# Magnetohydrodynamics in the expanding Universe

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We present a comprehensive study of the magnetohydrodynamic (MHD) modes and corresponding state eigenvectors in the early Universe. The fluid equations are formulated using the 3+1 formalism of Thorne and MacDonald, as utilized by us in our previous work. The equations are then solved in various regimes. The first of these is the ultrarelativistic limit, where we study the electron-positron plasma at high temperatures. Here, we find the conformal invariance property of the metric ensures results very similar to those of flat spacetime. We also investigate an electron-proton fluid before and after recombination, where a nonrelativistic limit suffices. Here the various frequencies redshift at different rates with respect to one another, which complicates the solutions considerably. A thorough discussion, both quantitative and qualitative, is given for all solutions obtained.

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## I. INTRODUCTION

The formulation of magnetohydrodynamics (MHD) in curved spacetime is a relatively new development in astrophysics. Along with other branches of plasma physics, although the flat spacetime results are very well understood, a blending of the theory with general relativity is really only just beginning to be applied in detail to specific systems, in particular to early Universe plasmas.

The development of relativistic MHD began in the 1950s with the work of such people as Harris, Zumino, and Khalatnikov [1–3]. Here the basic equations and dispersion relationships were derived, along with some theory of weak shocks. The work we undertake in this paper on an ultrarelativistic (UR) electron-positron plasma will be seen to closely correspond to these results. This type of plasma is thought to have existed from around $t = 10^{-3}$ to $t = 1$ s in the early Universe.

Of course our formalism will include a non-Minkowski background spacetime metric. There have been various attempts to formulate a fully general relativistic version of MHD in covariant form, and this area of research is already quite well established (e.g., Lichnerowicz [4]). We have adopted the 3+1 approach to this problem, which has received much less attention in the past. The 3+1 formalism of general relativity was largely developed by Thorne and MacDonald [5]. It has the advantage of expressing theory in a form more suited to intuitive thinking, and is particularly useful in the application of numerical relativity. The application of the 3+1 approach to general relativistic perfect MHD was first carried out by Sloan and Smarr [6], though our approach closely mirrors the more intuitively useful form of Zhang [7]. In this formalism, there exists a set of preferred fiducial observers (FIDO's), with respect to which fluid quantities such as mass density and velocity as well as electromagnetic fields are measured. The work of Zhang [7,8] as well as others (e.g., Thorne *et al.* [9]) has concentrated on the role of MHD in black hole magnetospheres.

The work we pursue involves plasmas in the expanding universe. In particular, we will be dealing with a spatially flat Friedmann-Robertson-Walker (FRW) spacetime, which is the simplest background, yet still illustrates how the curvature of spacetime can affect simple MHD results. Apparently the first to tackle plasma physics in this context were Holcomb and Tajima [10]. They investigated both the high frequency modes of an electron-positron plasma and the Alfvén waves. We extend the work done in the low frequency Alfvén region to include the displacement current, which is necessary in UR work, to solve the equations for any direction of propagation vector $\mathbf{k}$, and to include the equation of state for electron-positron pair creation, which becomes important at these temperatures.

We will also be interested in the behavior of MHD plasmas around the time of recombination, i.e., at $t \simeq 10^{13}$ s. Around this epoch we may treat the fluid as nonrelativistic (NR), which enables us to neglect such effects as displacement current. Even so, we find the solution of the problem becomes considerably more difficult mathematically, due to how the various quantities scale differently with respect to the factor $R$, which appears in the FRW metric. We consider both the radiation-dominated era (also known as the pre-recombination era) and the matter-dominated era (also known as the post-recombination era), within each of which $R$ has a different dependence on cosmic time, thus causing the MHD equations to differ fundamentally. To the best of our knowledge, MHD has not been previously studied in the pre-recombination era, while the only previous work in the post-recombination era is that of Holcomb [11]. Holcomb obtains the relevant equations, but only solves them in the $\mathbf{B}_0 \to 0$ and $T \to 0$ limits, where $\mathbf{B}_0$ is the background magnetic field and $T$ is the temperature. A much more satisfactory result would be to obtain the full Alfvén and magnetosound modes, which we attempt in this paper. The complicated coupled nature of the equations does not allow us to solve for arbitrary wave vector $\mathbf{k}$, but we do obtain full solutions for specific directions of **k**. We find that our solutions reduce to that of Holcomb in the appropriate limits. We also derive the full MHD state eigenvector in all cases, something that has received little attention in the past.

The formalism we adopt, as stated earlier, closely parallels that of Zhang, and in particular follows directly from that developed in our earlier work on high frequency plasma modes in the expanding Universe, both with and without the presence of an external magnetic field [12,13]. As in this work, we find that the various modes all redshift at the same rate in the UR limit, but at different rates around recombination, due to the way the different quantities scale with respect to $R$.

Note that in this paper we assume the background metric to be fixed, and we assume the matter density perturbations are small enough to have negligible effect on the metric. Thus the self-gravity of the matter is neglected, which precludes the formation of such phenomena as gravity waves. This seems like a reasonable first approximation, and judging by the complexity of the solutions, it seems unlikely that a more complicated model would be analytically solvable. An investigation of cosmological plasma modes in the high frequency limit, including gravitational interactions, has been attempted by Zimdahl [14]. Here a four-component fluid is investigated, two of which are charged, and one of which is assumed to play the role of dark matter. This approach is specifically designed to compare the effects of plasma oscillations on the well-established scalar perturbation theory of gravitational instabilities, leading to structure formation in the Universe. This is well beyond the scope of our current work, which deals specifically with pure MHD. Once the behavior of pure MHD fluids is well understood, it may be possible to extend our current work to see what role it plays in such important areas of study as galaxy formation. We also assume a small enough background magnetic field, so that the FRW metric is not perturbed, which would make the Universe anisotropic or alternatively, we could just consider regions of space smaller than the horizon, and assume magnetic fields are only coherent over these scales.

We proceed to tackle our problem as follows: In Sec. II we introduce the main equations of 3+1 plasma physics again, with particular reference to a FRW metric. The full theory of this has been developed in [5,7] and [10–13], and so we will mainly restrict ourselves to results, and not detailed derivations.

In Sec. III we place our equations in an UR setting, invoke an electron-positron equation of state, and solve the MHD equations in the most general case, comparing our results to the special relativistic results of [1]. We also obtain the MHD state vectors, which to our knowledge has only previously been done for the NR flat spacetime case, as found in most textbooks on MHD.

In Sec. IV we find the NR limit of our equations, assuming an electron-hydrogen ion plasma. This section just presents the formalism. In Sec. V we proceed to solve these equations in the pre-recombination era, something not attempted in previous studies on early Universe plasmas.

Finally in Sec. VI we investigate the NR equations in the post-recombination era, which proves to be by far the most mathematically challenging case. We are able to solve the equations for specific directions of the wave vector **k**, and also recover the special cases considered in [11]. Here the true nature of the phase velocities is only elucidated with some fairly lengthy asymptotic analysis.

## II. FORMALISM AND EQUATIONS

Although the formalism adopted here follows closely from that of [12,13], we emphasize that the presentation given in this section has been specifically fashioned for the study of MHD. Furthermore, several of the equations presented in this section are new. The sections that follow rely completely on the formalism developed herein. To be specific, we note that in our previous work we were considering high frequency oscillations, and consequently we adopted a two-fluid model of the plasma. The ions or positrons and electrons each had their own set of equations, and were linked via the charge and current densities

$$\rho_e = e(n_+ - n_-), \tag{2.1}$$

$$\mathbf{j} = e(n_+\mathbf{v}_+ - n_-\mathbf{v}_-), \tag{2.2}$$

where $n_\pm$ is the number density of each species, and $\mathbf{v}_\pm$ is the velocity of the bulk motion of each component of the plasma.

We now adopt the MHD one-fluid approach, where we assume the plasma approximation $n \equiv n_+ = n_-$ on scales larger than the Debye radius. Thus we are investigating low frequency waves, where the ions or positrons and electrons move as a single fluid without charge separation, and we may consequently define an overall mass density and average fluid velocity

$$\rho_m = \rho_{m+} + \rho_{m-}, \tag{2.3}$$

$$\mathbf{U} = \frac{\rho_{m+}\mathbf{v}_+ + \rho_{m-}\mathbf{v}_-}{\rho_{m+} + \rho_{m-}}. \tag{2.4}$$

Throughout this paper we will use unrationalized units, with $c = G = k_B = 1$, where $k_B$ is Boltzmann's constant.

If we begin with a covariant point of view, our one-fluid description is contained in the equations for the stress-energy tensor, along with the Maxwell equations. In particular, the perfect fluid component of the stress-energy tensor will contain just the averaged fluid velocity, rather than components for each species:

$$T^{\alpha\beta}_{\rm fl} = (\rho + p)U^\alpha U^\beta + p g^{\alpha\beta}, \tag{2.5}$$

where $U^\alpha$ is the four-velocity of the fluid, $\rho$ is the total (rest plus internal) energy density, and $p$ is the pressure. Including the electric field component

$$T^{\alpha\beta}_{\rm em} = \frac{1}{4\pi}(F^{\alpha\mu}F^{\beta}{}_{\mu} - \tfrac{1}{4}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}), \tag{2.6}$$

we have the usual stress-energy conservation law

$$T^{\alpha\beta}{}_{;\beta} = (T^{\alpha\beta}_{\rm fl} + T^{\alpha\beta}_{\rm em})_{;\beta} = 0, \tag{2.7}$$

and using

$$T^{\alpha\beta}_{\rm em\,;\beta} = -F^{\alpha\beta}J_\beta, \tag{2.8}$$

we may derive the energy and momentum conservation laws for the perfect fluid interacting with the electromagnetic field, as in [12].

Thus we perform the 3+1 split of spacetime [5], which involves choosing a set of FIDO's, whose time is measured by a global parameter, and whose world lines form a congruence. This global parameter increases smoothly as one moves forward in time, and labels spatial hypersurfaces, which are orthogonal to the FIDO world lines. These spatial hypersurfaces consequently foliate the spacetime. Together with the FIDO world lines come a collection of kinematic variables, which describe the acceleration, expansion, and twisting of the world lines as time progresses. These are discussed fully in [5], and appear in the general 3+1 equations. We will not repeat these equations here, but apply them to our particular spacetime of the expanding Universe.

The metric of our spacetime is the spatially flat FRW metric, usually written as

$$ds^2 = -dt^2 + R^2(t)(dx_1^2 + dx_2^2 + dx_3^2), \tag{2.9}$$

with $t$ being cosmological time. Our equations take their simplest form if we make the coordinate transformation to the new global time parameter

$$\eta = \int \frac{1}{R}dt, \tag{2.10}$$

$$ds^2 = R^2(\eta)(-d\eta^2 + dx_1^2 + dx_2^2 + dx_3^2). \tag{2.11}$$

Using this metric, our energy-momentum equations become

$$\frac{1}{R}\epsilon' + \frac{3R'}{R^2}\epsilon + \frac{1}{R}\boldsymbol{\partial}\cdot\hat{\mathbf{S}} + \frac{R'}{R^2}Tr(\overset{\leftrightarrow}{\hat{\mathbf{W}}}) = \hat{\jmath}\cdot\hat{\mathbf{E}}, \tag{2.12}$$

$$\frac{1}{R}\hat{\mathbf{S}}' + \frac{4R'}{R^2}\hat{\mathbf{S}} + \frac{1}{R}\boldsymbol{\partial}\cdot\overset{\leftrightarrow}{\hat{\mathbf{W}}} = \rho_e\hat{\mathbf{E}} + \hat{\jmath}\times\hat{\mathbf{B}}, \tag{2.13}$$

where

$$\epsilon = \gamma^2(\rho + p\hat{U}^2), \tag{2.14}$$

$$\hat{\mathbf{S}} = \gamma^2(\rho+p)\hat{\mathbf{U}}, \tag{2.15}$$

$$\overset{\leftrightarrow}{\hat{\mathbf{W}}} = \gamma^2(\rho+p)\hat{\mathbf{U}}\otimes\hat{\mathbf{U}} + p\,\overset{\leftrightarrow}{\mathbf{1}}, \tag{2.16}$$

$$\gamma = (1-\hat{U}^2)^{-1/2}. \tag{2.17}$$

The quantities denoted by carets refer to their values as measured by a FIDO observer in an orthonormal frame, and the derivatives are explicitly with respect to the coordinates $(\eta, \mathbf{x})$, denoted by $(', \boldsymbol{\partial})$.

There are also several other useful MHD equations we did not require in our previous work, and so we derive them here.

It is convenient in MHD to include an equation of mass conservation, and the mass density $\rho_m$ will be taken to be one of our fundamental MHD state variables. Taking the FIDO four-velocity to be $u^\alpha$, and introducing a spatial fluid velocity vector $v^\alpha$, defined as

$$v^\alpha \overset{\rm FIDO}{\to} (0, \gamma\mathbf{U}), \tag{2.18}$$

our fluid four-velocity can be written as

$$U^\alpha = \gamma u^\alpha + v^\alpha. \tag{2.19}$$

The covariant form of mass conservation

$$(\rho_m U^\alpha)_{;\alpha} = 0 \tag{2.20}$$

can then be written in 3+1 form as

$$D_\tau\rho_m + \theta\rho_m + \gamma^2\rho_m[\mathbf{U}\cdot(D_\tau\mathbf{U}) + \mathbf{U}\cdot(\mathbf{U}\cdot\nabla)\mathbf{U}] + \alpha^{-1}\nabla\cdot(\alpha\rho_m\mathbf{U}) = 0, \tag{2.21}$$

where the FIDO kinematic properties $\theta$ and $\alpha$, and derivative $D_\tau$ are discussed in [5]. These quantities are calculated in [12] for the FRW metric, so that (2.21) becomes

$$\bar{\rho}_m' + \boldsymbol{\partial}\cdot(\bar{\rho}_m\hat{\mathbf{U}}) + \gamma^2\bar{\rho}_m\hat{\mathbf{U}}\cdot[\hat{\mathbf{U}}' + (\hat{\mathbf{U}}\cdot\boldsymbol{\partial})\hat{\mathbf{U}}] = 0. \tag{2.22}$$

Here we have introduced the "conformalized" variable

$$\bar{\rho}_m = R^3\rho_m. \tag{2.23}$$

Introducing similar quantities for the electromagnetic field,

$$\bar{\mathbf{E}} = R^2\hat{\mathbf{E}}, \quad \bar{\mathbf{B}} = R^2\hat{\mathbf{B}}, \quad \bar{\jmath} = R^3\hat{\jmath}, \tag{2.24}$$

we may write down the Maxwell equations in a simple form [12]:

$$\bar{\mathbf{B}}' = -\boldsymbol{\partial}\times\bar{\mathbf{E}}, \tag{2.25}$$

$$\bar{\mathbf{E}}' = \boldsymbol{\partial}\times\bar{\mathbf{B}} - 4\pi\bar{\jmath}. \tag{2.26}$$

Finally we derive the law of entropy conservation for a perfect fluid. A straightforward textbook calculation shows that

$$U_\alpha T^{\alpha\beta}_{\rm fl\,;\beta} = -nT\frac{ds}{d\tau}, \tag{2.27}$$

where $T$ is the temperature and $s$ is the entropy per baryon. The derivative is with respect to $\tau$, the proper time of a fluid element. It can also be shown from (2.8) that

$$U_\alpha T^{\alpha\beta}_{\rm em\,;\beta} = \gamma(\hat{\jmath} - \rho_e\hat{\mathbf{U}})\cdot(\hat{\mathbf{E}} + \hat{\mathbf{U}}\times\hat{\mathbf{B}}), \tag{2.28}$$

so that (2.7) implies

$$\frac{ds}{d\tau} = \frac{\gamma}{nT}(\hat{\jmath} - \rho_e\hat{\mathbf{U}})\cdot(\hat{\mathbf{E}} + \hat{\mathbf{U}}\times\hat{\mathbf{B}}). \tag{2.29}$$

If we examine the covariant form of Ohm's law [7]

$$J^\mu + U^\mu U^\nu J_\nu = \sigma F^{\mu\nu}U_\nu, \tag{2.30}$$

where $\sigma$ is the conductivity, we deduce in the orthonormal frame

$$\hat{\jmath} + \gamma^2(\hat{\mathbf{U}}\cdot\hat{\jmath})\hat{\mathbf{U}} - \gamma^2\rho_e\hat{\mathbf{U}} = \sigma\gamma(\hat{\mathbf{E}} + \hat{\mathbf{U}}\times\hat{\mathbf{B}}). \quad (2.31)$$

Ideal MHD makes the assumption of infinite conductivity, so that

$$\hat{\mathbf{E}} + \hat{\mathbf{U}}\times\hat{\mathbf{B}} = 0, \quad (2.32)$$

and entropy is consequently conserved:

$$\frac{ds}{d\tau} = 0. \quad (2.33)$$

We now assume $\hat{\mathbf{U}} \ll 1$, and consider small perturbations around background quantities:

$$\bar{\rho}_m = \bar{\rho}_{m0} + \bar{\rho}_{m1}, \quad (2.34)$$
$$\bar{\mathbf{B}} = \bar{\mathbf{B}}_0 + \bar{\mathbf{B}}_1, \quad (2.35)$$
$$\rho = \rho_0 + \rho_1, \quad (2.36)$$
$$p = p_0 + p_1, \quad (2.37)$$
$$T = T_0 + T_1, \quad (2.38)$$
$$n = n_0 + n_1. \quad (2.39)$$

Here $\bar{\mathbf{E}}$ and $\bar{\jmath}$ are already first-order quantities. Let us also substitute (2.32) into the Maxwell equations (2.25), (2.26). Then we have the following set of equations, which form the basis of our work:

$$\bar{\rho}_m' + \boldsymbol{\partial}\cdot(\bar{\rho}_m\hat{\mathbf{U}}) = 0, \quad (2.40)$$

$$\bar{\mathbf{B}}' = \boldsymbol{\partial}\times(\hat{\mathbf{U}}\times\bar{\mathbf{B}}), \quad (2.41)$$

$$\bar{\jmath} = \frac{1}{4\pi}\left[\boldsymbol{\partial}\times\bar{\mathbf{B}} + (\hat{\mathbf{U}}\times\bar{\mathbf{B}})'\right], \quad (2.42)$$

$$\rho' + \frac{3R'}{R}(\rho+p) + (\rho+p)\boldsymbol{\partial}\cdot\hat{\mathbf{U}} = 0, \quad (2.43)$$

$$(\rho+p)'\hat{\mathbf{U}} + (\rho+p)\hat{\mathbf{U}}' + \frac{4R'}{R}(\rho+p)\hat{\mathbf{U}} + \boldsymbol{\partial}p = R\hat{\jmath}\times\hat{\mathbf{B}}, \quad (2.44)$$

$$s' = 0. \quad (2.45)$$

Here we have not yet explicitly made the perturbed substitutions, as we may find it more convenient to express the equations in terms of temperature and number density, depending on whether we are considering the UR or NR limits. Notice also that the energy and momentum equations (2.43), (2.44) are not yet expressed in terms of conformalized variables. This is due to the fact that $\rho$ and $p$ scale differently with respect to $R$ in the UR and NR limits; hence, we will make the explicit substitutions at appropriate times. The conformalized background quantities $\bar{\rho}_{m0}$ and $\bar{\mathbf{B}}_0$ and yet to be defined $\bar{T}_0$ and $\bar{n}_0$ are completely independent of time.

The above set has more variables than equations, and so it is not complete. To close the system we need to introduce an equation of state, relating the variables $\rho_m$, $\rho$, $p$, and possibly $s$. We do not specify this equation yet, as we require different equations of state for the UR and NR limits.

Note that we could also have derived this set of equations by initially assuming a two-fluid model, writing down the energy and momentum equations along with the mass conservation equation for each species, and then linearizing the equations and adding them together in an appropriate fashion. This type of derivation of the one-fluid MHD equations from two-fluid plasma equations has been done for simple NR flat spacetime MHD, as can be seen in many textbooks on the subject, a particularly good example of which is [15].

## III. ULTRARELATIVISTIC LIMIT

The UR limit places the plasma in the context of the very early Universe, where temperatures were of the order of $T \gtrsim 10^{10}$ K. Consequently particle pair creation and/or annihilation effects come into consideration, and so we model an equation of state around these ideas. As discussed at some length in [12] and [13], a quantum field theoretic description would be precise, but the task would be onerous and unnecessary for examining the basic MHD properties of the fluid. Leading order results may be successfully derived with a semiclassical treatment, and this approach may be compared to some of the previous work attempted in this area [10–13].

The energy density for an UR charged quantum pair plasma has been calculated from statistical mechanics arguments in [16–19]. We take only the leading order term from these lengthy expansions, neglecting the chemical potential, which makes the assumption of equal densities of particles and antiparticles. This is in keeping with the plasma approximation $n_+ = n_-$. It is found that

$$\rho = 2\frac{2S+1}{2\pi^2}\Gamma(4)\left\{\begin{matrix}\tau(4)\\ \zeta(4)\end{matrix}\right\}T^4 \equiv aT^4, \quad (3.1)$$

$$p = \frac{1}{3}\rho. \quad (3.2)$$

Note that we have an extra factor of 2 in our definition of $\rho$ compared to that found in [12] and [13]. This is due to the fact that we are considering the plasma as a single fluid, rather than in the previous papers where we wrote down equations for each particle species individually. The above expression is quite general, including descriptions of both fermions (information contained in the $\tau$ function) and bosons (contained in the $\zeta$ function), as well as the spin $S$ of the particles. Thus our formalism is applicable to different regimes in the early Universe, containing many different species of charged particle pairs.

In the radiation-dominated era, where $T \propto R^{-1}$ as in [12], we may define

$$\bar{T} = RT. \tag{3.3}$$

In contrast with the NR results, the particle number density $n$ is not proportional to $\rho$, but obeys the law $n \propto T^3$; hence, we define

$$\bar{n} = R^3 n. \tag{3.4}$$

We now apply the linearization procedure to our set of MHD equations, substituting for $\rho$ and $p$. The energy equation (2.43) yields the result

$$\bar{T}_1' + \frac{1}{3}\bar{T}_0 \boldsymbol{\partial} \cdot \hat{\mathbf{U}} = 0. \tag{3.5}$$

In the UR limit, we find it convenient to include $T$ as one of our state variables, but we can still relate this to particle number density via the mass conservation equation (2.40). We achieve a particle conservation equation

$$\bar{n}_1' + \bar{n}_0 \boldsymbol{\partial} \cdot \hat{\mathbf{U}} = 0, \tag{3.6}$$

and a combination of (3.5) and (3.6) gives us an adiabatic relation

$$\frac{\bar{T}_1'}{\bar{T}_0} = \frac{1}{3}\frac{\bar{n}_1'}{\bar{n}_0}. \tag{3.7}$$

The momentum equation (2.44) becomes, in the UR limit,

$$\hat{\mathbf{U}}' = \frac{3}{16\pi a \bar{T}_0^4}(\boldsymbol{\partial} \times \bar{\mathbf{B}}_1 + \hat{\mathbf{U}}' \times \bar{\mathbf{B}}_0) \times \bar{\mathbf{B}}_0 - \frac{\boldsymbol{\partial}\bar{T}_1}{\bar{T}_0}. \tag{3.8}$$

These equations, coupled with the Maxwell equations, constitute the UR MHD set. Notice that time is nowhere explicit in these equations, which facilitates a straightforward method of solution. Consequently, simple dispersion relations for the plasma modes are found, where all characteristic velocities scale in the same manner with respect to $R$. This is equivalent to saying that they redshift at the same rate.

We now choose the background magnetic field $\bar{\mathbf{B}}_0$ to lie in the $\hat{\mathbf{x}}_3$ direction. Our equations take the form

$$\frac{\partial \bar{B}_{1x_i}}{\partial \eta} = \bar{B}_0 \frac{\partial \hat{U}_i}{\partial x_3}, \quad i = 1,\, 2, \tag{3.9}$$

$$\frac{\partial \bar{B}_{1x_3}}{\partial \eta} = -\bar{B}_0 \left(\frac{\partial \hat{U}_1}{\partial x_1} + \frac{\partial \hat{U}_2}{\partial x_2}\right), \tag{3.10}$$

$$\left(1 + \frac{3}{4}\frac{\bar{B}_0^2}{4\pi a \bar{T}_0^4}\right)\frac{\partial \hat{U}_i}{\partial \eta} + \frac{1}{\bar{T}_0}\frac{\partial \bar{T}_1}{\partial x_i} = \frac{3}{4}\frac{\bar{B}_0}{4\pi a \bar{T}_0^4}\left(\frac{\partial \bar{B}_{1x_i}}{\partial x_3} - \frac{\partial \bar{B}_{1x_3}}{\partial x_i}\right), \quad i = 1,\, 2, \tag{3.11}$$

$$\frac{\partial \hat{U}_3}{\partial \eta} + \frac{1}{\bar{T}_0}\frac{\partial \bar{T}_1}{\partial x_3} = 0, \tag{3.12}$$

$$\frac{\partial \bar{T}_1}{\partial \eta} + \frac{1}{3}\bar{T}_0 \left(\frac{\partial \hat{U}_j}{\partial x_j}\right) = 0, \tag{3.13}$$

$$\frac{\partial s_1}{\partial \eta} = 0. \tag{3.14}$$

Here, as everywhere throughout the paper, summation over the index $j$ is implied.

We also have the Maxwell equation [12]

$$\boldsymbol{\partial} \cdot \bar{\mathbf{B}}_1 = 0, \tag{3.15}$$

which may be used as a boundary condition for $\bar{\mathbf{B}}_1$. This condition will become useful when we calculate the eigenvectors of the MHD states.

In the above form, it is not very clear what the characteristic phase velocities of the MHD modes are. We may combine the above equations into three velocity equations reminiscent to that of Harris [1]: namely,

$$\frac{\partial^2 \hat{U}_i}{\partial \eta^2} = V_1^2 \left[\frac{\partial^2 \hat{U}_i}{\partial x_3^2} + \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{U}_1}{\partial x_1} + \frac{\partial \hat{U}_2}{\partial x_2}\right)\right] + V_2^2 \frac{\partial^2 \hat{U}_j}{\partial x_3 \partial x_j}, \quad i = 1,\, 2, \tag{3.16}$$

$$\frac{\partial^2 \hat{U}_3}{\partial \eta^2} = V_3^2 \frac{\partial^2 \hat{U}_j}{\partial x_3 \partial x_j}. \tag{3.17}$$

We have introduced the characteristic phase velocities

$$V_1^2 = \frac{\bar{B}_0^2/4\pi}{4a\bar{T}_0^4/3 + \bar{B}_0^2/4\pi}, \qquad (3.18)$$

$$V_2^2 = \frac{4a\bar{T}_0^4/9}{4a\bar{T}_0^4/3 + \bar{B}_0^2/4\pi}, \qquad (3.19)$$

$$V_3^2 = \frac{1}{3}. \qquad (3.20)$$

$V_1$ is a relativistic Alfvén velocity, normally written as

$$V_1 = \frac{B_0}{\sqrt{4\pi h(1 + B_0^2/4\pi h)}}, \qquad (3.21)$$

with $h$ the enthalpy. Obviously in our case

$$h = \frac{4}{3}aT_0^4. \qquad (3.22)$$

$V_2$ is a combination of both magnetic and pressure effects, which in ordinary relativistic calculations may reduce to the speed of sound in the limit $c \to \infty$ [1]. Clearly in an UR calculation we cannot take this limit to recover the NR speed of sound. Finally, $V_3 = 1/3$ is the usual UR speed of sound. The general formula from [1] for $V_3$ is

$$V_3 = \frac{\Gamma p_0}{h}, \qquad (3.23)$$

with $\Gamma$ being defined as the gas constant. This is clearly equivalent to our result when it may be noted that in the high temperature limit $\Gamma \to 4/3$.

We also note the following relation that arises between the phase velocities when calculating the dispersion relation:

$$V_3^2 - V_2^2 = V_1^2 V_3^2. \qquad (3.24)$$

This formula will become useful in manipulating the dispersion relation and calculating the eigenvectors of the MHD states.

Having now identified the relevant MHD parameters, we may solve our system of equations to find dispersion relations and eigenstates for the various MHD modes.

Let us first define some more useful conformalized variables:

$$\bar{\omega} = R\omega, \quad \bar{\mathbf{k}} = R\mathbf{k}. \qquad (3.25)$$

Since time is not explicit, we may follow the usual Fourier transform procedure. Thus we assume harmonic space and time dependence $e^{i(\bar{\mathbf{k}}\cdot\mathbf{x} - \bar{\omega}\eta)}$ for the linearized quantities. The dispersion relation may be obtained from (3.16), (3.17), but we use our original set (3.9)–(3.14), so that we may simultaneously find the eigenstates.

Let us define the state vector

$$\mathbf{\Psi} = \left[\hat{U}_1,\, \hat{U}_2,\, \hat{U}_3,\, \bar{B}_{x_1},\, \bar{B}_{x_2},\, \bar{B}_{x_3},\, \bar{T},\, s\right], \qquad (3.26)$$

which has been perturbed:

$$\mathbf{\Psi} = \mathbf{\Psi}_0 + \mathbf{\Psi}_1, \qquad (3.27)$$

with

$$\mathbf{\Psi}_0 = \left[0,\, 0,\, 0,\, 0,\, 0,\, \bar{B}_0,\, \bar{T}_0,\, s_0\right]. \qquad (3.28)$$

The direction cosines

$$\begin{aligned} c_1 &\equiv \cos\theta_1 = \bar{k}_1/|\bar{\mathbf{k}}|, \\ c_2 &\equiv \cos\theta_2 = \bar{k}_2/|\bar{\mathbf{k}}|, \\ c_3 &\equiv \cos\theta_3 = \bar{k}_3/|\bar{\mathbf{k}}|, \end{aligned} \qquad (3.29)$$

and phase velocity

$$v = \frac{\bar{\omega}}{\bar{k}} \qquad (3.30)$$

may be defined and substituted into the system of equations. We obtain a corresponding system, written in matrix form as the eigenvalue equation

$$\left(\overleftrightarrow{\mathbf{A}} - v\,\overleftrightarrow{\mathbf{I}}\right)\mathbf{r} = 0, \qquad (3.31)$$

where

$$\overleftrightarrow{\mathbf{A}} = \begin{bmatrix} 0 & 0 & 0 & \frac{-V_1^2 c_3}{\bar{B}_0} & 0 & \frac{V_1^2 c_1}{\bar{B}_0} & \frac{(1-V_1^2)c_1}{\bar{T}_0} & 0 \\ 0 & 0 & 0 & 0 & \frac{-V_1^2 c_3}{\bar{B}_0} & \frac{V_1^2 c_2}{\bar{B}_0} & \frac{(1-V_1^2)c_2}{\bar{T}_0} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{c_3}{\bar{T}_0} & 0 \\ -\bar{B}_0 c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\bar{B}_0 c_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{B}_0 c_1 & \bar{B}_0 c_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\bar{T}_0 c_1}{3} & \frac{\bar{T}_0 c_2}{3} & \frac{\bar{T}_0 c_3}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (3.32)$$

and $\mathbf{r}$ is defined so that

$$\mathbf{\Psi}_1 = \mathbf{r}\, e^{i(\bar{\mathbf{k}}\cdot\mathbf{x} - \bar{\omega}\eta)}. \qquad (3.33)$$

The solution of this eigenvalue problem yields the following results.

### A. Dispersion relation

The characteristic equation of (3.31) gives the dispersion relation

$$v^2\left(v^2 - V_1^2c_3^2\right)\left[v^4 - \left(V_1^2 + V_2^2 + \frac{1}{3}V_1^2c_3^2\right)v^2 + \frac{1}{3}V_1^2c_3^2\right] = 0. \quad (3.34)$$

This corresponds to the dispersion relation found by Harris with $V_3^2 = \frac{1}{3}$. The $v = 0$ term corresponds to the entropy wave, the first set of parentheses corresponds to the Alfvén mode, and the second set of parentheses corresponds to the fast and slow magnetosound modes. If we were to take the limit $c \to \infty$ and identify $V_2 = V_3 = c_s$, the speed of sound, we would recover the usual NR results.

### B. Entropy wave $v = 0$

This wave is very simple, with no phase velocity. Hence it appears "frozen" in space. The eigenvector is

$$\mathbf{r}^S = [0, 0, 0, 0, 0, 0, 0, 1], \quad (3.35)$$

and so only the entropy is perturbed.

### C. Alfvén wave $v = \epsilon_A V_1|c_3|$

Here we have defined $\epsilon_A = \pm 1$ to indicate the direction of propagation of the wave. The eigenvector of the perturbed amplitudes turns out to be

$$\mathbf{r}^A = [\epsilon_A V_1 c_2, -\epsilon_A V_1 c_1, 0, -\epsilon_C B_0 c_2, \epsilon_C B_0 c_1, 0, 0, 0]. \quad (3.36)$$

Here $\epsilon_C = \pm 1$ denotes the sign of $c_3$, i.e., $\epsilon_C = \cos\theta_3/|\cos\theta_3|$. We see that as usual the Alfvén mode is purely transverse, with no perturbations along the direction of propagation; that is, there are no pressure effects involved.

### D. Magnetosound waves

These waves are a mixture of longitudinal and transverse components. We have two types, the fast magnetosound wave with phase velocity $\epsilon_F V_F$ and the slow magnetosound wave with phase velocity $\epsilon_S V_S$. Here, as usual, $\epsilon_F = \epsilon_S = \pm 1$ denotes the direction of propagation, and $V_F$ and $V_S$ are given by

$$V_F = \left(\frac{1}{2}\left\{V_1^2 + V_2^2 + \frac{1}{3}V_1^2c_3^2 + \left[\left(V_1^2 + V_2^2 + \frac{1}{3}V_1^2c_3^2\right)^2 - \frac{4}{3}V_1^2c_3^2\right]^{1/2}\right\}\right)^{1/2}, \quad (3.37)$$

$$V_S = \left(\frac{1}{2}\left\{V_1^2 + V_2^2 + \frac{1}{3}V_1^2c_3^2 - \left[\left(V_1^2 + V_2^2 + \frac{1}{3}V_1^2c_3^2\right)^2 - \frac{4}{3}V_1^2c_3^2\right]^{1/2}\right\}\right)^{1/2}. \quad (3.38)$$

If we denote

$$v = \begin{cases} V_F, \\ V_S, \end{cases} \quad \epsilon = \begin{cases} \epsilon_F, \\ \epsilon_S, \end{cases} \quad (3.39)$$

we may write down the eigenvector of the perturbed amplitudes for both modes as

$$\mathbf{r}^{F,S} = \left[\epsilon v\frac{V_1^2 - 1}{V_1^2 - v^2}c_1, \epsilon v\frac{V_1^2 - 1}{V_1^2 - v^2}c_2, \frac{\epsilon c_3}{v}, -\bar{B}_0\frac{V_1^2 - 1}{V_1^2 - v^2}c_1c_3\right.$$
$$\left. -\bar{B}_0\frac{V_1^2 - 1}{V_1^2 - v^2}c_2c_3, \bar{B}_0\frac{V_1^2 - 1}{V_1^2 - v^2}(1 - c_3^2), \bar{T}_0, 0\right]. \quad (3.40)$$

We see in conclusion that the UR waves resemble their NR counterparts in structure, though of course the inclusion of relativistic effects has complicated the relations somewhat. The important result is that all phase velocities scale in the same way and are in fact independent of $R$. Thus the various modes do not redshift differently with respect to one another.

## IV. NONRELATIVISTIC LIMIT

We now consider an electron-proton–helium nuclei plasma, as may have existed around the period of recombination. The age of the Universe at this epoch of time was approximately $10^5$ yr, and the temperatures were around 3000 K. Consequently $m \gg T$, and we may treat the plasma nonrelativistically. Around this time the mass-energy density of matter came to exceed that of radiation, and so we enter what is known as the matter-dominated era. We consider both the pre- and post-recombination periods. In the first case, thermal photons dominated the temperature and, hence, geometry,

$$T \sim R^{-1}, \quad R = \left(\frac{t}{t_i}\right)^{1/2} = \frac{\eta}{2t_i}, \quad (4.1)$$

while in the latter, the photons had decoupled from matter so that

$$T \sim R^{-2}, \quad R = \left(\frac{t}{t_i}\right)^{2/3} = \frac{\eta^2}{9t_i^2}. \tag{4.2}$$

Here we have introduced a fiducial time constant $t_i$, which is arbitrary. For simplicity we assume it to be of the same order as $t$, so that $R$ is of order unity. This will facilitate analyzing the asymptotics of our solutions, and comparing the "conformalized" variables to the physically measured ones.

Our equation of state now models that of a standard NR ideal gas. Thus

$$\rho = n(m + \tfrac{3}{2}T), \tag{4.3}$$

$$p = nT, \tag{4.4}$$

where we have of course assumed $T_+ = T_- = T$, and as always, the plasma approximation $n_+ = n_-$.

After substitution of (4.3) and (4.4), the energy equation (2.43) reduces to

$$R^2 T = \kappa (R^3 n)^{2/3}, \tag{4.5}$$

which gives

$$\frac{\bar{n}_1}{\bar{n}_0} = \frac{3}{2}\frac{T_1}{T_0} \tag{4.6}$$

after linearization. This is our adiabatic relation.

We perform the same substitutions into the momentum equation (2.44), also using (4.6) to eliminate $T_1$. In the NR case, we find the conformalized variable

$$\bar{\mathbf{U}} = R\hat{\mathbf{U}} \tag{4.7}$$

the most convenient to work with. The resulting momentum equation takes the form

$$\bar{\rho}_{m0}\bar{\mathbf{U}}' + \frac{5}{3}RT_0\boldsymbol{\partial}\bar{n}_1 = \bar{\jmath} \times \bar{\mathbf{B}}_0. \tag{4.8}$$

We may introduce the NR speed of sound, $c_S$, into this equation by noting

$$c_S^2 = \frac{\Gamma p_0}{\rho_{m0}} = \frac{5}{3}\frac{n_0 T_0}{n_0(m_+ + m_-)}. \tag{4.9}$$

We now take $\bar{\rho}_m$ to be one of the fundamental quantities in our MHD state vector rather than $\bar{T}$, which was used in the UR limit. This approach more closely parallels the usual textbook flat spacetime methods. Let us also neglect the displacement current term $(\hat{\mathbf{U}} \times \bar{\mathbf{B}}_0)'$ in the Maxwell equations, which is a valid approximation in the NR limit, if phase velocities are far less than the speed of light. We then have the resulting NR MHD set of equations:

$$\bar{\rho}_{m1}' + \frac{\bar{\rho}_{m0}}{R}\boldsymbol{\partial}\cdot\bar{\mathbf{U}} = 0, \tag{4.10}$$

$$\bar{\rho}_{m0}\bar{\mathbf{U}}' + Rc_S^2\boldsymbol{\partial}\bar{\rho}_{m1} = \frac{1}{4\pi}(\boldsymbol{\partial}\times\bar{\mathbf{B}}_1)\times\bar{\mathbf{B}}_0, \tag{4.11}$$

$$\bar{\mathbf{B}}_1' = \frac{1}{R}\boldsymbol{\partial}\times(\bar{\mathbf{U}}\times\bar{\mathbf{B}}_0), \tag{4.12}$$

$$s' = 0. \tag{4.13}$$

We now turn our attention to pre- and post-recombination each separately.

## V. PRE-RECOMBINATION

As mentioned above, in pre-recombination $T \sim R^{-1}$; hence, we may define the time-independent quantity

$$\bar{c}_S^2 = Rc_S^2. \tag{5.1}$$

It then follows that the sound velocity and NR Alfvén velocity

$$\bar{c}_A^2 = Rc_A^2 = \frac{\bar{B}_0^2}{4\pi\bar{\rho}_{m0}} \tag{5.2}$$

scale in the same manner, which makes pre-recombination considerably more easy to solve mathematically than post-recombination. Both cases, however, still have explicit time dependences in the equations, and so solutions cannot follow the simple exponential form of the UR case.

If we choose $\bar{\mathbf{B}}_0 = \bar{B}_0\hat{\mathbf{x}}_3$, we have the pre-recombination set of equations

$$\frac{\partial\bar{\rho}_{m1}}{\partial\eta} + \frac{2t_i}{\eta}\bar{\rho}_{m0}\left(\frac{\partial\bar{U}_j}{\partial x_j}\right) = 0, \tag{5.3}$$

$$\frac{\partial\bar{B}_{1x_i}}{\partial\eta} = \frac{2t_i}{\eta}\bar{B}_0\frac{\partial\bar{U}_i}{\partial x_3}, \quad i = 1, 2, \tag{5.4}$$

$$\frac{\partial\bar{B}_{1x_3}}{\partial\eta} = -\frac{2t_i}{\eta}\bar{B}_0\left(\frac{\partial\bar{U}_1}{\partial x_1} + \frac{\partial\bar{U}_2}{\partial x_2}\right), \tag{5.5}$$

$$\bar{\rho}_{m0}\frac{\partial\bar{U}_i}{\partial\eta} + \bar{c}_S^2\frac{\partial\bar{\rho}_{m1}}{\partial x_i} = \frac{\bar{B}_0}{4\pi}\left(\frac{\partial\bar{B}_{1x_i}}{\partial x_3} - \frac{\partial\bar{B}_{1x_3}}{\partial x_i}\right), \quad i = 1, 2, \tag{5.6}$$

$$\bar{\rho}_{m0}\frac{\partial\bar{U}_3}{\partial\eta} + \bar{c}_S^2\frac{\partial\bar{\rho}_{m1}}{\partial x_3} = 0, \tag{5.7}$$

$$\frac{\partial s}{\partial\eta} = 0. \tag{5.8}$$

To solve this set of equations, we may proceed in a similar manner to the UR limit, and eliminate all the variables except velocity, to obtain the coupled set of velocity equations

$$\frac{\partial^2 \bar{U}_i}{\partial \eta^2} = \frac{2t_i}{\eta}\bar{c}_A^2 \left[\frac{\partial \bar{U}_i}{\partial x_3} + \frac{\partial}{\partial x_i}\left(\frac{\partial \bar{U}_1}{\partial x_1} + \frac{\partial \bar{U}_2}{\partial x_2}\right)\right] + \frac{2t_i}{\eta}\bar{c}_S^2 \frac{\partial^2 \bar{U}_j}{\partial x_3 \partial x_j}, \quad i = 1,\, 2, \tag{5.9}$$

$$\frac{\partial^2 \bar{U}_3}{\partial \eta^2} = \frac{2t_i}{\eta}\bar{c}_S^2 \frac{\partial^2 \bar{U}_j}{\partial x_3 \partial x_j}. \tag{5.10}$$

If we assume an $e^{i\mathbf{k}\cdot\mathbf{x}}$ dependence for the spatial variables, these equations have the characteristic form

$$\frac{d^2 y}{d\eta^2} = -\frac{2t_i \bar{\omega}^2}{\eta} y. \tag{5.11}$$

It turns out that a solution to this equation is

$$y = \eta^{1/2} Z_{\pm 1}\left(\sqrt{8t_i \bar{\omega}^2 \eta}\right), \tag{5.12}$$

where $Z$ is any Bessel function. We choose Hankel functions, as these most resemble an exponential solution. The functions $H_1^{(1),(2)}$ and $H_{-1}^{(1),(2)}$ only differ by a constant phase factor, and so let us choose $H_1^{(1),(2)}$ as our solution. We may then postulate a solution to (5.9), (5.10) of the form

$$\bar{U}_i = \tilde{U}_i \eta^{1/2} H_1\left(\sqrt{8t_i \bar{\omega}^2 \eta}\right) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5.13}$$

where $H_1$ denotes $H_1^{(1)}$ and $H_1^{(2)}$, and substitute this in to derive a dispersion relation for $\bar{\omega}$. Instead, we will substitute back into our original set (5.3)–(5.8), so that we may simultaneously find the eigenstates.

First, we may use the property

$$\frac{dH_0(\lambda \eta^1/2)}{d\eta} = -\frac{1}{2}\lambda \eta^{-1/2} H_1(\lambda \eta^{1/2}), \tag{5.14}$$

to assume our MHD variables have the following form of solution:

$$\bar{B}_{1i} = \tilde{B}_{1x_i} H_0\left(\sqrt{8t_i \bar{\omega}^2 \eta}\right) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5.15}$$

$$\bar{\rho}_{m1} = \tilde{\rho}_{m1} H_0\left(\sqrt{8t_i \bar{\omega}^2 \eta}\right) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5.16}$$

where $H_0$ denotes $H_0^{(1)}$ and $H_0^{(2)}$. A substitution of (5.13), (5.15), and (5.16) into (5.3)–(5.8) leads to the matrix equation

$$\begin{bmatrix} i(2t_i)^{1/2}v & 0 & 0 & \frac{\bar{c}_A^2 c_3}{\bar{B}_0} & 0 & -\frac{\bar{c}_A^2 c_1}{\bar{B}_0} & -\frac{\bar{c}_S^2 c_1}{\bar{\rho}_{m0}} & 0 \\ 0 & i(2t_i)^{1/2}v & 0 & 0 & \frac{\bar{c}_A^2 c_3}{\bar{B}_0} & -\frac{\bar{c}_A^2 c_2}{\bar{B}_0} & -\frac{\bar{c}_S^2 c_2}{\bar{\rho}_{m0}} & 0 \\ 0 & 0 & i(2t_i)^{1/2}v & 0 & 0 & 0 & -\frac{\bar{c}_S^2 c_3}{\bar{\rho}_{m0}} & 0 \\ -(2t_i)^{1/2}\bar{B}_0 c_3 & 0 & 0 & iv & 0 & 0 & 0 & 0 \\ 0 & -(2t_i)^{1/2}\bar{B}_0 c_3 & 0 & 0 & iv & 0 & 0 & 0 \\ (2t_i)^{1/2}\bar{B}_0 c_1 & (2t_i)^{1/2}\bar{B}_0 c_2 & 0 & 0 & 0 & iv & 0 & 0 \\ (2t_i)^{1/2}\bar{\rho}_{m0} c_1 & (2t_i)^{1/2}\bar{\rho}_{m0} c_2 & (2t_i)^{1/2}\bar{\rho}_{m0} c_3 & 0 & 0 & 0 & iv & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & iv \end{bmatrix} \mathbf{r} = 0, \tag{5.17}$$

where

$$\mathbf{r} = \left[\tilde{U}_1,\, \tilde{U}_2,\, \tilde{U}_3,\, \tilde{B}_{1x_1},\, \tilde{B}_{1x_2},\, \tilde{B}_{1x_3},\, \tilde{\rho}_{m1},\, \tilde{s}\right]. \tag{5.18}$$

From this equation we obtain the following MHD modes.

### A. Dispersion relation

We achieve the standard NR flat spacetime dispersion relation, with the exception that the phase velocities are conformalized quantities:

$$v^2\left(v^2 - \bar{c}_A^2 c_3^2\right)\left[v^4 - v^2(\bar{c}_A^2 + \bar{c}_S^2) + \bar{c}_A^2 \bar{c}_S^2 c_3^2\right] = 0, \tag{5.19}$$

where all variables have the same definition as in the UR case. Thus a FIDO observer would measure phase velocities redshifted by the factor $R$.

### B. Entropy wave $v = 0$

As usual, we have the eigenvector

$$\mathbf{r}^S = [0, 0, 0, 0, 0, 0, 0, 1]. \tag{5.20}$$

### C. Alfvén wave $v = \epsilon_A \bar{c}_A |c_3|$

This mode is very similar to the UR results (and of course flat spacetime results), with the relativistic Alfvén velocity being replaced by its NR counterpart. Defining

$$\mathbf{\Psi}_1 = \left[\bar{U}_1,\ \bar{U}_2,\ \bar{U}_3,\ \bar{B}_{1x_1},\ \bar{B}_{1x_2},\ \bar{B}_{1x_3},\ \bar{\rho}_{m1},\ s\right], \tag{5.21}$$

we write down the eigenstate as

$$\mathbf{\Psi}_1^A = \begin{bmatrix} \frac{\epsilon_A \bar{c}_A}{(2t_i)^{1/2}} c_2 \eta^{1/2} H_1\left(\sqrt{8t_i\bar{\omega}_A^2\eta}\right) \\ -\frac{\epsilon_A \bar{c}_A}{(2t_i)^{1/2}} c_1 \eta^{1/2} H_1\left(\sqrt{8t_i\bar{\omega}_A^2\eta}\right) \\ 0 \\ -i\epsilon_C \bar{B}_0 c_2 H_0\left(\sqrt{8t_i\bar{\omega}_A^2\eta}\right) \\ i\epsilon_C \bar{B}_0 c_1 H_0\left(\sqrt{8t_i\bar{\omega}_A^2\eta}\right) \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{i\bar{\mathbf{k}}\cdot\mathbf{x}}, \tag{5.22}$$

where $\bar{\omega}_A^2 = \bar{c}_A^2 \bar{k}^2 c_3^2$. In the above, $\epsilon_A = +1$ corresponds to the $H^{(1)}$ mode, and $\epsilon_A = -1$ corresponds to the $H^{(2)}$ mode, analogously to the flat spacetime $\pm e^{\pm i\omega t}$ results; i.e., the velocity of the medium is reversed for waves propagating in the opposite direction.

### D. Magnetosound waves $v = \epsilon_F V_F,\ \epsilon_S V_S$

Once again we have fast and slow magnetosound modes, with

$$V_F = \left\{ \frac{1}{2}\left[\bar{c}_A^2 + \bar{c}_S^2 + \left((\bar{c}_A^2 + \bar{c}_S^2)^2 - 4\bar{c}_A^2\bar{c}_S^2c_3^2\right)^{1/2}\right]\right\}^{1/2}, \tag{5.23}$$

$$V_S = \left\{ \frac{1}{2}\left[\bar{c}_A^2 + \bar{c}_S^2 - \left((\bar{c}_A^2 + \bar{c}_S^2)^2 - 4\bar{c}_A^2\bar{c}_S^2c_3^2\right)^{1/2}\right]\right\}^{1/2}. \tag{5.24}$$

The eigenstate is a mixture of transverse and longitudinal components, and is given by

$$\mathbf{\Psi}_1^{F,S} = \begin{bmatrix} \frac{\epsilon v}{(2t_i)^{1/2}} \frac{\bar{c}_S^2 c_1}{v^2 - \bar{c}_A^2} \eta^{1/2} H_1\left(\sqrt{8t_i\bar{\omega}^2\eta}\right) \\ \frac{\epsilon v}{(2t_i)^{1/2}} \frac{\bar{c}_S^2 c_2}{v^2 - \bar{c}_A^2} \eta^{1/2} H_1\left(\sqrt{8t_i\bar{\omega}^2\eta}\right) \\ \frac{\epsilon}{(2t_i)^{1/2}} \frac{\bar{c}_S^2}{v} \eta^{1/2} H_1\left(\sqrt{8t_i\bar{\omega}^2\eta}\right) \\ -i\frac{\bar{B}_0\bar{c}_S^2}{v^2 - \bar{c}_A^2} c_1 c_3 H_0\left(\sqrt{8t_i\bar{\omega}^2\eta}\right) \\ -i\frac{\bar{B}_0\bar{c}_S^2}{v^2 - \bar{c}_A^2} c_2 c_3 H_0\left(\sqrt{8t_i\bar{\omega}^2\eta}\right) \\ i\frac{\bar{B}_0\bar{c}_S^2}{v^2 - \bar{c}_A^2} (1 - c_3^2) H_0\left(\sqrt{8t_i\bar{\omega}^2\eta}\right) \\ i\bar{\rho}_{m0} H_0\left(\sqrt{8t_i\bar{\omega}^2\eta}\right) \\ 0 \end{bmatrix} e^{i\bar{\mathbf{k}}\cdot\mathbf{x}}, \tag{5.25}$$

where

$$\bar{\omega}^2 = \begin{cases} V_F^2\bar{k}^2, \\ V_S^2\bar{k}^2, \end{cases} \qquad \epsilon = \begin{cases} \epsilon_F, \\ \epsilon_S. \end{cases} \tag{5.26}$$

The same relation as above also exists between $\epsilon_F$, $\epsilon_S$ and $H^{(1)}$, $H^{(2)}$.

## VI. POST-RECOMBINATION

In the post-recombination case, $T \sim R^{-2}$, and so the time-independent sound velocity must be defined as

$$\bar{c}_S^2 = R^2 c_S^2. \tag{6.1}$$

Now $c_S$ scales differently to $c_A$; hence, the equations take a more complicated form. Thus, choosing $\bar{\mathbf{B}}_0 = \bar{B}_0\hat{\mathbf{x}}_3$ again, we may write down the post-recombination set of equations from (4.10)–(4.13) as follows:

$$\frac{\partial\bar{\rho}_{m1}}{\partial\eta} + \frac{9t_i^2}{\eta^2}\bar{\rho}_{m0}\left(\frac{\partial\bar{U}_j}{\partial x_j}\right) = 0, \tag{6.2}$$

$$\frac{\partial\bar{B}_{1x_i}}{\partial\eta} = \frac{9t_i^2}{\eta^2}\bar{B}_0\frac{\partial\bar{U}_i}{\partial x_3}, \quad i = 1, 2, \tag{6.3}$$

$$\frac{\partial\bar{B}_{1x_3}}{\partial\eta} = -\frac{9t_i^2}{\eta^2}\bar{B}_0\left(\frac{\partial\bar{U}_1}{\partial x_1} + \frac{\partial\bar{U}_2}{\partial x_2}\right), \tag{6.4}$$

$$\bar{\rho}_{m0}\frac{\partial\bar{U}_i}{\partial\eta} + \frac{9t_i^2}{\eta^2}\bar{c}_S^2\frac{\partial\bar{\rho}_{m1}}{\partial x_i} = \frac{\bar{B}_0}{4\pi}\left(\frac{\partial\bar{B}_{1x_i}}{\partial x_3} - \frac{\partial\bar{B}_{1x_3}}{\partial x_i}\right), \quad i = 1, 2, \tag{6.5}$$

$$\bar{\rho}_{m0}\frac{\partial\bar{U}_3}{\partial\eta} + \frac{9t_i^2}{\eta^2}\bar{c}_S^2\frac{\partial\bar{\rho}_{m1}}{\partial x_3} = 0, \tag{6.6}$$

$$\frac{\partial s}{\partial\eta} = 0. \tag{6.7}$$

It is impossible to obtain a set of second-order velocity equations from the above. The equations must be differentiated several times to eliminate $\bar{\mathbf{B}}_1$ and $\bar{\rho}_{m1}$. The simplest set obtainable is

$$\frac{\partial}{\partial\eta}\left(\eta^3\frac{\partial^2\bar{U}_i}{\partial\eta^2}\right) = (9t_i^2\bar{c}_S)^2\left(\frac{1}{\eta}\frac{\partial}{\partial\eta}-\frac{3}{\eta^2}\right)\frac{\partial^2\bar{U}_j}{\partial x_i\partial x_j} + 9t_i^2\bar{c}_A^2\frac{\partial}{\partial\eta}\eta\left[\frac{\partial^2\bar{U}_i}{\partial x_3^2}+\frac{\partial}{\partial x_i}\left(\frac{\partial\bar{U}_1}{\partial x_1}+\frac{\partial\bar{U}_2}{\partial x_2}\right)\right], \quad i=1,\,2, \tag{6.8}$$

$$\frac{\partial^2\bar{U}_3}{\partial\eta^2}+\frac{2}{\eta}\frac{\partial\bar{U}_3}{\partial\eta}-\frac{(9t_i^2\bar{c}_S)^2}{\eta^4}\frac{\partial^2\bar{U}_j}{\partial x_3\partial x_j}=0. \tag{6.9}$$

We now have an eighth-order differential equation, and so would expect some spurious solutions resulting from differentiating to eliminate variables. There is no clear or systematic approach to solve this system; hence, we will tackle the problem first by taking some limiting cases, and then by examining specific directions of $\bar{\mathbf{k}}$.

### A. $\bar{c}_A \to 0$ limit

Rather than using (6.8), (6.9) and carrying around spurious solutions, we return to the original equations (6.2)–(6.6), set $\bar{\mathbf{B}}_0 = 0$, and obtain the simple system

$$\frac{\partial^2\bar{U}_i}{\partial\eta^2}+\frac{2}{\eta}\frac{\partial\bar{U}_i}{\partial\eta}-\frac{(9t_i^2\bar{c}_S)^2}{\eta^4}\frac{\partial^2\bar{U}_j}{\partial x_i\partial x_j}=0, \quad i=1,\,2,\,3. \tag{6.10}$$

If we assume solutions of the form

$$\bar{U}_i = \tilde{U}_i e^{i(\bar{\mathbf{k}}\cdot\mathbf{x}-9t_i^2\bar{\omega}/\eta)}, \tag{6.11}$$

this system reduces to a simple algebraic matrix form

$$\begin{bmatrix} v^2-\bar{c}_S^2c_1^2 & -\bar{c}_S^2c_1c_2 & -\bar{c}_S^2c_1c_3 \\ -\bar{c}_S^2c_1c_2 & v^2-\bar{c}_S^2c_2^2 & -\bar{c}_S^2c_2c_3 \\ -\bar{c}_S^2c_1c_3 & -\bar{c}_S^2c_2c_3 & v^2-\bar{c}_S^2c_3^2 \end{bmatrix}\begin{bmatrix}\tilde{U}_1\\ \tilde{U}_2\\ \tilde{U}_3\end{bmatrix}=0. \tag{6.12}$$

By taking the determinant, it is simple to deduce the dispersion relationship

$$v^4\left(v^2-\bar{c}_S^2\right)=0. \tag{6.13}$$

Thus we simply obtain pure sound waves $v = \pm\bar{c}_S$, as one would expect. The solutions $v = 0$ of multiplicity 4 merely correspond to the fact that the Alfvén and slow magnetosound velocities vanish in this limit. This result corresponds to that found by Holcomb [11], if we examine the argument of the exponential

$$\frac{9t_i^2\bar{k}\bar{c}_S}{\eta}=\frac{3t_i^{4/3}\bar{k}\bar{c}_S}{t^{1/3}}. \tag{6.14}$$

This is the argument of the Holcomb solution.

We find the locally measured FIDO frequency $\omega_F$ by differentiating the argument with respect to $t$. Then

$$\omega_F=\left(\frac{t}{t_i}\right)^{-4/3}\bar{k}\bar{c}_S = R^{-2}(Rk)(Rc_S). \tag{6.15}$$

Thus the locally measured frequency is just the redshifted product of $k$ and a phase velocity, as we have been finding throughout our work.

### B. $\bar{c}_S \to 0$ limit

Although unrealistic for an early Universe plasma, we momentarily consider a pressureless gas, in order to compare our results to that of Holcomb. We derive the set of equations

$$\frac{\partial^2\bar{U}_i}{\partial\eta^2}=\frac{9t_i^2\bar{c}_A^2}{\eta^2}\left[\frac{\partial^2\bar{U}_i}{\partial x_3^2}+\frac{\partial}{\partial x_i}\left(\frac{\partial\bar{U}_1}{\partial x_1}+\frac{\partial\bar{U}_2}{\partial x_2}\right)\right], \quad i=1,\,2. \tag{6.16}$$

With no pressure, we only expect longitudinal modes, as is confirmed by the fact that the equation for $\bar{U}_3$ becomes trivial.

Making the usual spatial dependence assumption of $e^{i\bar{\mathbf{k}}\cdot\mathbf{x}}$, we find (6.16) resembles the Euler equation

$$\frac{d^2y}{d\eta^2}+\frac{9t_i^2\bar{\omega}^2}{\eta^2}y=0. \tag{6.17}$$

By solving this equation, we may assume solutions for $\bar{U}_i$ of the form

$$\bar{U}_i=\tilde{U}_i\eta^{1/2\pm\sqrt{D}/2}e^{i\bar{\mathbf{k}}\cdot\mathbf{x}}, \tag{6.18}$$

where $D = 1-36t_i^2\bar{\omega}^2$. Substituting this back into (6.16), we obtain the dispersion relation

$$v^4-\bar{c}_A^2(1+c_3^2)v^2+\bar{c}_A^4c_3^2=0. \tag{6.19}$$

Thus the slow magnetosound mode vanishes, the fast becomes just $v^2=\bar{c}_A^2$, and the usual Alfvén mode $v^2=\bar{c}_A^2c_3^2$ is retained. A substitution of $t$ for $\eta$ will show that our solutions agree with that of Holcomb. Note that $D$ can take on a range of values of differing sign. Thus we have three separate cases to consider.

(i) $D<0$. Here the general solution may be written

$$\bar{U}_i \sim \eta^{1/2}\left[c_{i1}\exp\left(i\frac{1}{2}\sqrt{36t_i^2\bar{\omega}^2-1}\ln\eta\right)+c_{i2}\exp\left(-i\frac{1}{2}\sqrt{36t_i^2\bar{\omega}^2-1}\ln\eta\right)\right], \tag{6.20}$$

where

$$\bar{\omega}^2 = \begin{cases} \bar{k}^2\bar{c}_A^2, \\ \bar{k}^2\bar{c}_A^2 c_3^2, \end{cases} \tag{6.21}$$

and $c_{i1}$ and $c_{i2}$ are arbitrary constants of integration. We have an oscillatory solution, with an unusual logarithmic time dependence. The FIDO measured frequency turns out to be

$$\omega_F = \frac{\sqrt{36t_i^2\bar{\omega}^2-1}}{6t}. \tag{6.22}$$

We may assume for the present that $\bar{\omega}^2 t_i^2 \gg 1$. (We will present numerical estimates to justify this assumption later.) Then we may write

$$\omega_F = \bar{\omega}\left(\frac{t}{t_i}\right)^{-1}\left(1-\frac{(t/t_i)^2}{36t^2\bar{\omega}^2}\right)^{1/2} = \omega\left(1-\frac{1}{36t^2\omega^2}\right)^{1/2}. \tag{6.23}$$

Thus the usual flat spacetime frequencies

$$\omega = \begin{cases} kc_A \\ kc_A c_3 \end{cases} \tag{6.24}$$

have been shifted by a time dependent correction factor, as was found in many cases in [12,13]. This same effect would have been observed in the pre-recombination solutions had we examined the asymptotics of the Bessel functions, though this was rather unnecessary, considering we had derived the full solution and dispersion relations, and the frequencies were obtained by inspection in an obvious manner.

(ii) $D = 0$. In this case the solution takes the special form

$$\bar{U}_i \sim \eta^{1/2}(c_{i1} + c_{i2}\ln\eta). \tag{6.25}$$

(iii) $D > 0$. Here we get nonoscillatory solutions

$$\bar{U}_i \sim \eta^{1/2}\left(c_{i1}\eta^{\sqrt{D}/2} + c_{i2}\eta^{-\sqrt{D}/2}\right). \tag{6.26}$$

The last two cases show that for a certain frequency range, the modes are evanescent; i.e., there are no propagating waves. They simply decay in time, which becomes clear if we take note of the fact that the physically measured quantity is $\hat{\mathbf{U}} \sim \eta^{-2}\bar{\mathbf{U}}$.

The question still remains at what point the modes become evanescent rather than oscillatory. If we take the usual value of $\rho_m$ around recombination, namely, $\rho_m \sim 10^9$ particles per cubic meter, we find $c_A \sim 10^2 B$, where $B$ is the magnetic field measured in gauss. It then follows that

$$D \sim 1 - 10^{17}\frac{B^2}{l^2}, \tag{6.27}$$

where $l$ the wavelength is measured in meters, and we have taken $t_i \sim 10^{13}$ s. This relation shows that for all physically reasonable type of MHD modes, $D$ is far less than zero, and so we generally expect only oscillatory modes to exist in the post-recombination era. These numerics also justify our assumption made when calculating the FIDO frequency.

In summary, though, the unusual logarithmic behavior of the modes with time, as well as the possibility of the existence of evanescent modes, show us that the expanding Universe significantly alters the properties of a MHD fluid.

Now that we have examined the limiting cases, we will let $\mathbf{k}$ take specific directions to enable the equations to take a more tractable form. We will always assume the spatial dependence of the solutions follows an $e^{i\mathbf{k}\cdot\mathbf{x}}$ form.

### C. k parallel to B

#### 1. *Longitudinal mode*

Here only $U_3$ is involved. The equation takes the simple form

$$\frac{\partial^2\bar{U}_3}{\partial\eta^2} + \frac{2}{\eta}\frac{\partial\bar{U}_3}{\partial\eta} + \frac{(9t_i^2\bar{k}\bar{c}_S)^2}{\eta^4}\bar{U}_3 = 0. \tag{6.28}$$

Solutions to this are identical to that discussed in the $\bar{c}_A \to 0$ limit, hence we need no further discussion. We have simple sound waves propagating along the direction of the magnetic field.

#### 2. *Transverse mode*

Here we must examine the $\bar{U}_1$ and $\bar{U}_2$ components. The equations reduce to

$$\frac{d}{d\eta}\eta^3\left(\frac{d^2\bar{U}_i}{d\eta^2} + \frac{9t_i^2\bar{k}^2\bar{c}_A^2}{\eta^2}\bar{U}_i\right) = 0, \quad i = 1, 2. \tag{6.29}$$

Thus the differential equations may be written as inhomogeneous second-order types. The homogeneous solutions recover the results of $\bar{c}_S \to 0$ (though of course only the $v^2 = \bar{c}_A^2$ solution). This is logical, since the transverse modes contain no sound component.

The third inhomogeneous solution has the form $\bar{U}_i = \eta^{-1}$. This may be rejected when we substitute back into our original equations, and find it is not a compatible solution. Here we have a spurious solution alluded to earlier, caused by differentiating the original equations a number of times.

### D. k perpendicular to B

We are finally left with the most difficult but important case. The solution to this mode, which as far as we know has not been achieved before, is the most interesting, as it shows how $\bar{c}_A$ and $\bar{c}_S$ combine in a dispersion relation, when they scale differently. This does lead to some intricate forms of solution though.

Since we only require $\mathbf{k}$ to be perpendicular to $\mathbf{B}$, we have some freedom in our choice of specific direction. To facilitate a solution, we need to reduce the coupled equations into a single ordinary differential equation (ODE), and so let us take $\bar{\mathbf{k}} = \bar{k}\hat{\mathbf{x}}_1$, when $\bar{U}_2$ will vanish. We obtain the ODE

$$\frac{d^3\bar{U}}{d\eta^3} + \frac{3}{\eta}\frac{d^2\bar{U}}{d\eta^2} + \frac{1}{\eta^2}\left[9t_i^2\bar{k}^2\bar{c}_A^2 + \frac{(9t_i^2\bar{k}\bar{c}_S)^2}{\eta^2}\right]\frac{d\bar{U}}{d\eta} + \frac{1}{\eta^3}\left[9t_i^2\bar{k}^2\bar{c}_A^2 - 3\frac{(9t_i^2\bar{k}\bar{c}_S)^2}{\eta^2}\right]\bar{U} = 0, \tag{6.30}$$

where now we may take $\bar{U} = \bar{U}_1$.

To manipulate this equation into a more recognizable form, we transform it to normal form, where the second highest derivative is removed [20]. Thus we let

$$\bar{U}(\eta) = u(\eta)v(\eta), \tag{6.31}$$

and try to eliminate $d^2u/d\eta^2$ by choosing $v$ in an appropriate fashion. It turns out $v(\eta)$ is given by

$$v(\eta) = \exp\left[-\frac{1}{3}\int\frac{3}{\eta}d\eta\right] = \frac{1}{\eta}, \tag{6.32}$$

and $u(\eta)$ is given by the differential equation

$$\frac{d^3u}{d\eta^3} + \frac{1}{\eta^2}\left[9t_i^2\bar{k}^2\bar{c}_A^2 + \frac{(9t_i^2\bar{k}\bar{c}_S)^2}{\eta^2}\right]\frac{du}{d\eta} - 4\frac{(9t_i^2\bar{k}\bar{c}_S)^2}{\eta^5}u = 0. \tag{6.33}$$

If we make the variable substitution $\zeta = \eta^{-2}$, we obtain

$$\frac{d^3u}{d\zeta^3} + \frac{9}{2\zeta}\frac{d^2u}{d\zeta^2} + \frac{1}{4\zeta}\left[(9t_i^2\bar{k}\bar{c}_S)^2 + \frac{12 + 9t_i^2\bar{k}^2\bar{c}_A^2}{\zeta}\right]\frac{du}{d\zeta} + \frac{(9t_i^2\bar{k}\bar{c}_S)^2}{2\zeta^2}u = 0. \tag{6.34}$$

We finally have transformed our equation into a recognizable form, if one considers the following.

The generalized hypergeometric functions and equations are discussed at length in a variety of books, e.g., [21,22]. In particular, (6.34) resembles the differential equation for the function $u = {}_1F_2(a;\, b_1,\, b_2;\, \kappa\zeta)$, which has three linearly independent solutions; namely,

$$u = \begin{cases} {}_1F_2(a;\, b_1,\, b_2;\, \kappa\zeta), \\ \zeta^{1-b_1}\,{}_1F_2(1+a-b_1;\, 2-b_1,\, 1+b_2-b_1;\, \kappa\zeta), \\ \zeta^{1-b_2}\,{}_1F_2(1+a-b_2;\, 1+b_1-b_2,\, 2-b_2;\, \kappa\zeta). \end{cases} \tag{6.35}$$

Here the $b_i$ may not be negative integers or zero, or differ from each other by a negative integer or zero. The general differential equation takes the form

$$\frac{d^3u}{d\zeta^3} + \frac{1+b_1+b_2}{\zeta}\frac{d^2u}{d\zeta^2} + \left(\frac{b_1b_2}{\zeta^2} - \frac{\kappa}{\zeta}\right)\frac{du}{d\zeta} - \frac{a\kappa}{\zeta^2}u = 0. \tag{6.36}$$

By comparing this to (6.34), we may deduce the following set of solutions for $\bar{U}$:

$$\bar{U} = \begin{cases} y_1 = \eta^{-1}\,{}_1F_2\left(2;\, \tfrac{7}{4}+\tfrac{1}{4}\sqrt{D},\, \tfrac{7}{4}-\tfrac{1}{4}\sqrt{D};\, -\tfrac{\lambda^2}{\eta^2}\right), \\ y_2 = \eta^{1/2+\sqrt{D}/2}\,{}_1F_2\left(\tfrac{5}{4}-\tfrac{1}{4}\sqrt{D};\, \tfrac{1}{4}-\tfrac{1}{4}\sqrt{D},\, 1-\tfrac{1}{2}\sqrt{D};\, -\tfrac{\lambda^2}{\eta^2}\right), \\ y_3 = \eta^{1/2-\sqrt{D}/2}\,{}_1F_2\left(\tfrac{5}{4}+\tfrac{1}{4}\sqrt{D};\, \tfrac{1}{4}+\tfrac{1}{4}\sqrt{D},\, 1+\tfrac{1}{2}\sqrt{D};\, -\tfrac{\lambda^2}{\eta^2}\right), \end{cases} \tag{6.37}$$

where

$$D = 1 - 36t_i^2\bar{k}^2\bar{c}_A^2, \quad \lambda = \frac{9}{2}t_i^2\bar{k}\bar{c}_S. \tag{6.38}$$

First, we will take some limiting cases again, to see if these rather complicated solutions conform with our earlier work.

In the $\bar{c}_S \to 0$ limit, equivalent to $\lambda \to 0$, the ${}_1F_2$ functions merely reduce to unity; hence, we remain with the $\eta^{1/2\pm\sqrt{D}/2}$ solutions found earlier, along with the spurious $\eta^{-1}$ solution, which we disregard.

The $\bar{c}_A \to 0$ or $D \to 1$ limit turns out to be more difficult to analyze. If we examine (6.36), it is clear that $D \to 1$ is equivalent to taking $b_1 \to a$, and hence is a singular point of the equation. In general terms, taking this limit is equivalent to considering the equation

$$\frac{d}{d\zeta}\zeta^{a+1}\left(\frac{d^2u}{d\zeta^2} + \frac{b_2}{\zeta}\frac{du}{d\zeta} - \frac{\kappa u}{\zeta}\right) = 0. \tag{6.39}$$

The expression in parentheses is the general equation for a ${}_0F_1\,(b_2;\, \kappa\zeta)$, and has only two linearly independent solutions. It is apparent though that $y_2$ will contain a parameter equal to zero, which is undefined for generalized hypergeometric functions, and must consequently correspond to the third inhomogeneous solution of (6.39). The three solutions turn out to be

$$y_1 \to \sin\left(\frac{2\lambda}{\eta}\right), \tag{6.40}$$

$$y_2 \to -\frac{\eta}{2\lambda} + \sin\left(\frac{2\lambda}{\eta}\right) \text{ci}\left(\frac{2\lambda}{\eta}\right) - \cos\left(\frac{2\lambda}{\eta}\right) \text{si}\left(\frac{2\lambda}{\eta}\right), \tag{6.41}$$

$$y_3 \to \cos\left(\frac{2\lambda}{\eta}\right), \tag{6.42}$$

where ci and si are the sine and cosine integral functions

$$\text{ci}(x) = -\int_x^\infty \frac{\cos t}{t} dt, \tag{6.43}$$

$$\text{si}(x) = -\int_x^\infty \frac{\sin t}{t} dt. \tag{6.44}$$

We may disregard $y_2$ upon substitution into our original equations, which leaves us with the simple sinusoidallike solutions, which agree with (6.11).

It remains to decide which two of our solutions are the physically correct ones. To facilitate isolating the correct solutions, as well as to later analyzing the asymptotics of our solutions, we may reexpress the ${}_1F_2$ functions in terms of more well-known functions. Using various identities [21,23], we find

$$y_1 = \frac{(2\lambda)^{-1/2}}{16}(9-D)(1+\sqrt{D})\eta^{-1/2}\left[\frac{\eta}{2\lambda}s_{1/2,\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right) - s_{-1/2,1+\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right)\right], \tag{6.45}$$

$$y_2 = \lambda^{\sqrt{D}/2}\Gamma\left(1-\frac{1}{2}\sqrt{D}\right)\eta^{-1/2}\left[\eta J_{-\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right) - \frac{4\lambda}{1-\sqrt{D}}J_{1-\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right)\right], \tag{6.46}$$

$$y_3 = \lambda^{-\sqrt{D}/2}\Gamma\left(1+\frac{1}{2}\sqrt{D}\right)\eta^{-1/2}\left[\eta J_{\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right) - \frac{4\lambda}{1+\sqrt{D}}J_{1+\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right)\right], \tag{6.47}$$

where $s_{\mu,\nu}(z)$ is the Lommel function, $J_\nu(z)$ is the standard Bessel function of the first kind [24], and $\Gamma(z)$ is the gamma function.

In determining which solutions solve (6.2)–(6.6), we require the integrals

$$\begin{aligned} I_1 &\equiv \int \eta^{-3}\,{}_1F_2\left(2;\ \frac{7}{4}+\frac{1}{4}\sqrt{D},\ \frac{7}{4}-\frac{1}{4}\sqrt{D};\ -\frac{\lambda^2}{\eta^2}\right)d\eta \\ &= \frac{9-D}{32\lambda^2}\left[\frac{1-D}{4}\left(\frac{2\lambda}{\eta}\right)^{1/2} s_{-3/2,\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right) - 1\right], \end{aligned} \tag{6.48}$$

$$\begin{aligned} I_2 &\equiv \int \eta^{-3/2+\sqrt{D}/2}\,{}_1F_2\left(\frac{5}{4}-\frac{1}{4}\sqrt{D};\ \frac{1}{4}-\frac{1}{4}\sqrt{D},\ 1-\frac{1}{2}\sqrt{D};\ -\frac{\lambda^2}{\eta^2}\right)d\eta \\ &= -\frac{2\lambda^{\sqrt{D}/2}}{1-\sqrt{D}}\Gamma\left(1-\frac{1}{2}\sqrt{D}\right)\eta^{-1/2}J_{-\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right), \end{aligned} \tag{6.49}$$

$$\begin{aligned} I_3 &\equiv \int \eta^{-3/2-\sqrt{D}/2}\,{}_1F_2\left(\frac{5}{4}+\frac{1}{4}\sqrt{D};\ \frac{1}{4}+\frac{1}{4}\sqrt{D},\ 1+\frac{1}{2}\sqrt{D};\ -\frac{\lambda^2}{\eta^2}\right)d\eta \\ &= -\frac{2\lambda^{-\sqrt{D}/2}}{1+\sqrt{D}}\Gamma\left(1+\frac{1}{2}\sqrt{D}\right)\eta^{-1/2}J_{\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right). \end{aligned} \tag{6.50}$$

If we were to assume a general solution of the form

$$\bar{U} = a_1 y_1 + a_2 y_2 + a_3 y_3, \tag{6.51}$$

with $a_1$, $a_2$, and $a_3$ being arbitrary constants, this solution must satisfy the equations [taken from (6.2)–(6.6)]

$$\bar{B}_{1x_3} = -i9t_i^2\bar{B}_0\bar{k}(a_1I_1 + a_2I_2 + a_3I_3), \tag{6.52}$$

$$\bar{\rho}_{m1} = -i9t_i^2\bar{\rho}_{m0}\bar{k}(a_1I_1 + a_2I_2 + a_3I_3), \tag{6.53}$$

$$\frac{d}{d\eta}(a_1y_1 + a_2y_2 + a_3y_3) = -9t_i^2\bar{k}^2\left(\bar{c}_A^2 + \frac{9t_i^2}{\eta^2}\bar{c}_S^2\right)(a_1I_1 + a_2I_2 + a_3I_3). \tag{6.54}$$

Upon examining the derivatives of $y_1$, $y_2$, and $y_3$, it is found that $y_2$ and $y_3$ exactly solve these equations, but that

$$\frac{dy_1}{d\eta} = -9t_i^2\bar{k}^2\left(\bar{c}_A^2 + \frac{9t_i^2}{\eta^2}\bar{c}_S^2\right)\frac{9-D}{32\lambda^2}\left[\frac{1-D}{4}\left(\frac{2\lambda}{\eta}\right)^{1/2}s_{-3/2,\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right) - 1 + \frac{9t_i^2\bar{c}_S^2/\eta^2}{\bar{c}_A^2 + 9t_i^2\bar{c}_S^2/\eta^2}\right]; \tag{6.55}$$

hence, we must conclude that $y_1$ is a spurious solution.

It may be noted that, in the limit $\bar{c}_A \to 0$,

$$\left(\frac{2\lambda}{\eta}\right)^{1/2} s_{-3/2,\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right) \to \cos\left(\frac{2\lambda}{\eta}\right), \tag{6.56}$$

and the constants of integration may conspire, so that due to the singular nature of the differential equation at this point, $y_1$ and $y_3$ do indeed become the correct solutions, whereas $y_2$ is spurious, as was found earlier.

Having determined the correct solutions as well as their integrals, we are now in a position to write down the MHD state vector for the $\mathbf{k} \perp \mathbf{B}_0$ post-recombination case. To solve our initial equations (6.8), (6.9), we assumed $\mathbf{k}$ lay in the $\hat{\mathbf{x}}_1$ direction, and that $\bar{U}_2 = 0$ for simplicity. Now that the solution is known, we may generalize and take $\bar{\mathbf{k}} = \bar{k}_1\hat{\mathbf{x}}_1 + \bar{k}_2\hat{\mathbf{x}}_2$, so that $\bar{U}_2 \neq 0$. We then assume solutions of the form

$$\left\{\begin{matrix}\bar{U}_1\\ \bar{U}_2\end{matrix}\right\} = \left\{\begin{matrix}\tilde{U}_1\\ \tilde{U}_2\end{matrix}\right\}\Gamma\left(1+\epsilon\frac{1}{2}\sqrt{D}\right)\lambda^{-\epsilon\sqrt{D}/2}\eta^{-1/2}\left[\eta J_{\epsilon\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right) - \frac{4\lambda}{1+\epsilon\sqrt{D}}J_{1+\epsilon\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right)\right]e^{i\bar{\mathbf{k}}\cdot\mathbf{x}}, \tag{6.57}$$

$$\left\{\begin{matrix}\bar{\rho}_{m1}\\ \bar{B}_{1x_3}\end{matrix}\right\} = \left\{\begin{matrix}\tilde{\rho}_{m1}\\ \tilde{B}_{1x_3}\end{matrix}\right\}\frac{2\lambda^{-\epsilon\sqrt{D}/2}}{1+\epsilon\sqrt{D}}\Gamma\left(1+\epsilon\frac{1}{2}\sqrt{D}\right)\eta^{-1/2}J_{\epsilon\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right)e^{i\bar{\mathbf{k}}\cdot\mathbf{x}}, \tag{6.58}$$

where $\epsilon = \pm 1$ denotes waves propagating in both directions. This once again leads to a simple algebraic matrix, whose eigenvectors correspond to the appropriate MHD state:

$$\mathbf{\Psi}_1 = \begin{bmatrix} c_1\left(1+\epsilon\sqrt{D}\right)\eta^{-1/2}\left[\eta J_{\epsilon\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right) - \frac{4\lambda}{1+\epsilon\sqrt{D}}J_{1+\epsilon\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right)\right] \\ c_2\left(1+\epsilon\sqrt{D}\right)\eta^{-1/2}\left[\eta J_{\epsilon\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right) - \frac{4\lambda}{1+\epsilon\sqrt{D}}J_{1+\epsilon\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right)\right] \\ 0 \\ 18t_i^2\bar{k}\bar{B}_0\eta^{-1/2}J_{\epsilon\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right) \\ 18t_i^2\bar{k}\bar{\rho}_{m0}\eta^{-1/2}J_{\epsilon\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right) \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{i\bar{\mathbf{k}}\cdot\mathbf{x}}. \tag{6.59}$$

It remains to determine at what phase velocity these modes propagate. The nature of our solutions does not make this immediately apparent. Information is contained both in the argument and order of the Bessel functions. To obtain the FIDO measured velocities, we must examine the asymptotics of our solutions, gaining information from the leading order terms.

Consider the fluid velocity, which has the most complicated form. Let us take a linear combination of our two solutions in order to express them in terms of Hankel functions, which most resemble simple exponential solutions. Thus we consider the linear combinations

$$i\csc\left(\frac{1}{2}\sqrt{D}\pi\right)\left(e^{-i\sqrt{D}\pi/2}y_3 - y_2\right), \tag{6.60}$$

$$i\csc\left(\frac{1}{2}\sqrt{D}\pi\right)\left(y_2 - e^{i\sqrt{D}\pi/2}y_3\right), \tag{6.61}$$

ignore the constant prefactors, and use some Bessel recursion relations to obtain the corresponding equivalent expression for $\bar{U}$:

$$\bar{U} = \eta^{1/2}\left[H^{(i)}_{\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right) - 2\eta\frac{d}{d\eta}H^{(i)}_{\sqrt{D}/2}\left(\frac{2\lambda}{\eta}\right)\right], \quad i = 1, 2. \tag{6.62}$$

The method of examining the asymptotics follows from that of previous papers [12,13], whose solutions had the same type of functional form. A discussion of the values $D$ could realistically take was made earlier. Assuming a proton mass in the speed of sound expression, it can be deduced $c_S \sim 10^4$ m/s, and hence

$$\frac{9t_i^2\bar{k}\bar{c}_S}{\eta} \sim \frac{10^{20}}{l}, \tag{6.63}$$

where $l$ is the wavelength measured in meters. Hence we wish to expand Bessel functions of large argument and order, assuming wavelengths of the oscillations to be far less than the radius of the Universe around recombination. Using the asymptotic expansions found in Watson [24], pp. 262–268, we calculate the first terms for $\bar{U}$. The expansion parameter $\nu\tanh\gamma \gg 1$ found in [24] corresponds to

$$\bar{k}t_i\left(\bar{c}_A^2 + \frac{9t_i^2}{\eta^2}\bar{c}_S^2\right)^{1/2} = R^{1/2}kt_i\left(c_A^2 + c_S^2\right)^{1/2} \gg 1. \tag{6.64}$$

Also requiring the individual frequencies of sound and Alfvén waves to be far greater than the reciprocal age of the Universe, i.e.,

$$kc_At_i \gg 1, \quad kc_st_i \gg 1, \tag{6.65}$$

we may, after a lengthy calculation, write down the asymptotic expansion for $\bar{U}$ to first order:

$$\bar{U} \sim M^{1/2}\left\{1+\frac{13}{1152}\frac{1}{M^2}-\frac{149}{1728}\frac{A^2}{M^4}+\frac{889}{5184}\frac{A^4}{M^6}+O(M^{-3})\right\}$$
$$\times\exp\left\{\pm i\left[3M-3A\,\text{arcsinh}\left(\frac{A}{S}\right)-\frac{1}{12M}+\frac{1}{24A}\text{arcsinh}\left(\frac{A}{S}\right)+\frac{1}{24A}\frac{AM+A^2}{AM+M^2}-\frac{7}{72}\frac{A^2}{M^3}+O(M^{-2})\right]\right\}. \tag{6.66}$$

Here

$$A=\bar{k}t_i\bar{c}_A,\quad S=\frac{3t_i}{\eta}\bar{k}t_i\bar{c}_S,\quad M=(A^2+S^2)^{1/2} \tag{6.67}$$

are our three expansion parameters, all of the same order. This accounts for the number of individual terms required to completely specify $\bar{U}$ to first order.

The FIDO velocity, obtained by differentiating the argument of the exponential with respect to $t$, can now be calculated. We display only the leading order term, as the first order term only becomes more unwieldy than that found above, and contains no new interesting information. Thus

$$\omega_F=\bar{k}\left(\frac{t}{t_i}\right)^{-1}\left(\bar{c}_A^2+\left(\frac{t}{t_i}\right)^{-2/3}\bar{c}_S^2\right)^{1/2}\left[1+O(v^{-2}k^{-2}t^{-2})\right]$$
$$=k\left(c_A^2+c_S^2\right)^{1/2}\left[1+O(v^{-2}k^{-2}t^{-2})\right], \tag{6.68}$$

where $v$ is a generic expression for a combination of velocities $c_A$ and $c_S$.

Thus we have demonstrated that a FIDO observer does measure a redshifted magnetosound phase velocity, where the time-independent quantities combine in such a way that the whole expression scales the same way with respect to $R$. Notice that we also have the characteristic time-dependent correction terms due to spacetime curvature. The usual flat spacetime results may be recovered by taking the limit $t\to\infty$, where we may think of infinite time corresponding to infinite radius of the Universe. This may be visualized as describing flat spacetime.

### E. General solution

Using knowledge gained in the preceding work, we may deduce the general Alfvén solution for arbitrary direction of $\mathbf{k}$ with respect to $\mathbf{B}_0$. Thus as with the usual Alfvén mode structure, assume $\bar{U}_3=0$, and now take

$$\bar{U}_i\propto\eta^{1/2\pm\sqrt{D}/2}e^{i\bar{\mathbf{k}}\cdot\mathbf{x}},\quad i=1,\,2, \tag{6.69}$$

with $D=1-36t_i^2\bar{k}^2\bar{c}_A^2c_3^2$. A substitution of this into the general velocity equations (6.8), (6.9) shows that they do indeed prove to be solutions.

To find the corresponding eigenvector, we assume the same form for all quantities, and following the usual procedure deduce

$$\Psi_1^A=\begin{bmatrix}\bar{k}\bar{c}_A^2c_2c_3\eta^{1/2\pm\sqrt{D}/2}\\-\bar{k}\bar{c}_A^2c_1c_3\eta^{1/2\pm\sqrt{D}/2}\\0\\-i\bar{B}_0\left(\frac{1}{2}\pm\frac{1}{2}\sqrt{D}\right)c_2\eta^{-1/2\pm\sqrt{D}/2}\\i\bar{B}_0\left(\frac{1}{2}\pm\frac{1}{2}\sqrt{D}\right)c_1\eta^{-1/2\pm\sqrt{D}/2}\\0\\0\\0\end{bmatrix}e^{i\bar{\mathbf{k}}\cdot\mathbf{x}}. \tag{6.70}$$

Here we see a similar structure to previous general results, with the exception of the presence of $\sqrt{D}$, which has been seen to complicate all solutions in the post-recombination era. Making the assumption $36t_i^2\bar{k}^2\bar{c}_A^2c_3^2\gg1$, we may derive an identical structure to that found in the pre-recombination era, distinguished only by the time dependences.

The general magnetosound modes prove to be a lot more difficult to obtain. To give an idea of the complexity involved, if we examine the leading order term of the FIDO frequency found for the $\mathbf{k}\perp\mathbf{B}_0$ case, we may integrate this with respect to $t$ to recover the leading asymptotic form of the solution. For the general case, we postulate the expression for the leading order FIDO velocity to be

$$v=\left(\frac{t}{t_i}\right)^{-1}\left[\frac{1}{2}\left(\bar{c}_A^2+\left(\frac{t}{t_i}\right)^{-2/3}\bar{c}_S^2\pm\left\{\left[\bar{c}_A^2+\left(\frac{t}{t_i}\right)^{-2/3}\bar{c}_S^2\right]^2-4\left(\frac{t}{t_i}\right)^{-2/3}\bar{c}_A^2\bar{c}_S^2c_3^2\right\}^{1/2}\right)\right]^{1/2}. \tag{6.71}$$

This whole expression just scales as $R^{-1}$, as must be expected. The integral of this expression cannot be expressed in terms of known functions; hence, we are unable even to write down the leading order asymptotic form, let alone obtain the full solution.

Nevertheless, we have extracted all the major properties of post-recombination MHD modes. Most importantly, we have demonstrated how $\bar{c}_A$ and $\bar{c}_S$ combine when they scale differently with respect to $R$. The full solution would only really be required for aesthetic completeness. We have to all intents and purposes fully solved the problem.

In closing this section, we remark that the post-recombination modes will evolve differently depending on the scale of the mode compared to the Jeans length, and if they are sonic or Alfvén. In particular, modes with a sonic behavior (sonic or magnetosonic modes) on scales larger than the Jeans length are unstable to gravitational collapse.

## VII. CONCLUSION

We have continued the work of [12] and [13], and formulated the equations for MHD in the early Universe. In particular, we have managed to solve the equations completely in the three major eras possible to study, namely, the very early Universe, where a semiclassical UR treatment was required, and both before and after recombination, where a NR treatment sufficed. We have found, as has previous work in this field, that in the UR limit all modes redshift in the same manner as that of a free photon. In the NR limit, however, different frequencies redshift at different rates, leading to complicated solutions, requiring some intricate mathematics. We have managed to obtain the MHD eigenstates in all cases, and clearly elucidated all the various modes possible. The locally measured frequencies resemble that of flat spacetime, though due to the expansion of the Universe, they do decay in time. These effects are only apparent at lower orders in the calculations.

Possible extensions to this work could include formulating the equations for nonperfect MHD, including such effects as viscosity and finite conductivity. Nonlinear modes such as shocks could also be investigated. Judging by the complexity of the solutions obtained in this paper, it seems likely that more complicated models would not be analytically solvable, and may require extensive numerical solution. Our simple approach has kept intact the basic essence of MHD theory, highlighting the similarities and occasional differences with standard flat spacetime MHD, which complicated numerical procedures may mask. Thus we may conclude that our approach has been successful in its endeavors.

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