Stochastic dynamics of relativistic turbulence

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We apply generalized Kolmogorov scaling to continuous time random walks, coupled in space and time, to obtain anomalous diffusion laws for relativistic turbulent media. Richardson's law for the mean square separation of two particles that are initially close together $\langle R^2\rangle \sim t^3$ is recovered in the nonrelativistic limit, while the ultrarelativistic limit is characterized by a different power law $\langle R^2\rangle \sim t^2$. Intermediate velocities are treated numerically, showing a smooth transition from one regime to the other.

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No complete theory of turbulence exists, despite the apparent simplicity of the Navier-Stokes equations and the important fact that almost all problems involving fluids have some physically relevant set of parameters in which the motion is turbulent. Nevertheless, there are general principles that apply to most turbulent flows to some degree and particular flows to a good approximation. An example is the observation by Kolmogorov [1] and others that for large Reynolds number there is a range of scales between the largest motions that are driven by external forces and boundary conditions and the smallest eddies at which viscous dissipation becomes important, called the inertial subrange, in which the fluid motions are to some degree homogeneous, isotropic, and scale invariant in a statistical sense. This is a better approximation for flow through a grid than shear flows. Kolmogorov's theory predicts that the rate of energy dissipated per unit volume over a scale ϵ_R is independent of R. This leads to velocities that scale as $v(R)\sim R^{1/3}$ and an energy spectrum that scales as $E(k)\sim k^{-5/3}$. These scaling laws are found to be valid for a large number of different types of flow, as long as the Reynolds number is sufficiently large.

One remarkable feature of turbulent flows is enhanced diffusion, in which the mean square separation of two particles placed initially close together is proportional to a power of the time

$$\langle R^2\rangle \sim t^{\gamma}. \tag{1}$$

The value $\gamma=1$ corresponds to ordinary molecular diffusion governed by Brownian motion, while turbulent flows are better described by Richardson's law $\gamma=3$. The motion of the particles cannot now be described by a random walk that has a probability distribution with finite moments, because the central limit theorem applies, leading to $\gamma=1$. Shlesinger *et al.* [2–4] have modeled the diffusion of the particles using Lévy walks [5], which have infinite moments. Their approach fits well with the Kolmogorov theory, since the power laws that characterize the Lévy walks are statistically homogeneous, isotropic, and scale invariant and Richardson's law is recovered.

This analysis is, however, restricted to nonrelativistic velocities and fails when the scales become so large that the characteristic velocities approach the velocity of light. Relativistic turbulence phenomena are important in astrophysics, particularly in the study of the accretion disks and jets of active galactic nuclei [6], and slightly relativistic turbulent velocities also occur in cosmology [7]. In this Brief Report we discuss the full relativistic turbulent diffusion problem, using a generalization of the Lévy walk. In the limit of small velocities we recover Richardson's law, but for velocities approaching the velocity of light we obtain a different diffusion law $\gamma=2$.

In order to understand the intricacies of the relativistic problem, we must first understand the physical arguments and mathematical details that are used in Refs. [2–4] to tackle nonrelativistic turbulent diffusion. The Lévy walk is a random walk where each step consists of a pause of time t, followed by a jump of distance R. These are not independent random variables; a step that moves a large distance takes a long time. The physically relevant parameter that determines the exact relation between these variables is the velocity associated with the scale R, that is,

$$R=v(R)t. \tag{2}$$

This type of walk is called a continuous time random walk since the emphasis is on the time rather than the number of steps. The probability distribution for each step of length R is scale invariant in the limit of large R,

$$p(R)=R^{-1-\beta}. \tag{3}$$

For $\beta<2$ this distribution has an infinite second moment and does not satisfy the conditions of the central limit theorem; a large number of steps governed by such a probability distribution does not tend to a Gaussian, but rather a Lévy stable law [10]. This means that the ordinary diffusion law $\gamma=1$ need not hold. It is necessary to couple the time and space probability distributions (for example, as above) to ensure that the moments $\langle R^2\rangle$ are not infinite at finite times. The calculations leading from the Kolmogorov scaling law for $v(R)$ to the Richardson law $\gamma=3$ may be obtained from [2,3]. The result is

$$\gamma=\begin{cases} 3, & \beta<1/3 \\ (7-3\beta)/2, & 1/3<\beta<5/3 \\ 1, & \beta>5/3. \end{cases} \tag{4}$$

Note that Richardson's law applies only to the smallest values of β, while the largest values give the same result as Brownian motion.

In their discussion, Shlesinger *et al.* included the effects of intermittency, the effect that the turbulent dissipation is not observed to be homogeneously distributed. Mandelbrot [8] has argued that the dissipation is concentrated on a fractal set of dimension d_f. This has the effect of changing the scaling law for $v(R)$, and hence γ, although these effects are very small in practice [2]. The presence of a fractal structure is in accord with the scale invariance proposed by Kolmogorov, since fractals are scale invariant. While it is not disputed that turbulent dissipation is inhomogeneous at small scales, recent experimental evidence [9] finds no indication that, for sufficiently large Reynolds numbers there is any deviation from Kolmogorov scaling in the inertial subrange. For this reason we have not attempted to include the extra complication of intermittency.

Now we see how these results can be generalized to include the full relativistic dynamics. The Kolmogorov theory is nonrelativistic and includes among its assumptions the Galilean expression for relative velocities and the incompressibility of the fluid. Neither of these properties holds for the relativistic regime, in which it is necessary to use a compressible fluid with a given equation of state in order to preserve causality. However, we expect the main features of the energy cascade to remain unchanged: assuming that, as in the nonrelativistic regime, the dynamics is that of a highly damped system, the rate of energy dissipation (per unit mass) over a scale R is given by the kinetic energy density $\Gamma(R)-1$ divided by the characteristic time $R/v(R)$. Here $\Gamma(R)$ is the usual relativistic factor $[1-v(R)^2]^{-1/2}$, with $c=1$. Noting that, at least by energy conservation, the energy lost by the largest scales is equal to that dissipated at the smallest, it is reasonable to assume that, as in the nonrelativistic regime, the rate of dissipation

$$\epsilon \sim [\Gamma(R)-1]\frac{v(R)}{R} \tag{5}$$

is a constant over all scales. This leads to an equation for $v(R)$,

$$v^4+2rv^3+r^2v^2-2rv-r^2=0\ , \tag{6}$$

where r is proportional to ϵR and is thus a scaled distance.

This equation has a complicated solution in general, but in the nonrelativistic and ultrarelativistic limits (respectively) we obtain

$$v=\begin{cases}(2r)^{1/3}, & r\ll 1\\ 1-r^{-2}/2, & r\gg 1.\end{cases} \tag{7}$$

Note that the Kolmogorov scaling law $v(r)\sim r^{1/3}$ is recovered in the nonrelativistic limit. In the intermediate region we may evaluate $v(r)$ numerically using Newton's method.

Before proceeding further, let us consider some of the physics of Eq. (5), in particular how the Kolmogorov theory can be generalized to a relativistic framework. We have assumed that the turbulent medium is homogeneous and isotropic in a statistical sense, although the scale invariance of the nonrelativistic theory no longer holds. The physical picture of the Kolmogorov approach is of eddies within eddies, but this may not be appropriate for the relativistic regime, as the largest velocities approach the velocity of light, making it difficult for such causally coherent structures to form. In addition, the Lorentz contraction in moving from one reference frame to another may at first sight appear to destroy the isotropy that we require. Thus we need an intuitive model of isotropic turbulence. It is clear that the Friedmann-Robertson-Walker universe is both relativistic and isotropic as seen from any point in space. A cosmology that has, in addition, random fluctuations in density and velocity would thus be a clear example of homogeneous and isotropic relativistic turbulence. Coherent structures such as eddies could form up to horizon scales, or greater, if a mechanism such as inflation has caused correlations on superhorizon scales. An example of isotropic turbulence in a bounded domain would be a region of fluid buffeted by a large number of randomly oriented relativistic jets. Thus isotropic and homogeneous turbulence is certainly possible in the relativistic domain.

Another important difference that affects relativistic descriptions of a system is the lack of a universal definition of simultaneity. It is no longer possible to simply state that $\langle r^2(t)\rangle$ is the scaled mean square displacement between two particles as a function of time, unless there is an agreed frame of reference in which to make measurements of distance and time. To be specific, we choose the momentarily comoving rest frame of the particles when they are initially close together. This calculation depends only on the distribution of relative velocities and not on the distances between the particles. In addition, it is approximate, in that dimensionless factors of order unity are ignored, while the important results are the power law exponents. Thus it does not matter from the point of view of this calculation whether we are estimating the relative separation of two particles or one particle and a point fixed in the fluid. Because the results are approximate in that small dimensionless factors have been neglected, the issue of which frame of reference is used is not crucial.

To summarize what we have so far, the relativistic turbulent diffusion process is modeled by a random walk that consists of a pause of time t followed by a jump of scaled distance r. The probability distribution for r is given by Eq. (3) in the large r limit. Its direction is an independent random variable with uniform probability distribution. The time t is given by Eqs. (2) and (6). It is difficult to evaluate $\langle r^2(t)\rangle$ analytically, along the lines of Eq. (4), so we will investigate the ultrarelativistic limit exactly and then treat the full problem numerically.

In the ultrarelativistic limit ($r\gg 1$, $v\approx 1$) the velocity is independent of r, that is, a power law with zero exponent, so we are back to a Lévy walk, which can be analyzed in a manner similar to Ref. [3]. The analytic result is

$$\gamma=\begin{cases}2, & \beta<1\\ 3-\beta, & 1<\beta<2\\ 1, & \beta>2.\end{cases} \tag{8}$$

The exponent corresponding to small β, which is physically the most reasonable, is $\gamma=2$. Note that even if we had in-

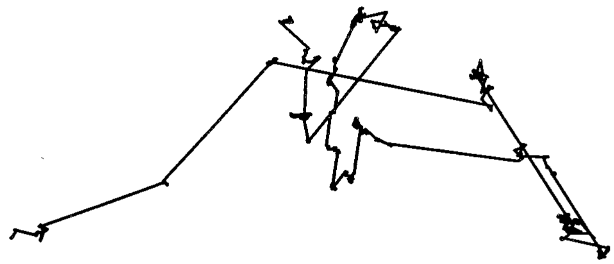

FIG. 1. Two-dimensional projection of a $\beta=1$ random walk.

cluded the effects of intermittency, the velocity law would be unchanged, so there would be no difference in the results.

In the intermediate region ($r\approx 1$) we use numerical methods. The exact probability distribution we use, following Eq. (3), is

$$p(r)=\begin{cases}\dfrac{\beta}{l(\beta+1)}, & r<l\\[2mm] \dfrac{\beta}{l(\beta+1)}\left(\dfrac{r}{l}\right)^{-1-\beta}, & r>l,\end{cases} \tag{9}$$

where l is the characteristic length of a step, made smaller than the values of r of interest. The results are independent of l under this condition. An example of such a random walk, with $\beta=1$, is shown in Fig. 1. Each instance of a walk is sampled at specified times and r^2 is averaged over an ensemble of such walks and plotted as a function of time for different β. The results are shown in Fig. 2.

The lines given for low β are particularly curved, as indicated by the limiting power laws given in Eqs. (4) and (8). The amount of computer time needed for each of the curves shown in Fig. 2 varied with β (increasing by several orders of magnitude from $\beta=0.1$ to $\beta=10$), the range of $\langle r^2\rangle$, and the sample size. Thus, the larger values of β took days to run, even though the sample size was reduced, causing the statistical scatter in these curves. The limiting power laws were estimated by making a fit to ten data points at either end of the lines shown in Fig. 2 and the results, along with the theoretical values given in Eqs. (4) and (8), are shown in Fig. 3.

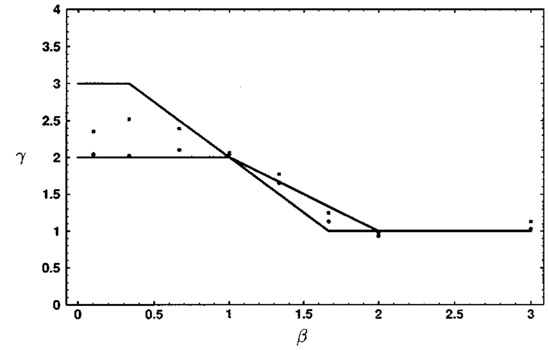

FIG. 3. Nonrelativistic (square) and ultrarelativistic (circle) scaling exponents, evaluated by fitting a straight line to the small and large t sections of Fig. 2. The theoretical exponents are given by the solid lines.

It is interesting to note that, although the nonrelativistic exponent is less than the ultrarelativistic exponent for $1<\beta<2$, the fitted exponents do not show this, except for $\beta=2$. This and the other deviations from the theoretical values at small β can be explained by the finite range $-5<\ln\langle r^2\rangle<5$ and the finite sample size. In particular, the nonrelativistic small β values displayed in Fig. 3 were strongly affected by the above choice of range. We have extended the range of the variables for the specific case $\beta=0.1$ (see Fig. 4), for further clarity. The fitted exponents are now 2.96, and 2.04, very close to the theoretical values, which are 3 and 2 for the nonrelativistic and ultrarelativistic limits, respectively.

We note that, although we have made several approximations in obtaining the curves for velocities in the intermediate region, the general shape of the curves and the point at which the effects are felt are quite insensitive to these approximations and, in particular, the exponent of 2 for the ultrarelativistic limit is exact within the Lévy walk approach.

This work is a basis for studies of particular relativistic turbulent systems. It could be extended in a number of ways, depending on the nature of the problem. We have not included effects due to magnetic fields or the fact that relativ-

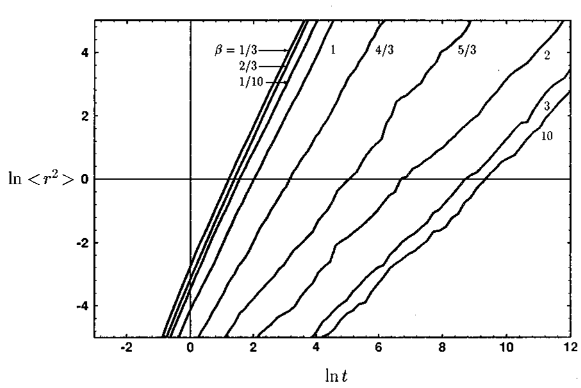

FIG. 2. Main numerical results: r^2 is averaged over 50–10 000 walks (depending on β) and plotted as a function of time on a log-log plot for values of β in the range 0.1–10.

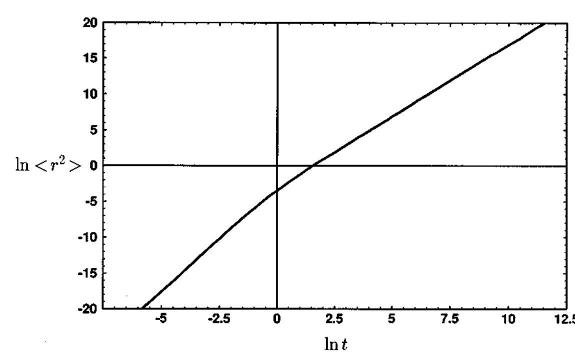

FIG. 4. Plot for $\beta=0.1$ in Fig. 2, extended to a larger range of times. The fitted exponents are 2.96 in the nonrelativistic limit and 2.04 in the ultrarelativistic limit, very close to the theoretical values.

istic jets are quite anisotropic, at least on large scales. Nevertheless, for large Reynolds number, we expect that these homogeneous, isotropic calculations should provide an understanding of anomalous diffusion in the presence of relativistic velocities.

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