COUNTING POINTS IN DEFINABLE SETS AND THE THREE EXPONENTIALS CONJECTURE

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Abstract. We show how a special case of Wilkie's conjecture would imply a real version of the three exponentials conjecture.

1. Introduction

In their 2006 paper [PW06], Pila and Wilkie proved a far-reaching result concerning the paucity of rational points in the transcendental parts of sets definable in o-minimal structures. In the years since that result, numerous applications have been found to the area of Diophantine geometry. In that same paper, Wilkie conjectured an improvement to his and Pila's result for those sets definable in a particular o-minimal structure: the real exponential field. This conjectured improvement is expected to have consequences in the area of transcendental number theory rather than Diophantine geometry. In this note we show how a special case of Wilkie's conjecture for a set in $\mathbb{R}^4$ would establish a real version of an open problem in transcendental number theory known as the three exponentials conjecture.

For the purposes of this note it suffices to know that Wilkie's conjecture deals with a class of subsets of $\mathbb{R}^n$ that can be written as a projection $\pi^m(V)$ where π^m is the projection map onto the first m coordinates and V is the zero set of an exponential polynomial, i.e.

$$V = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \, : \, P(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) = 0\}$$

where $P \in \mathbb{R}[t_1, \ldots, t_n, y_1, \ldots, y_n]$ is a real polynomial in $2n$ variables. Subsets of the above form – that is projections of the zero sets of exponential polynomials – will be referred to as definable in $\mathbb{R}_{\text{exp}}$.

To state Wilkie's conjecture we need a way of measuring the size of algebraic numbers, and so we define the multiplicative height of an algebraic number α as being $H(\alpha) = \exp(h(\alpha))$ where h is the logarithmic height

$$h(\alpha) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} D_v \log \max\{1, |\alpha|_v\},$$

where K is any number field containing α, M_K is the set of places of K, and D_v is the local degree at $v \in M_K$. Although this is a rather complicated definition, we only really need the fact that for $n \in \mathbb{Z}$ we have $H(\alpha^n) = H(\alpha)^{|n|}$ (see, for example, [Wal00]).

Given a set $X \subseteq \mathbb{R}^n$ and a number field $F \subset \mathbb{R}$ we set $X(F) = X \cap F^n$ and are interested in the density of points in $X(F)$ with respect to height, videlicet the

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cardinality of the set

$$X(F,T) = \{(x_1,\ldots,x_n) \in X(F) \,:\, H(x_i) \le T, 1 \le i \le n\}.$$

This cardinality we denote

$$N(X,F,T) = \mathrm{card}(X(F,T)).$$

Loosely speaking, then, Wilkie's conjecture posits an upper bound for $N(X,F,T)$ when X is definable in $\mathbb{R}_{\exp}$. There is a catch though. By taking the zero polynomial we can see that the set $\mathbb{R}^n$ is definable in $\mathbb{R}_{\exp}$, and so if we work in full generality then any upper bound for $N(X,F,T)$ will just be an upper bound on the number of elements of F^n of bounded height. In fact it is not just $\mathbb{R}^n$ that is the problem: any semi-algebraic subset contained in our set X may contain many algebraic points, that is $\gg T^{\delta}$ algebraic points of height less than T for some positive constant δ. But while most of the algebraic points in a set X may be found in the semi-algebraic subsets of X, these are – at least if one is studying transcendental number theory – the least interesting algebraic points. To remedy this we define the algebraic part of a set X to be union of all connected semi-algebraic subsets of X of positive dimension (so not the points). We denote this union X^{alg}, and then ignore it and concentrate on counting the density of points in $X^{\mathrm{trans}} = X \setminus X^{\mathrm{alg}}$.

Above we alluded to the fact that a set with $\gg T^{\delta}$ algebraic points should be considered to contain 'many' algebraic points. Thus it seems reasonable to consider a set with $\ll T^{\varepsilon}$ algebraic points to contain 'few' algebraic points, where ε may be any arbitrarily small positive number. Indeed, the Pila–Wilkie theorem says that for a broad class of well behaved sets, one has, for any $\varepsilon > 0$,

$$N(X^{\mathrm{trans}},F,T) \ll_{F,\varepsilon,X} T^{\varepsilon}.$$

This mooted class of well behaved sets includes those definable in $\mathbb{R}_{\exp}$ and many more besides. For some of these sets one does not seem to be able to do better than an upper bound of T^{ε}. But Wilkie conjectured otherwise for the case we are interested in.

Conjecture 1.1 (Wilkie's conjecture)**.** *Let X be a definable set in $\mathbb{R}_{\exp}$ and $F \subset \mathbb{R}$ be a number field of degree f. Then there are constants $c_1(X,f)$ and $c_2(X)$ such that, for any $T \ge e$,*

$$N(X^{\mathrm{trans}},F,T) \le c_1(X,f)(\log T)^{c_2(X)}.$$

At present proofs of this conjecture are only known when the dimension of X is 1 or for surfaces in $\mathbb{R}^3$ that can be represented as the images of a particularly well-behaved kind of function. In this note we will consider a set not covered by this list. Let $X_3 \subset \mathbb{R}^4$ be the set

$$X_3 = \{(x,y,z,t) \in \mathbb{R}^3_{>0} \times \mathbb{R}^{\times} \,:\, (\log x)(\log y) = t \log z\}.$$

In this paper we will show that a proof of Wilkie's conjecture for X_3 with a particularly low exponent would prove the real case of a conjecture in transcendental number theory known as the three exponentials conjecture. Following that we show that the same correlation holds if we can prove Wilkie's conjecture with an even lower exponent for all the fibres of X_3 if it is viewed as a definable family parametrised by t.

2. The algebraic part

Wilkie's conjecture concerns itself only with the transcendental parts of sets. To apply it, then, one must know the algebraic part of the set in question, otherwise one is apt to draw mistaken conclusions about the transcendental part of the set. The set X_3 is a three-fold in $\mathbb{R}^4$ and so to find all of X_3^{alg} we should look for semi-algebraic curves, surfaces, and three-folds. In practice we need only look for the curves: by fixing one of the coordinates in any algebraic surface contained in X_3 we get an algebraic curve contained in X_3, thus by finding all the algebraic curves we will have found every point in X_3^{alg}. Recall after all that we are not that interested in interesting representations of X_3^{alg} in terms of a few high dimensional objects, merely knowing which points in X_3 are in X_3^{alg} will suffice.

Lemma 2.1. *For $p, q \in \mathbb{R}$ let*

$$\Gamma_{1,p,q} = \{(e^q, s^p, s, pq) \,:\, s > 0\},$$

and

$$\Gamma_{2,p,q} = \{(s^p, e^q, s, pq) \,:\, s > 0\}$$

and for any algebraic function f let

$$\Gamma_{3,f} = \{(1, f(s), 1, s) \,:\, s \in f^{-1}(\mathbb{R}_{>0}) \setminus \{0\}\}$$

and

$$\Gamma_{4,f} = \{(f(s), 1, 1, s) \,:\, s \in f^{-1}(\mathbb{R}_{>0}) \setminus \{0\}\}.$$

Then

$$X_3^{\mathrm{alg}} = \bigcup_{\substack{p\in\mathbb{Q}\\ q\in\mathbb{R}\\ pq\neq 0}} \Gamma_{1,p,q} \cup \Gamma_{2,p,q} \bigcup_{\text{algebraic } f} \Gamma_{3,f} \cup \Gamma_{4,f}.$$

Proof. Let Γ be an arc of an algebraic curve contained in X_3. If t is constant on Γ then we should consider four cases: either x is constant on Γ, y is constant on Γ, z is constant on Γ, or none of the other three variables is constant on Γ. In the first couple of cases, if x (respectively y) is constant on Γ then Γ is part of $\Gamma_{1,p,q}$ (respectively $\Gamma_{2,p,q}$) for some p and q. If z is constant then $(\log x)(\log y)$ is constant on Γ, say equal to c. This would mean that

$$\exp\left(\frac{c}{\log y}\right)$$

was an algebraic function of x, which it clearly is not. The final case is when x, y, and z all vary. Here we may take y and z to be algebraic functions of x on some domain, and then by analytic continuation extend this relation to all sufficiently large, possibly complex, x. We can take these functions $y(x)$ and $z(x)$ to be given by Puiseux series in x, say

$$z(x) = a_0 x^{\alpha} + \ldots$$

and

$$y(x) = b_0 x^{\beta} + \ldots.$$

Then

$$x = \exp\left(\frac{\tau \log a + \tau\alpha \log x + \tau \log(1 + \ldots)}{\log b + \beta \log x + \log(1 + \ldots)}\right)$$

which tends to $\tau\alpha/\beta$ as $|x| \to \infty$, a contradiction.

That deals with the case that t is constant. So now suppose that t varies on Γ. If z is identically equal to 1 on Γ then we are in the situation of $\Gamma_{3,f}$ or $\Gamma_{4,f}$ for some algebraic function f. Otherwise x, y, and z will be algebraic functions of t that we can analytically continue to $|t| \to \infty$. As before, these functions can be given by Puiseux series, say

$$x(t) = at^{\alpha} + \dots,$$
$$y(t) = bt^{\beta} + \dots,$$

and

$$z(t) = ct^{\gamma} + \dots.$$

Then, since

$$t = \frac{\log x(t) \log y(t)}{\log z(t)}$$

will hold for all t we should have

$$t = \frac{\log(at^{\alpha} + \dots) \log(bt^{\beta} + \dots)}{\log(ct^{\gamma} + \dots)}$$
$$\to \frac{\alpha\beta}{\gamma} \log t$$

as $|t| \to \infty$, a contradiction. □

3. The real three exponentials conjecture

In the 1960s, Lang and Ramachandra independently proved a result that established the transcendence of at least one of six values of the exponential function, given the right conditions on the exponents.[Lan66][Ram68] Their result became known as the six exponentials theorem. They also conjectured an improved version: one should be able to eschew two of the exponentials and still guarantee the transcendence of at least one of the remaining four. This four exponentials conjecture remains an open problem. Variations on these two statements abound. Most of these variations fit into a neat system of implications, but sitting by itself in the corner is the three exponentials conjecture.

Conjecture 3.1 (Three exponentials conjecture)**.** *Let λ_1, λ_2, λ_3 be three complex numbers such that e^{λ_1}, e^{λ_2}, and e^{λ_3} are algebraic, and let $\gamma \neq 0$ be an algebraic number. If $\lambda_1\lambda_2 = \gamma\lambda_3$ then in fact $\lambda_1\lambda_2 = \gamma\lambda_3 = 0$.*

An equivalent formulation that looks more related to the aforementioned version of the six exponentials theorem is the following.

Conjecture 3.2 (Three exponentials conjecture – exponential formulation)**.** *Let x_1, x_2, and y be non-zero complex numbers and γ be a non-zero algebraic number. Then at least one of the following three numbers is transcendental:*

$$e^{x_1 y}, \quad e^{x_2 y}, \quad e^{\gamma x_1 / x_2}.$$

A currently unknown consequence of this conjecture would be the transcendence of e^{π^2}, a number that at present isn't even known to be irrational. To prove its transcendence from the conjecture one takes, in the logarithm version, $\lambda_1 = i\pi$, $\lambda_2 = -i\pi$, and $\gamma = 1$. If e^{π^2} is algebraic we can take $\lambda_3 = \pi^2$ and contradict the conjecture.

The results in this paper relate to a weaker version of the conjecture that only deals with real numbers.

Conjecture 3.3 (Real three exponentials conjecture)**.** *Let λ_1, λ_2, λ_3 be three real numbers such that e^{λ_1}, e^{λ_2}, and e^{λ_3} are algebraic, and let $\gamma \neq 0$ be a real algebraic number. If $\lambda_1\lambda_2 = \gamma\lambda_3$ then in fact $\lambda_1\lambda_2 = \gamma\lambda_3 = 0$.*

This can also be reformulated in terms of the transcendence of one of three exponentials by making the obvious changes to conjecture 3.2. This result implies real versions of classical results in transcendence theory such as the Gelfond–Schneider theorem and Hermite's theorem.

All this is very well, but what does it have to do with Wilkie's conjecture? The answer is as follows.

Theorem 3.4. *Suppose that for any number field $F \subset \mathbb{R}$ of degree $[F : \mathbb{Q}] = f$ we have*

$$N(X_3^{\text{trans}}, F, T) \ll_f (\log T)^2.$$

Then conjecture 3.3 is true. Conversely, if conjecture 3.3 is true then $X_3^{\text{trans}}(F) = \emptyset$.

This suggests a nice little repulsion result. Wilkie's conjecture doesn't forbid X_3 from containing lots of algebraic points, as long as their height grows slowly enough. But combined with transcendence properties of the exponential function, if $X_3^{\text{trans}}(\mathbb{R}_{\text{alg}})$ isn't empty then there is a lower bound on the density of its algebraic points.

Proof. First suppose that conjecture 3.3 is false, so we have three real numbers $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^\times$ such that $e^{\lambda_1}, e^{\lambda_2}, e^{\lambda_3} \in \mathbb{R}_{\text{alg}}$, and a real algebraic number $\gamma \neq 0$ such that $\lambda_1\lambda_2 = \gamma\lambda_3$. Let $x_1 = e^{\lambda_1}$, $y_1 = e^{\lambda_2}$, $z_1 = e^{\lambda_3}$, and $t_1 = \gamma$. Let F be the number field obtained by adjoining these numbers to $\mathbb{Q}$, so $F = \mathbb{Q}(x_1, y_1, z_1, t_1)$. Then we have that

$$(\log x_1)(\log y_1) = \lambda_1\lambda_2 = \gamma\lambda_3 = t_1 \log z_1.$$

And so $(x_1, y_1, z_1, t_1) \in X_3(F)$. We can raise each of these numbers to integer powers without expelling them from F, and so for any integers $a, b, c \in \mathbb{Z}$ we also have

$$(x_1^a, y_1^b, z_1^c, abt_1/c) \in X_3(F).$$

We now need to check two things: that by varying a, b, and c we get enough algebraic points to confound the $(\log T)^2$ upper bound, and that the points we get aren't contained in X_3^{alg}. The second of these issues is easy to deal with.

Recall that X_3^{alg} consists of the four families $\Gamma_{1,p,q}$, $\Gamma_{2,p,q}$ for $p \in \mathbb{Q}$, $q \in \mathbb{R}$, and $pq \neq 0$, and $\Gamma_{3,f}$ and $\Gamma_{4,f}$ for algebraic functions f. In the first two families two of the coordinates are of the form (e^q, pq) for our nonzero rational parameter p and real parameter q. But by Hermite's theorem at least one of these two numbers is transcendental, and so either x_1, y_1, or z_1 can't be of this form, and nor can any of their powers. In the latter two families of curves either the x or y coordinate is fixed at 1, but this would correspond to either λ_1 or λ_2 being zero, which we ruled out by hypothesis. So our points and the derived points cannot lie in X_3^{alg}.

We now show that varying a, b, c gives $\gg (\log T)^3$ points in $X_3^{\text{trans}}(F, T)$. As just stated, neither x_1, y_1, nor z_1 can be 1, so, since we are in $\mathbb{R}_{>0}$, raising them to different powers always results in different numbers. Now we check how many different values of a, b, and c we can use without exceeding height T. First let $\eta_1 = h(x_1)$, $\eta_2 = h(y_1)$, $\eta_3 = h(z_1)$, and $\eta_4 = H(t_1)$, where h is the logarithmic height defined in the introduction and $H = e^h$ is the multiplicative height. Recall that $H(\alpha^n) \le |n| H(\alpha)$ for algebraic α and integral n, and so if we want $H(x_1^a) \le T$ then it suffices to have $|a| \le \log T / \log H(x_1)$, and similarly for y_1^b and z_1^c. And so we can take

$$|a| \le \frac{\log T}{\eta_1}, \quad |b| \le \frac{\log T}{\eta_2}, \quad |c| \le \frac{\log T}{\eta_3}.$$

But we still need to check that the height in the t-coordinate doesn't grow too quickly. The multiplicative height H is sub-multiplicative, so we have

$$H\left(\frac{ab}{c}t_1\right) \le H\left(\frac{ab}{c}\right) H(t_1) = \max\{|ab|, |c|\}\eta_4.$$

In particular, if a, b, c range over the domains specified above then the above height is $\ll (\log T)^2$ which is definitely less than T after some small, fixed value of T depending on the η_is. So if we let $\eta = \max_i \eta_i$ then we can let $|a|, |b|, |c|$ vary between 1 and $\lfloor (\log T)/\eta \rfloor$ and keep all our points of height at most T. Hence we have

$$N(X_3^{\text{trans}}, F, T) \gg (\log T)^3.$$

We now prove the converse implication. So suppose we have a point $(x, y, z, t) \in X_3^{\text{trans}}$. We show this gives a counterexample to conjecture 3.3. By definition this point gives $\lambda_1 = \log x$, $\lambda_2 = \log y$, $\lambda_3 = \log z$, and $\gamma = t$ that satisfy the hypotheses of conjecture 3.3 but such that $\lambda_1\lambda_2 = \gamma\lambda_3$. And so we are done unless $\lambda_1\lambda_2 = \gamma\lambda_3 = 0$. Since $t = \gamma \neq 0$ this only leaves the possibility that $\lambda_3 = 0$ so that $z_1 = 1$ and $\lambda_1 = 0$ or $\lambda_2 = 0$ so that x or y is 0. But these two possibilities both lie in X_3^{alg}, specifically the families referred to as $\Gamma_{3,f}$ and $\Gamma_{4,f}$. This contradicts $(x, y, z, t) \in X_3^{\text{trans}}$, and so we are done. □

The set X_3 tackles the three exponentials conjecture all at once, as it were. An alternative approach would be to try to prove the conjecture individually for each value of γ. This corresponds to proving Wilkie's conjecture for the algebraic fibres of X_3 if we view it as a definable family parametrised by t. So we define

$$X_{3,\gamma} = \{(x, y, z) \in \mathbb{R}^3_{>0} \,:\, (\log x)(\log y) = \gamma \log z\}.$$

The case $\gamma = 1$ is the surface considered in [Pil10], where it is shown that

$$N(X_{3,1}^{\text{trans}}, F, T) \ll (\log T)^{44+\varepsilon}.$$

We show below how proving $N(X_{3,\gamma}^{\text{trans}}, F, T) \ll \log T$ for all algebraic $\gamma \neq 0$ would prove conjecture 3.3.

Theorem 3.5. *Let $\gamma \neq 0$ be real and algebraic. If for any number field $F \subset \mathbb{R}$ of degree f we have*

$$N(X_{3,\gamma}^{\text{trans}}, F, T) \ll_f \log T$$

then if λ_1, λ_2, and λ_3 are nonzero logarithms of positive, real algebraic numbers then $\lambda_1\lambda_2 \neq \gamma\lambda_3$.

Proof. This is essentially the same as the previous proof. $X_{3,\gamma}^{\text{alg}}$ is comprised of the sections of X_3^{alg} under $t = \gamma$. In particular, for any $(x, y, z) \in X_{3,\gamma}^{\text{alg}}$ either $x = 1$ or $y = 1$ or x is transcendental or y is transcendental. So if we have nonzero logarithms of algebraic numbers $\lambda_1, \lambda_2, \lambda_3$ such that $\lambda_1\lambda_2 = \gamma\lambda_3$ then letting $x_1 = e^{\lambda_1}, y_1 = e^{\lambda_2}, z_1 = e^{\lambda_3}$ gives us a point in $X_{3,\gamma}^{\text{trans}}(F)$ for $F = \mathbb{Q}(x_1, y_1, z_1)$. As above we can

then take integral powers of the exponents, this time parametrised by two integers, so we get
$$(x_1^a, y_1^b, z_1^{ab}) \in X_{3,\gamma}^{\text{trans}}(F)$$
for any integers a and b. This time we can't simply vary a and b between $-\lfloor \log T \rfloor$ and $\lfloor \log T \rfloor$ since then the height of the third coordinate, z^{ab} would grow about as quickly as $T^{\log T}$, faster than the T permitted. So we need $|a|, |b| \ll \log T$ and $|ab| \ll \log T$. If we take the lattice points $(a, b) \in \mathbb{Z}^2$ below the curve $xy = \log T$ then our conditions are met. The number of such points is bounded by
$$\int_1^{\log T} \frac{\log T}{x} dx = \log T \log \log T.$$
And so taking all such a and b gives us
$$N(X_{3,\gamma}^{\text{trans}}, F, T) \gg \log T \log \log T.$$
$\square$

Wilkie's conjecture has already been established for these sets in [JT12], but unfortunately the exponent is somewhat larger than 1.

4. Further work

Just as the set X_3 provides a tangible link between Wilkie's conjecture and the three exponentials conjecture, so too can one find sets $X_4 \subset \mathbb{R}^4$ and $X_{\#4} \subset \mathbb{R}^8$ that link Wilkie's conjecture to the four exponentials conjecture and the sharp four exponentials conjecture respectively. Unfortunately these sets are somewhat unwieldy to work with, making even finding their algebraic parts difficult. Moreover proving a logarithmic bound for their algebraic points doesn't currently seem possible. Finally, the sets corresponding to these conjectures don't have a group structure, so as with X_3 one will presumably have to prove a logarithmic bound with exceedingly small exponent to prove the conjectures, whereas a group structure would likely lead to any exponent being sufficient to bound the number of counterexamples to these conjectures.

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