Circle Packings, Conformal Mappings, and Probability

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LMS Prospects in Mathematics
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Outline

- Mathematics and PhD opportunities at the University of Bristol
- Some connections between probability and complex analysis
- One of my current research topics: Circle Packing
    - The Koebe-Andreev-Thurston theorem
    - Some applications
    - A recent PhD project on circle packings
    - Some open problems

Maths at the University of Bristol

The School of Mathematics has 60 permanent academic members of staff and 48 postdocs. There are also mathematicians in the engineering and computer science departments.

There are typically about 70 PhD students in maths at a time.

See *www.maths.bris.ac.uk* for lots of information about the department.

Admissions information for postgraduate study is at *www.maths.bris.ac.uk/study/admissions_postgrad*

Funding to do your PhD in Bristol

For PhD places funded by the EPSRC through the Maths DTA (doctoral training account), we encourage applications by mid-February, and in January if possible, as we usually make early offers to strong candidates.

DTA funding covers the tuition fees and provides a tax-free stipend which for 2013/14 is £13,726 per annum. Students typically earn another £1600 to £1700 per year by giving tutorials and marking undergraduate work.

Some members of staff hold their own research grants for specific projects, which can include funding for PhD students.

The University of Bristol Postgraduate Research Scholarship is another route for PhD funding – also competitive.

Taught Course Centre

Bristol is part of a Taught Course Centre (TCC) with Oxford, Reading, Warwick, Bath, and Imperial College. Graduate-level courses are given by videoconference.

If you are an EPSRC-funded student then you are required to take 100 hours of taught courses, and the TCC is designed to make it easier for you to use this time to learn background material relevant to your PhD research.

Pure maths research themes

- Number theory: analytic, algebraic, combinatorial and computational
- Combinatorics
- Ergodic theory and Dynamical systems
- Logic and set theory
- Analysis
- Group theory: geometric group theory and representation theory

Applied maths research themes

- Fluid Dynamics
- Dynamical systems and statistical mechanics
- Quantum chaos
- Random matrix theory
- Quantum information
- Materials science
- Scientific computing
- Complex systems

Statistics research themes

- Applied probability
- Bayesian modelling
- Behavioural biology
- Multiscale methods
- Monte Carlo methods
- Non-parametric regression
- Optimization under uncertainty
- Statistical bioinformatics
- Statistical signal processing
- Analysis of time series

Statistics and Probability at Bristol

Bristol's statistics department has recently expanded due to a multi-million pound grant, the **SuSTaIn** initiative - **S**tatistics **u**nderpinning **S**cience, **T**echnology **a**nd **In**dustry.

SuSTaIn funds students for the *MSc in Statistics*, which is a fully taught one-year course.

The majority of MSc students take it as the first year of a 1+3 year MSc/PhD programme. It includes four one-week residential courses with postgraduate students from other UK universities, provided by the Academy for PhD training in Statistics.

Further information:
www.maths.bris.ac.uk/study/admissions_postgrad/stats/

Interdisciplinary Doctoral Training Centres

- Bristol Centre for Complexity Sciences - a new DTC.
  - *www.bris.ac.uk/bccs/postgraduate-study/phd/*
  - BCCS will recruit 10-15 PhD students each year into a multidisciplinary PhD programme
  - Departments involved: Applied Mathematics and Statistics, Engineering Mathematics and Computer Science
  - Four year PhD course. The first year is fully taught.
- Centre for Doctoral Training in Communications:
  - 10 fully funded PhD places each year across a range of disciplines including mathematics.
  - Separate application procedure: see *www.bristol.ac.uk/cdt-communications*
  - To find out more, contact Dr Olly Johnson (reader in Statistics)

Nanoscience and Quantum Information

NSQI is an interdisciplinary research centre housed in a new building designed for nanoscience experiments that are very sensitive to vibration.

- *www.bristol.ac.uk/nsqi-centre/research*
- The quantum information group in the maths department currently has three permanent staff, four postdocs and three PhD students.

Engineering Mathematics

Engineering Mathematics is an applied maths department separate from the School of Mathematics.

www.bris.ac.uk/engineering/departments/engineering-mathematics/

- Applied nonlinear mathematics
- Intelligent systems
- Dynamics and control
- Robotics (Bristol and UWE hosted the Robot world cup in 2012)
- Complexity
- Mathematical neuroscience
- Predictive life sciences

Maths at the University of Bristol

- Maths in the Computer Science department:
  - mathematical cryptography
  - algorithms
- Heilbronn Institute for Mathematical Research
  - A partnership between GCHQ and the University of Bristol
  - No PhD students, but lots of postdocs

How to apply

First figure out which members of staff are potential PhD supervisors for the subject you are interested in. The postgraduate study booklets that you can download from the website give some details of possible PhD projects.

Get in touch with potential supervisors to find out more about their possible PhD projects. It will help you through the application process if you have the support of a potential supervisor.

Also contact the postgraduate admissions tutor for the group that you are interested in (pure, applied or probability and statistics).

See the website *www.maths.bris.ac.uk* for details of how to apply.

Apply as soon as you can! We make early offers to strong candidates.

My research

Most of my research has been in geometric function theory, the branch of analysis that deals with conformal mappings. In the last four years I have also been collaborating with colleagues in the probability group at Bristol.

Our current project is about a *forest fire* model that displays self-organized criticality. We are working to understand the local limit of this process, from the point of view of a single particle. A random metric space called the Brownian continuum random tree appears as a scaling limit.

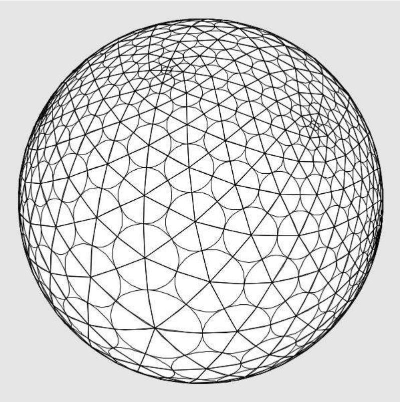
Complex Analysis meets Probability

- (classical) Conformal invariance of (the path of) Brownian motion in the plane
- Conformal invariance of critical bond percolation on the triangular lattice (Smirnov)
- Counting self-avoiding walks of length n in the triangular lattice using discrete holomorphic observables (Smirnov and Duminil-Copin)
- SLE curves - a parametrized family of random curves with a conformal Markov property. They are constructed using the Löwner differential equation (from geometric function theory) driven by a Brownian motion. They describe scaling limits of interfaces in various 2-d statistical mechanics models. (Schramm, Werner, Lawler)
- CLE: a *loop soup* with a similar conformal invariance property. Related to the Gaussian free field and to critical percolation. (Sheffield, Werner)

My research on Circle Packing

For the rest of this talk I will focus on a topic from geometric function theory. This is a discrete analogue of conformal mapping, called *circle packing*. It has lots of pictures, so it is fun to explain.

A circle packing filling the 2-sphere



The fundamental theorem of circle packing

The Koebe-Andreev-Thurston Theorem

For any triangulation T of the sphere, there exists a packing of the sphere by discs with disjoint interiors, whose pattern of tangencies is T.
The packing is unique up to Möbius maps.

Consequences

The Koebe-Andreev-Thurston theorem gives easy proofs of two theorems about finite planar graphs:

- Any finite planar graph can be drawn in the plane with straight non-crossing edges.
- The Miller-Tarjan separator theorem: any finite planar graph with n vertices can be cut into roughly equal-sized parts by removing $O(\sqrt{n})$ vertices.

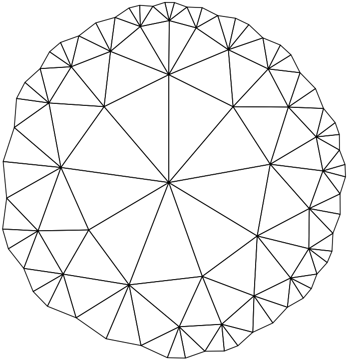
An equivalent statement

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

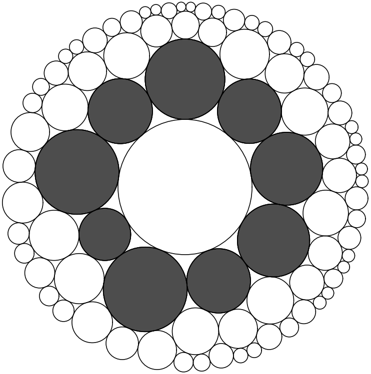
The Koebe-Andreev-Thurston Theorem

For any finite simply-connected triangulation T, there exists a packing of $\mathbb{D}$ by discs with disjoint interiors, whose pattern of tangencies is T, where the discs corresponding to the boundary (outside) vertices of T are internally tangent to the unit circle.
The packing is unique up to $\mathrm{Aut}(\mathbb{D})$.

Example: a simply-connected triangulation T



The circle packing of T in $\mathbb{D}$



How can we prove the Koebe-Andreev-Thurston theorem?

A nice way: using *hyperbolic geometry*.

The unit disc $\mathbb{D}$ has a famous Riemannian metric called the *Poincaré metric*, or *hyperbolic metric*. This is the unique complete Riemannian metric of constant curvature -1 on $\mathbb{D}$ that is *conformal*. The Poincaré metric is given for $z = x + iy$ by

$$\frac{4(dx^2 + dy^2)}{(1 - |z|^2)^2} .$$

The Poincaré metric gives a model of the *hyperbolic geometry* discovered by Gauss, Bolyai and Lobachevsky. There are analogues of all the familiar trigonometric formulae from Euclidean geometry, so for example we can compute the angles of a hyperbolic triangle from the side lengths.

What has hyperbolic geometry got to do with circles?

Euclidean circles inside $\mathbb{D}$ are also hyperbolic circles, but maybe with a different centre might be different.

Geodesics in the Poincaré metric are lines through 0 and arcs of circles that meet the unit circle orthogonally.

Circles that are internally tangent to the unit circle are called *horocycles*; they are limits of sequences of hyperbolic circles whose centres tend to a point of the unit circle and whose radii tend to infinity. Horocycles are curves of constant curvature 1 with respect to hyperbolic geometry.

Idea of a proof of Koebe-Andreev-Thurston

The idea is to show first that there's a consistent set of hyperbolic radii. This means that the sum of the angles at each interior vertex is exactly 2π.

Once we achieve this then we can lay the circles out one by one. Because the triangulation is simply-connected all the circles touch as they should (this is called monodromy).

We get the consistent set of radii as the (vertex-wise) infimum of the family of possible radius specifications for which the angle sum at each interior vertex is at most 2π. This is called a *Perron* argument, because it mimics Perron's famous construction of harmonic functions as the pointwise supremum of a family of subharmonic functions. The method relies strongly on the negative curvature of hyperbolic geometry, and does not work in Euclidean or spherical geometry.

Proofs

The Koebe-Andreev-Thurston theorem has many other proofs:

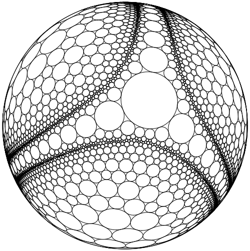
- Koebe proved it as a limiting case of his generalization of the Riemann mapping theorem to multiply-connected regions.
- Andreev and Thurston rediscovered it independently in connection with constructing polyhedra in 3-dimensional hyperbolic space, with specified dihedral angles. Their version generalized Koebe's theorem by allowing the discs to overlap at specified angles, up to $\pi/2$.
- Thurston gave an algorithm to compute circle packings, using 2-dimensional hyperbolic geometry. It can be viewed as finding the minimum of a certain convex function on $\mathbb{R}^{n-1}$, where n is the number of circles in the packing.
- Oded Schramm gave several huge generalizations, each with a totally different proof.

Circle packing as a numerical tool

Thurston's algorithm approximates a circle packing given only the triangulation as initial data. This makes circle packing a versatile tool for numerical conformal mapping and more general boundary value problems.

Thurston's algorithm was designed for a proof. It is beautifully simple, but slow in practice. But we now have numerical methods that are much more efficient than Thurston's algorithm, so that circle packings with millions of circles can be computed to high precision.

Numerical conformal welding by circle packing



Discrete conformal maps

Thurston suggested that the triangulation could be thought of as a discrete conformal structure, and the mapping between two different circle packings of the same triangulation could be thought of as a *discrete conformal map*.

Rodin and Sullivan proved Thurston's conjecture that discrete conformal maps approximate genuine conformal maps.

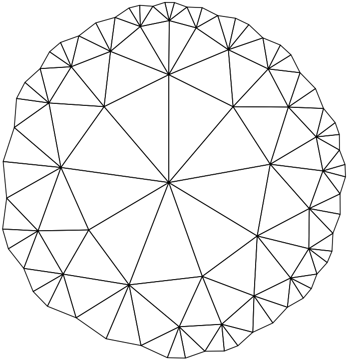
Branched circle packings

Circle packings behave like conformal maps.

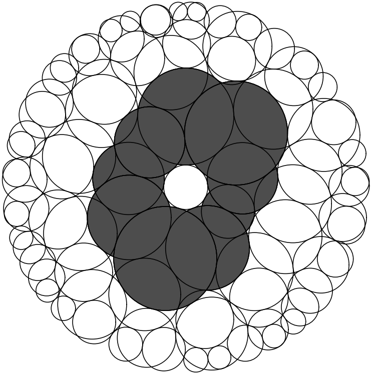
We would like to approximate more general analytic maps by circle packings, but we have to allow *branching*.

To get branching, we can allow the chain of neighbours of some specified circle to wrap around twice or more before joining up. For this to be possible the corresponding vertex must have at least five neighbours.

Example: a combinatorial disc



The same triangulation packed in $\mathbb{D}$ with branching



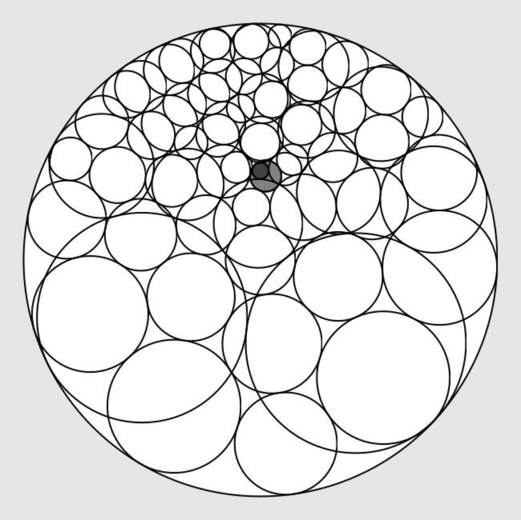
A recent PhD project

There are only finitely many places to put the branch point - at a vertex.

To get the correct dimension for the Teichmuller space of branched circle packings of a given triangulation T, we would like to be able to move the branch point continuously in the 2-complex given by the triangulation.

I collaborated with James Ashe and Ken Stephenson at the University of Tennessee, Knoxville, to work out how to do this. James wrote his PhD thesis about his contribution to the problem, and got his PhD last year.

Moving the branch point



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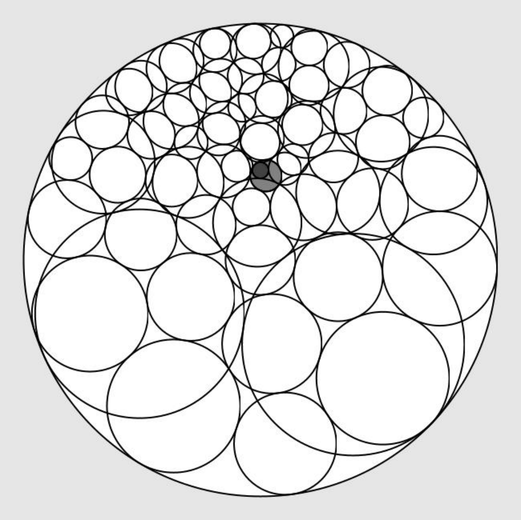

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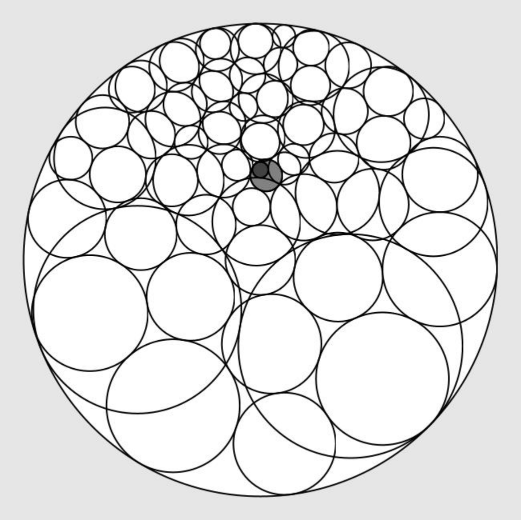

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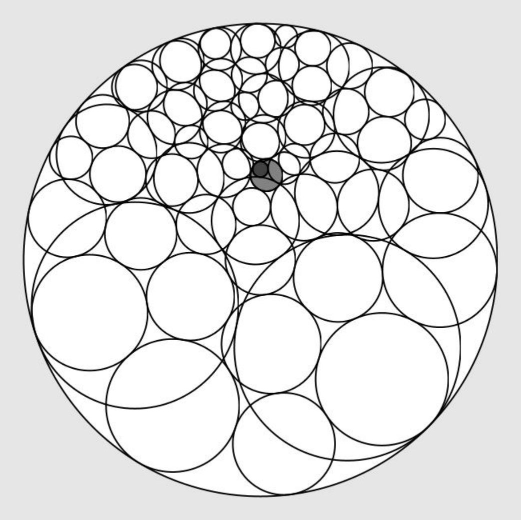

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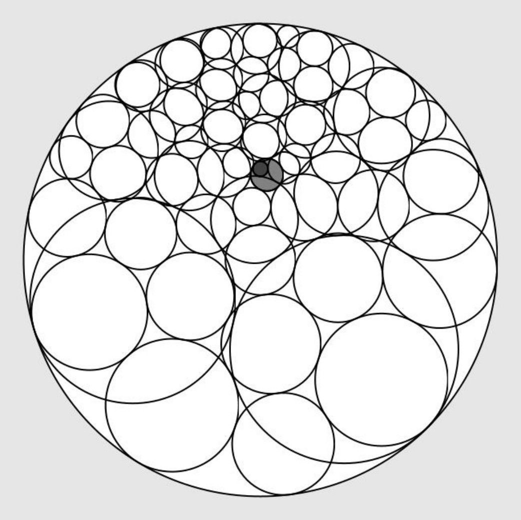

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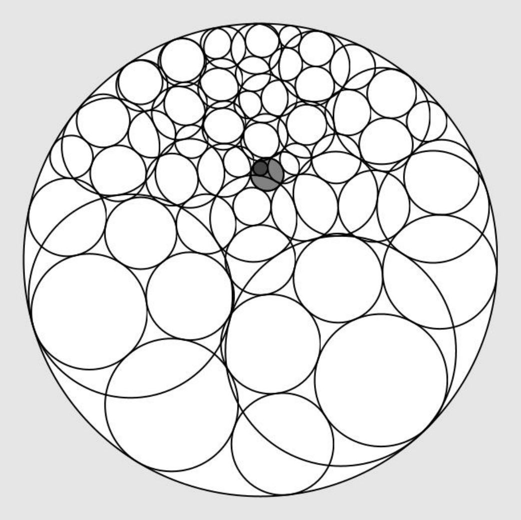

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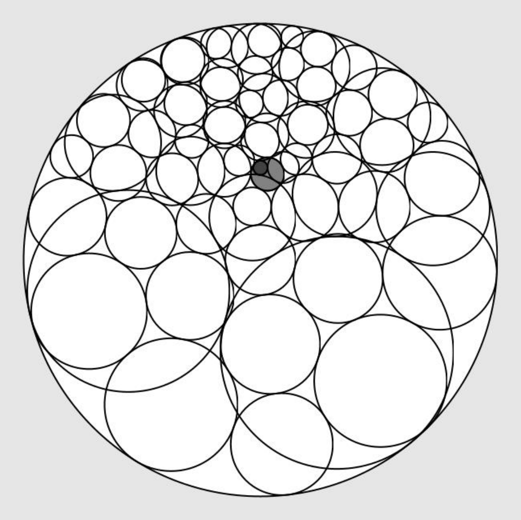

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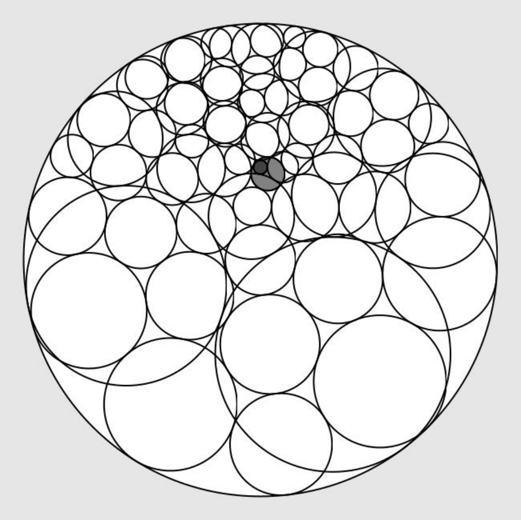

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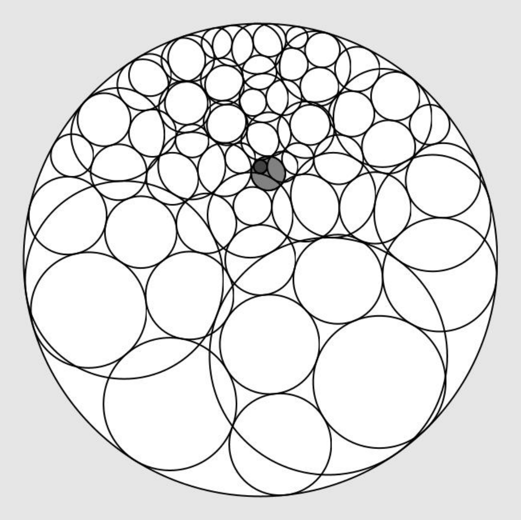

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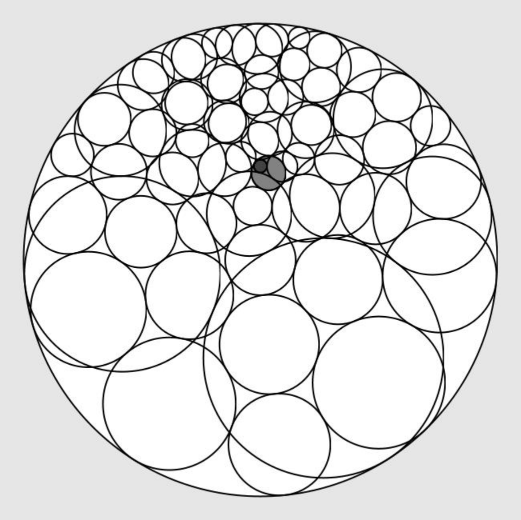

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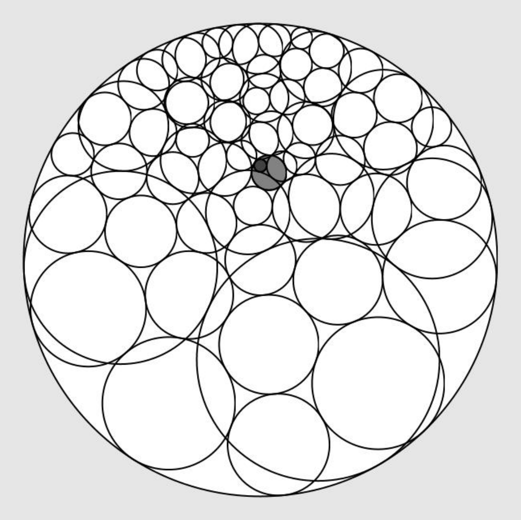

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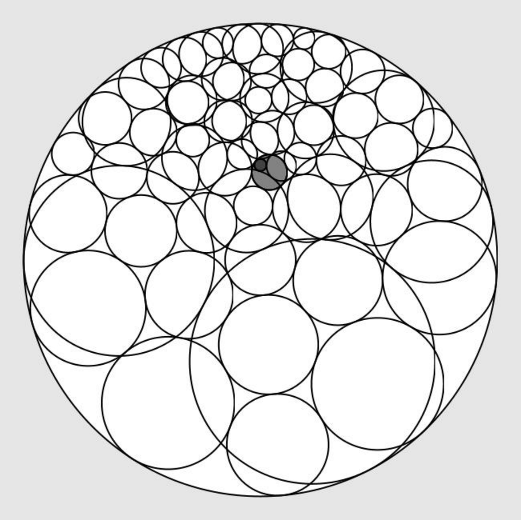

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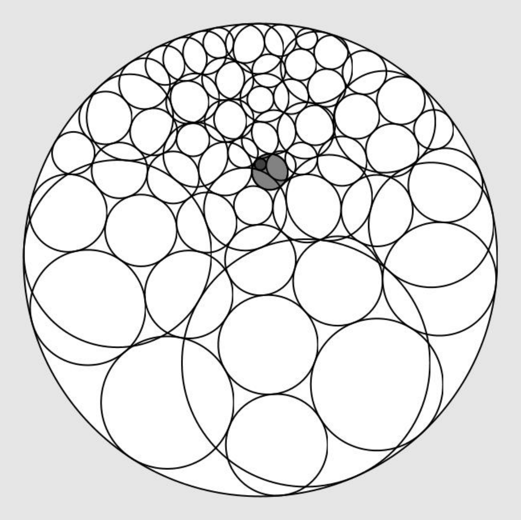

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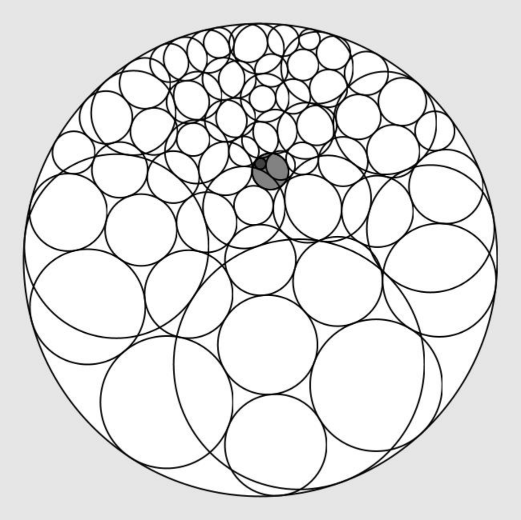

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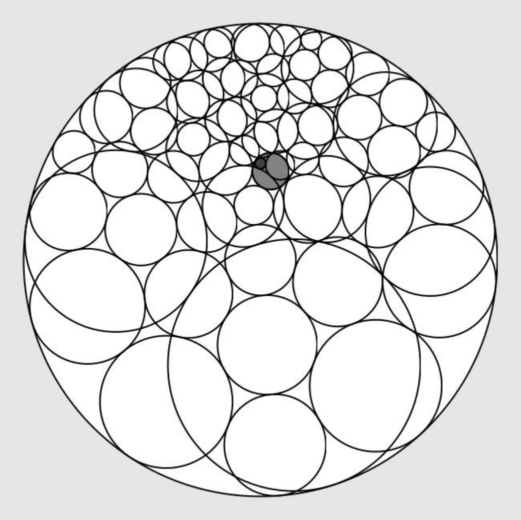

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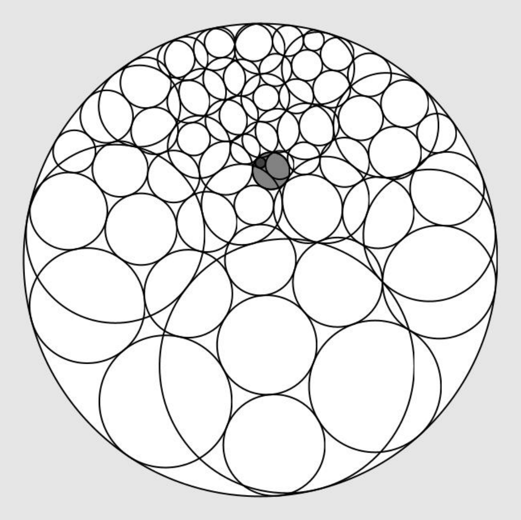

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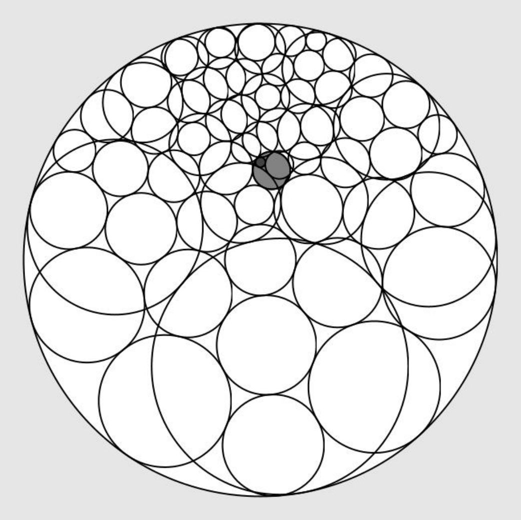

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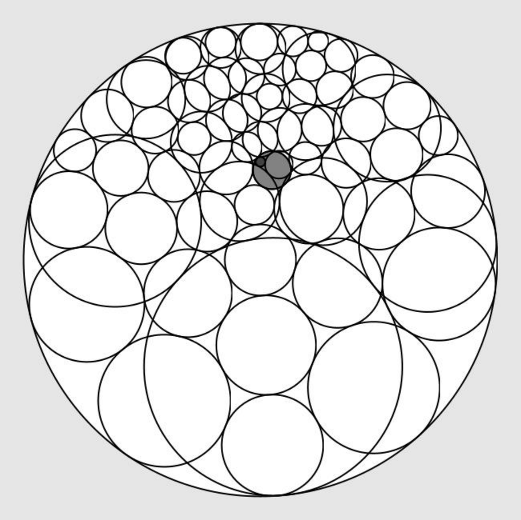

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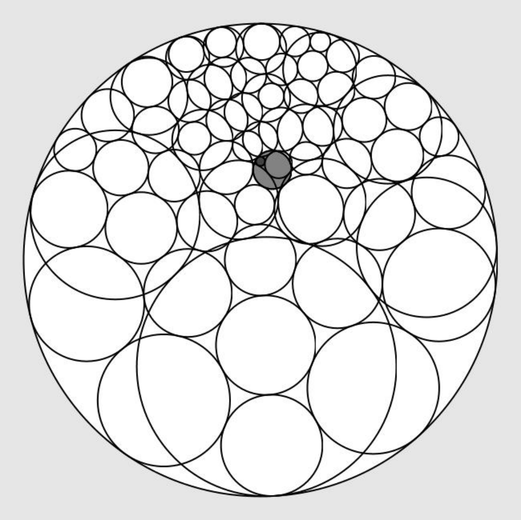

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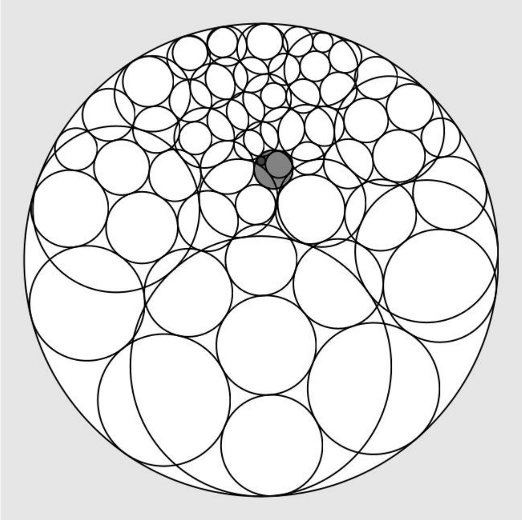

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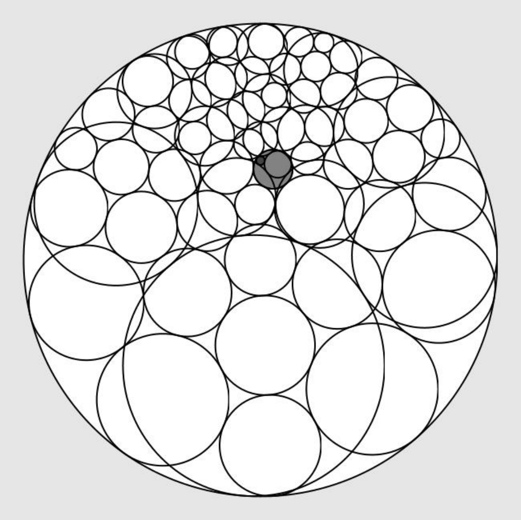

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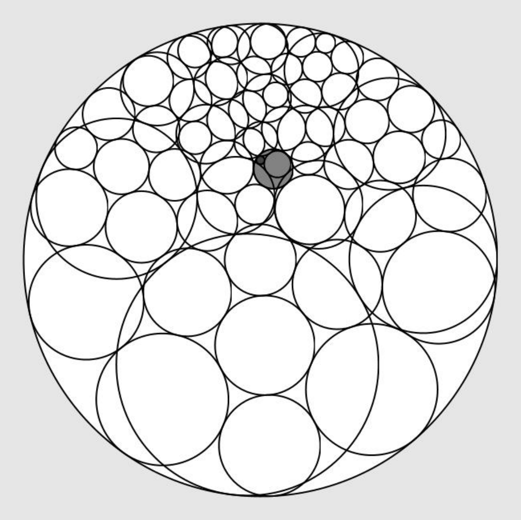


Two classical problems about rational functions

Suppose $d \geq 3$ and let $n = 2d - 2$.

1. Given points $w_1, \ldots, w_n \in \mathbb{C}$, find a rational function of degree d whose critical values are $w_1, \ldots, w_n$.
2. Given points $z_1, \ldots, z_n \in \mathbb{C}$, find a rational function of degree d whose critical points are $z_1, \ldots, z_n$.

Problem 1 generically has $(2d-2)!d^{d-3}/d!$ solutions.
(Hurwitz, Crescimanno and Taylor, Goulden)

Problem 2 generically has $\binom{2d-2}{d-1}/d$ solutions.
(Goldberg)

There is a *holomorphic correspondence* between the two parameter spaces.

Discrete rational maps – problem 1

The following covering construction solves the analogue of problem 1 for circle packings.

Given a triangulation of the sphere and its corresponding circle packing, take any finite branched covering of the underlying complex. If it is connected and has genus 0 then it is also a triangulation of the sphere and has its own circle packing.

We get a pair of circle packings and a simplicial map between them. We call this triple a *discrete rational map*.

Discrete rational maps – problem 2

An open problem

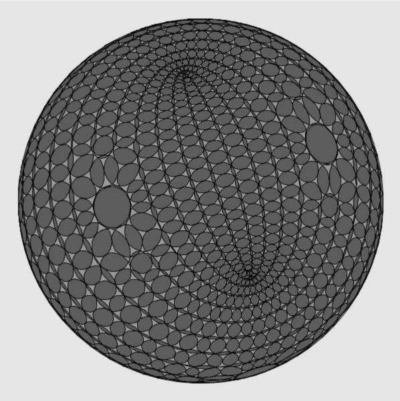
Given a triangulation of the sphere with specified branching, can it be realised as a circle packing on the sphere? If so, in how many ways?

There is no known packing algorithm that is *proven* always to solve this problem, except in certain special cases.

There are at least three unpublished methods that work experimentally in many other cases. But will they always work?

We have constructed examples to show that the solution is in general not unique up to Möbius maps, in the case of discrete rational maps of any degree at least 3.

A circle packing version of an elliptic function



The type problem and random walks

Any simply-connected infinite triangulation with no boundary can either be circle-packed to fill the open unit disc (the hyperbolic case), or circle packed to fill the Euclidean plane (the parabolic case). Which is it? You can decide if you can find out whether a certain random walk on the triangulation is transient or recurrent. This was proved in two PhD theses in the 1990s by McCaughan and Dubejko.

Further reading on circle packing

Ken Stephenson, *Introduction to Circle Packing*, CUP (2005).