbf{x}} + \bar{E}_y\hat{\mathbf{y}} \;,\; \bar{\mathbf{k}} = \bar{k}\hat{\mathbf{x}} \;,\; \bar{\mathbf{B}}_0 = \bar{B}_0\hat{\mathbf{z}} \;, \tag{4.63}$$

and derive the equations

Pre recombination

$$\bar{v}_x'' + \frac{5}{3}\frac{\bar{T}_0}{m}\frac{2t_i\bar{k}^2}{\eta}\bar{v}_x = -\frac{e}{m}\left(\bar{E}_x' + \frac{2t_i\bar{B}_0}{\eta}\bar{v}_y' - \frac{2t_i\bar{B}_0}{\eta^2}\bar{v}_y\right)$$
$$\bar{v}_y'' = -\frac{e}{m}\left(\bar{E}_y' - \frac{2t_i\bar{B}_0}{\eta}\bar{v}_x' + \frac{2t_i\bar{B}_0}{\eta^2}\bar{v}_x\right)$$
$$\bar{E}_x'' = 4\pi e\frac{2t_i\bar{n}_0}{\eta}\left(\bar{v}_x' - \frac{1}{\eta}\bar{v}_x\right)$$
$$\bar{E}_y'' = -\bar{k}^2\bar{E}_y + 4\pi e\frac{2t_i\bar{n}_0}{\eta}\left(\bar{v}_y' - \frac{1}{\eta}\bar{v}_y\right) \; , \qquad (4.64)$$

Post recombination

$$\bar{v}_x'' + \frac{2}{\eta}\bar{v}_x' + \frac{5}{3}\frac{\bar{T}_0}{m}\frac{81t_i^4\bar{k}^2}{\eta^4}\bar{v}_x = -\frac{e}{m}\left(\bar{E}_x' + \frac{2}{\eta}\bar{E}_x + \frac{9t_i^2\bar{B}_0}{\eta^2}\bar{v}_y'\right)$$
$$\bar{v}_y'' + \frac{2}{\eta}\bar{v}_y' = -\frac{e}{m}\left(\bar{E}_y' + \frac{2}{\eta}\bar{E}_y - \frac{9t_i^2\bar{B}_0}{\eta^2}\bar{v}_x'\right)$$
$$\bar{E}_x'' = 4\pi e\frac{9t_i^2\bar{n}_0}{\eta^2}\left(\bar{v}_x' - \frac{2}{\eta}\bar{v}_x\right)$$
$$\bar{E}_y'' = -\bar{k}^2\bar{E}_y + 4\pi e\frac{9t_i^2\bar{n}_0}{\eta^2}\left(\bar{v}_y' - \frac{2}{\eta}\bar{v}_y\right) \; . \qquad (4.65)$$

2. *Right- and left-hand circularly polarized waves*

This mode consists of $\bar{\mathbf{k}} \parallel \bar{\mathbf{B}}_0$, $\bar{\mathbf{E}} \perp \bar{\mathbf{B}}_0$ oscillations. We let

$$\bar{\mathbf{E}} = \bar{E}_x\hat{\mathbf{x}} + \bar{E}_y\hat{\mathbf{y}} \; , \; \bar{\mathbf{k}} = \bar{k}\hat{\mathbf{z}} \; , \; \bar{\mathbf{B}}_0 = \bar{B}_0\hat{\mathbf{z}} \; , \qquad (4.66)$$

and derive the equations

Pre recombination

$$\bar{v}_x'' = -\frac{e}{m}\left(\bar{E}_x' + \frac{2t_i\bar{B}_0}{\eta}\bar{v}_y' - \frac{2t_i\bar{B}_0}{\eta^2}\bar{v}_y\right)$$
$$\bar{v}_y'' = -\frac{e}{m}\left(\bar{E}_y' - \frac{2t_i\bar{B}_0}{\eta}\bar{v}_x' + \frac{2t_i\bar{B}_0}{\eta^2}\bar{v}_x\right)$$
$$\bar{E}_x'' = -\bar{k}^2\bar{E}_x + 4\pi e\frac{2t_i\bar{n}_0}{\eta}\left(\bar{v}_x' - \frac{1}{\eta}\bar{v}_x\right)$$
$$\bar{E}_y'' = -\bar{k}^2\bar{E}_y + 4\pi e\frac{2t_i\bar{n}_0}{\eta}\left(\bar{v}_y' - \frac{1}{\eta}\bar{v}_y\right) \; , \qquad (4.67)$$

Post recombination

$$\bar{v}_x'' = -\frac{2}{\eta}\bar{v}_x' - \frac{e}{m}\left(\bar{E}_x' + \frac{2}{\eta}\bar{E}_x + \frac{9t_i^2\bar{B}_0}{\eta^2}\bar{v}_y'\right)$$
$$\bar{v}_y'' = \frac{2}{\eta}\bar{v}_y' - \frac{e}{m}\left(\bar{E}_y' + \frac{2}{\eta}\bar{E}_y - \frac{9t_i^2\bar{B}_0}{\eta^2}\bar{v}_x'\right)$$
$$\bar{E}_x'' = -\bar{k}^2\bar{E}_x + 4\pi e\frac{9t_i^2\bar{n}_0}{\eta^2}\left(\bar{v}_x' - \frac{2}{\eta}\bar{v}_x\right)$$
$$\bar{E}_y'' = -\bar{k}^2\bar{E}_y + 4\pi e\frac{9t_i^2\bar{n}_0}{\eta^2}\left(\bar{v}_y' - \frac{2}{\eta}\bar{v}_y\right) \; . \qquad (4.68)$$

The solutions of the above systems of equations require an extensive numerical study which is beyond the scope of the present paper. We defer such an analysis to future papers and suffice here merely to list the relevant equations. Note that both the z component of the velocity equation and the z component of (4.62) now yield non-zero solutions which are incompatible; hence, we must have $\bar{v}_z = 0$. This is once again due to the fact that we are treating a one-component plasma. We can only have non-zero velocities for a two-component plasma such as the electron-positron system in the UR limit.

V. CONCLUSION

Using the formalism established by [6] and [8], we have analyzed the various possible plasma modes in the presence of an external constant magnetic field in the spatially flat Robertson-Walker metric, in both the UR and NR limits, using a fluid model. This completes the program for studying high frequency linear oscillations using a semi-classical approach.

At UR temperatures ($T \gg m$) we found that all plasma modes redshift at the same rate as that of a free photon. The fact that we treated an electron-positron plasma simplified the observable effects, predicting only linearly polarized waves.

At NR temperatures ($T \ll m$) we disagree with results found in [7], showing that the various plasma modes redshift at different rates. We have shown that in all the cases solved of pre- and post-recombination, the locally measured frequencies we obtain resemble the flat spacetime counterparts with time-dependent corrections, which disappear as we take the flat spacetime limit. The expansion of the Universe has been shown to cause the amplitude of oscillation of the various modes to decay in time in quite an intricate fashion.

There remains considerable work to be done in related topics. A full solution of all the transverse modes in the NR limit is yet to be undertaken. This will require numerical techniques. Other linear plasma modes such as the magnetohydrodynamic low frequency Alfvén waves and two stream instabilities may also be studied. We intend to discuss these in future papers.

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