Algebraic Number Theory – Lecture 11

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"The total number of Dirichlet's publications is not large: jewels are not weighed on a grocery scale."

– Carl Friedrich Gauss

## 1. Dirichlet characters

A *Dirichlet character* (mod $m$) for $m \in \mathbb{N}$ is a homomorphism

$$\chi : (\mathbb{Z}/m\mathbb{Z})^\times \longrightarrow S = \{z \in \mathbb{C} \,:\, |z| = 1\}.$$

It is called *primitive* if it does not arise from the composite

$$(\mathbb{Z}/m\mathbb{Z})^\times \longrightarrow (\mathbb{Z}/m'\mathbb{Z})^\times \xrightarrow{\chi'} S$$

of a Dirichlet character $\chi'$ (mod $m'$) for any proper divisor $m'$ of $m$. For Dirichlet characters in general, the greatest common divisor of all these proper divisors $m'$ is called the *conductor* $f$ of $\chi$.

Given $\chi$ we define a multiplicative function

$$\chi(n) = \begin{cases} \chi(n \bmod m) & \text{for } \operatorname{hcf}(n,m) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The *trivial character* mod $m$ is given by

$$\chi_t(n) = \begin{cases} 1 & \text{for } \operatorname{hcf}(m,n) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

**Definition.** For a Dirichlet character $\chi$ we form the *Dirichlet $L$-series*

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for $s \in \mathbb{C}$ with $\Re(s) > 1$.

**Proposition.** *The series $L(\chi, s)$ converges absolutely and uniformly in the domain $\Re(s) \geqslant 1 + \delta$ for any $\delta > 0$. It therefore represents an analytic function on the half plane $\Re(s) > 1$. We have Euler's identity:*

$$L(\chi, s) = \prod_p \frac{1}{1 - \chi(p)p^{-s}}.$$

The Dirichlet $L$-series satisfy a functional equation that relates the variable $s$ to a function with the variable $1 - s$ contained in it. This property also holds for a larger class of $L$-series, called Hecke $L$-series. In order to state the theorem about the functional equation, and in order to understand the proof (which hinges on the integral representation of $L(\chi, s)$) we need to see that $L(\chi, s)$ is in fact the Mellin transform of a theta series.

**Definition.** We define the *exponent* $p \in \{0, 1\}$ of $\chi$ by

$$\chi(-1) = (-1)^p.$$

We also define

$$\chi((n)) = \chi(n) \left(\frac{n}{|n|}\right)^p$$

which defines a multiplicative function on the semigroup of all ideals $(n)$ that are relatively prime to $m$. This function is called the Größencharakter.

Dirichlet characters (mod $m$) have the following properties:

(1) $\chi(n + m) = \chi(n)$ for any $n$;
(2) $\chi(xy) = \chi(x)\chi(y)$ for any $x$ and $y$;
(3) $\chi(1) = 1$;
(4) if $a \equiv b \pmod{m}$ then $\chi(a) = \chi(b)$.

## 2. Completed $L$-series

Consider the integral

$$\Gamma(\chi, s) = \Gamma(\tfrac{s+p}{2}) = \int_0^\infty e^{-y} y^{(s+p)/2} \frac{dy}{y}.$$

Making the substitution $y \mapsto \pi n^2 y/m$ we get

$$\left(\frac{m}{\pi}\right)^{\frac{s+p}{2}} \Gamma(\chi, s) \frac{1}{n^s} = \int_0^\infty n^p e^{-\pi n^2 y/m} y^{(s+p)/2} \frac{dy}{y}.$$

Multiplying this by $\chi(n)$ and summing over all $n \in \mathbb{N}$ we get

$$\left(\frac{m}{\pi}\right)^{\frac{s+p}{2}} \Gamma(\chi, s) L(\chi, s) = \sum_{n=1}^\infty \int_0^\infty \chi(n) n^p e^{-\pi n^2 y/m} y^{(s+p)/2} \frac{dy}{y}.$$

However,

$$\sum_{n=1}^\infty \int_0^\infty \left| \chi(n) n^p e^{-\pi n^2 y/m} y^{(s+p)/2} \frac{dy}{y} \right| \leqslant \left(\frac{m}{\pi}\right)^{\frac{\Re(s)+p}{2}} \Gamma\left(\frac{\Re(s)+p}{2}\right) \zeta(\Re(s)) < \infty,$$

so we may swap the order of summation and integration and get

$$\left(\frac{m}{\pi}\right)^{\frac{s+p}{2}} \Gamma(\chi, s) L(\chi, s) = \int_0^\infty \sum_{n=1}^\infty \chi(n) n^p e^{-\pi n^2 y/m} y^{(s+p)/2} \frac{dy}{y}. \tag{1}$$

The series under the integral in (1),

$$g(y) = \sum_{n=1}^\infty \chi(n) n^p e^{-\pi n^2 y/m},$$

arises from the theta series

$$\theta(\chi, z) = \sum_{n \in \mathbb{Z}} \chi(n) n^p e^{\pi i n^2 z/m}.$$

Because $\chi(n) n^p = \chi(-n)(-n)^p$,

$$\theta(\chi, z) = \chi(0) + 2 \sum_{n=1}^\infty \chi(n) n^p e^{\pi i n^2 z/m}.$$

So

$$g(y) = \frac{1}{2}(\theta(\chi, iy) - \chi(0))$$

with $\chi(0) = 1$ if $\chi$ is the trivial character, and 0 otherwise. In the case $m = 1$ we get

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z},$$

which is Jacobi's theta function.

We see

$$L_\infty(\chi, s) = \left(\frac{m}{\pi}\right)^{s/2} \Gamma(\chi, s)$$

as the "Euler factor" at the infinite prime.

**Definition.** The *completed $L$-series* of a character $\chi$ is given by

$$\Lambda(\chi, s) = L_\infty(\chi, s) L(\chi, s)$$

for $\Re(s) > 1$.

**Proposition.** *The function $\Lambda(\chi, s)$ has the integral representation*

$$\Lambda(\chi, s) = \frac{c(\chi)}{2} \int_0^\infty (\theta(\chi, iy) - \chi(0)) y^{(s+p)/2} \frac{dy}{y}$$

*where*

$$c(\chi) = \left(\frac{\pi}{m}\right)^{p/2}.$$

For a continuous function $f : \mathbb{R}_{>0}^\times \to \mathbb{C}$ on the (multiplicative) group of positive real numbers, we define the *Mellin transform* to be the improper integral

$$L(f, s) = \int_0^\infty (f(y) - f(\infty)) y^s \frac{dy}{y},$$

provided the limit $f(\infty) = \lim_{y \to \infty} f(y)$ and the integral exist.

**Theorem** (Mellin principle)**.** *Let $f, g : \mathbb{R}_{>0}^\times \to \mathbb{C}$ be continuous functions such that*

$$f(y) = a_0 + \mathcal{O}(\exp(-cy^\alpha)), \qquad g(y) = b_0 + \mathcal{O}(\exp(-cy^\alpha))$$

*for $y \to \infty$ with positive constants $c, \alpha$. If these functions satisfy the equation*

$$f(1/y) = C y^k g(y)$$

*for $k \in \mathbb{R}, k > 0$, and $C \in \mathbb{C}, C \neq 0$, then*

(1) *the integrals $L(f, s)$ and $L(g, s)$ converge absolutely and uniformly if $s$ varies in an arbitrary compact domain contained in $\{s \in \mathbb{C} : \Re(s) > k\}$. They are therefore holomorphic functions on $\{s \in \mathbb{C} : \Re(s) > k\}$ and they admit holomorphic continuations to $\mathbb{C} \setminus \{0, k\}$.*
(2) *They have simple poles at $s = 0$ and $s = k$ with residues*

$$\operatorname{Res}_{s=0} L(f, s) = -a_0, \quad \operatorname{Res}_{s=k} L(f, s) = C b_0$$
$$\operatorname{Res}_{s=0} L(g, s) = -b_0, \quad \operatorname{Res}_{s=k} L(g, s) = C^{-1} a_0.$$

(3) *They satisfy $L(f, s) = C L(g, k - s)$.*

**Proposition.** *Let $a, b, \mu \in \mathbb{R}$, $\mu > 0$. Then the series*

$$\theta_\mu(a,b,z) = \sum_{g \in \mu\mathbb{Z}} e^{\pi i (a+g)^2 z + 2\pi i b g}$$

*converges absolutely and uniformly in the domain $\Im(z) \geqslant \delta$ for every $\delta > 0$, and for $z \in \mathbb{H}$ one has the transformation formula*

$$\theta_\mu(a,b,-1/z) = e^{-2\pi i a b} \frac{\sqrt{z/i}}{\mu} \theta_{1/\mu}(-b,a,z).$$

If we differentiate this $p$ times with respect to $a$, recalling that $p \in \{0,1\}$, we get

$$\frac{d^p}{da^p} \theta_\mu(a,b,z) = (2\pi i)^p z^p \theta_\mu^p(a,b,z)$$

with

$$\theta_\mu^p(a,b,z) = \sum_{g \in \mu\mathbb{Z}} (a+g)^p e^{\pi i (a+g)^2 z + 2\pi i b g}$$

and

$$\frac{d^p}{da^p} e^{-2\pi i a b} \theta_{1/\mu}(-b,a,z) = (2\pi i)^p e^{-2\pi i a b} \theta_{1/\mu}^p(-b,a,z).$$

Applying the differentiation $d^p/da^p$ to the transformation formula in the previous proposition we get the following.

**Corollary.** *For $a, b, \mu \in \mathbb{R}$, $\mu > 0$, one has the transformation formula*

$$\theta_\mu^p(a,b,-1/z) = (i^p e^{2\pi i a b} \mu)^{-1} (z/i)^{p+\frac{1}{2}} \theta_{1/\mu}^p(-b,a,z).$$

**Definition.** For $n \in \mathbb{Z}$ the Gauss sum $\tau(\chi, n)$ associated to the Dirichlet character $\chi \pmod{m}$ is defined to be the complex number

$$\tau(\chi,n) = \sum_{v=0}^{n-1} \chi(v) e^{2\pi i v n/m}.$$

If $n = 1$ we just write $\tau(\chi)$.

**Proposition.** *For a primitive Dirichlet character $\chi \pmod{m}$ one has*

$$\tau(\chi,n) = \overline{\chi}(n)\tau(\chi)$$

*and*

$$|\tau(\chi)| = \sqrt{m}.$$

**Proposition.** *If $\chi$ is a primitive Dirichlet character mod $m$ then we have the transformation formula*

$$\theta(\chi,-1/z) = \frac{\tau(\chi)}{i^p\sqrt{m}} (z/i)^{p+\frac{1}{2}} \theta(\overline{\chi}, z).$$

**Theorem.** *If $\chi$ is a nontrivial primitive Dirichlet character, then the completed $L$-series $\Lambda(\chi,s)$ has an analytic continuation to the whole complex plane $\mathbb{C}$ and satisfies the functional equation*

$$\Lambda(\chi,s) = W(\chi)\Lambda(\overline{\chi}, 1-s)$$

*where*

$$W(\chi) = \frac{\tau(\chi)}{i^p\sqrt{m}}$$

*has absolute value 1.*