Algebraic Number Theory – Lecture 12

Lee Butler

To see a World in a Grain of Sand
And a Heaven in a Wild Flower,
Hold Infinity in the palm of your hand
And Eternity in an hour.

– William Blake

1. Some algebra

Let K be a number field and consider the sum

$$\sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} \tag{1}$$

where the sum is over all ideals $\mathfrak{a} \subseteq \mathcal{O}_K$. Recall that the norm of an ideal is

$$N(\mathfrak{a}) = |\mathcal{O}_K/\mathfrak{a}|.$$

One of the questions you might ask about the series (1) is: does it converge? Well, consider the partial sums

$$\sum_{N(\mathfrak{a})\leqslant x} \frac{1}{N(\mathfrak{a})^s}.$$

If these are bounded then the series must converge. Write $s = \sigma + i\tau$, then, since the ideals form a UFD,

$$\begin{aligned}
\sum_{N(\mathfrak{a})\leqslant x} \frac{1}{N(\mathfrak{a})^\sigma} &\leqslant \sum_{\substack{N(\mathfrak{p}_i)\leqslant x \\ e_i \geqslant 0}} \frac{1}{(N(\mathfrak{p}_1)^{e_1}\cdots)^\sigma} \\
&= \prod_{N(\mathfrak{p})\leqslant x} \left(1 + \frac{1}{N(\mathfrak{p})^\sigma} + \frac{1}{N(\mathfrak{p})^{2\sigma}} + \dots\right) \\
&= \prod_{N(\mathfrak{p})\leqslant x} \left(1 - \frac{1}{N(\mathfrak{p})^\sigma}\right)^{-1}.
\end{aligned}$$

Andrew assured us that for each prime ideal $\mathfrak{p}$ there is a unique prime number p such that $N(\mathfrak{p}) = p^f$ for some $f \in \mathbb{N}$. How many prime ideals can correspond to the same prime number p? Well $N(\mathfrak{p}) = p^f$ if and only if

$$p\mathcal{O}_K = \mathfrak{p}^e \mathfrak{p}_2^{e_2} \cdots \mathfrak{p}_r^{e_r}$$

and f is the inertia degree of $\mathfrak{p}$ over p. So from Andrew's lecture on Hilbert ramification and the fundamental identity therein, at most $[K:\mathbb{Q}]$ prime ideals can correspond to the same prime number p. So

$$\sum_{N(\mathfrak{a})\leqslant x} \frac{1}{N(\mathfrak{a})^\sigma} \leqslant \prod_{p\leqslant x} \left(1 - \frac{1}{p^\sigma}\right)^{-[K:\mathbb{Q}]}.$$

The product on the right converges for $\sigma > 1$ and so the series (1) converges.

Since we bounded a finite sum by an infinite one above we lost equality. But consider the infinite product

$$\prod_{\mathfrak{p}}\left(1+\frac{1}{N(\mathfrak{p})^s}+\frac{1}{N(\mathfrak{p})^{2s}}+\dots\right)$$

$$=1+\frac{1}{N(\mathfrak{p}_1)^s}+\frac{1}{N(\mathfrak{p}_2)^s}+\dots+\frac{1}{N(\mathfrak{p}_1)^{2s}}+\frac{1}{(N(\mathfrak{p}_1)N(\mathfrak{p}_2))^s}+\dots+\frac{1}{N(\mathfrak{p}_1)^{3s}}+\frac{1}{(N(\mathfrak{p}_1)^2N(\mathfrak{p}_2))^s}+\dots$$

$$=1+\frac{1}{N(\mathfrak{p}_1)^s}+\frac{1}{N(\mathfrak{p}_2)^s}+\dots+\frac{1}{N(\mathfrak{p}_1^2)^s}+\frac{1}{N(\mathfrak{p}_1\mathfrak{p}_2)^s}+\dots+\frac{1}{N(\mathfrak{p}_1^3)^s}+\frac{1}{N(\mathfrak{p}_1^2\mathfrak{p}_2)^s}+\dots$$

Since the ideals in $\mathcal{O}_K$ form a UFD each denominator is a unique ideal in $\mathcal{O}_K$, and each ideal in $\mathcal{O}_K$ appears as a unique denominator. So

$$\sum_{\mathfrak{a}}\frac{1}{N(\mathfrak{a})^s}=\prod_{\mathfrak{p}}\left(1-\frac{1}{N(\mathfrak{p})^s}\right)^{-1}.$$

We call this function on s the *Dedekind zeta function of* K, $\zeta_K(s)$.

Example. The Dedekind zeta function on $\mathbb{Q}$ is

$$\sum_{n\in\mathbb{N}}\frac{1}{N(n)^s}=\sum_{n=1}^{\infty}\frac{1}{n^s}=\zeta(s).$$

2. Analytic stuff

The Dedekind zeta function of K contains a vast amount of information about K. Hecke proved in 1917 that

$$\lim_{s\to 1}(s-1)\zeta_K(s)=\frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{w\sqrt{|d_K|}},$$

where

- r_1, r_2 are respectively the number of real and pairs of complex embeddings $K \hookrightarrow \mathbb{C}$,
- h_K is the class number of K,
- d_K is the discriminant of K,
- w is the number of roots of unity in K, and
- R_K is the regulator of K, which we'll define another time[1].

We're not going to prove this formula. Instead we'll prove a special case of it, modulo knowing about the regulator and roots of unity. We need some analysis first, though.

Lemma 1. *Let* $(a_m)_{m\in\mathbb{N}}$ *be a sequence in* $\mathbb{C}$ *and suppose that*

$$A(x)=\sum_{m\leqslant x}a_m=\mathcal{O}(x^\delta)$$

for some $\delta\geqslant 0$*. Then*

$$\sum_{m=1}^{\infty}\frac{a_m}{m^s}$$

converges for $\Re(s)>\delta$ *and in this half plane we have*

$$\sum_{m=1}^{\infty}\frac{a_m}{m^s}=s\int_1^{\infty}\frac{A(x)}{x^{s+1}}\,dx.$$

[1] It's $\det(\log|\sigma_i(\varepsilon_j)|)$ where $\varepsilon_1,\dots,\varepsilon_{r_1}$ are fundamental units and $\sigma_1,\dots,\sigma_{r_1}$ are the real embeddings.

Proof. Telescoping the sum we get

$$\sum_{m=1}^{M} \frac{a_m}{m^s} = \sum_{m=1}^{M} (A(m) - A(m-1)) m^{-s}$$

$$= A(M)M^{-s} + \sum_{m=1}^{M-1} A(m) \left(m^{-s} - (m+1)^{-s}\right).$$

Since

$$m^{-s} - (m+1)^{-s} = s \int_m^{m+1} \frac{1}{x^{s+1}}\, dx$$

and $A(x)$ is a step function, we get

$$\sum_{m=1}^{M} \frac{a_m}{m^s} = \frac{A(M)}{M^s} + s \int_1^M \frac{A(x)}{x^{s+1}}\, dx.$$

For $\Re(s) > \delta$, since $A(x) = \mathcal{O}(x^\delta)$ we have

$$\lim_{M\to\infty} \frac{A(M)}{M^s} = 0,$$

whereas the integral will converge. So the partial sums converge for $\Re(s) > \delta$ and

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s} = s \int_1^{\infty} \frac{A(x)}{x^{s+1}}\, dx.$$

□

Lemma 2. *The number of pairs of integers (a, b) with $a > 0$ satisfying*

$$a^2 + Db^2 \leqslant x$$

is

$$\frac{\pi x}{2\sqrt{D}} + \mathcal{O}(\sqrt{x}).$$

Proof. Exercise. □

We can now prove the following.

Theorem. *Let $K = \mathbb{Q}(\sqrt{-D})$ where $D > 0$ is square-free and $-D \not\equiv 1 \pmod 4$. Suppose K has class number 1. Then $(s-1)\zeta_K(s)$ extends analytically to $\Re(s) > \frac{1}{2}$ and*

$$\lim_{s\to 1} (s-1)\zeta_K(s) = \frac{\pi}{\sqrt{|d_K|}}.$$

Proof. Since $h_K = 1$, by Andrew's lecture $\mathcal{O}_K$ is a PID, so any ideal $\mathfrak{a} \subseteq \mathcal{O}_K$ is of the form $(a+b\sqrt{-D})$ with, say, $a > 0$. From Jobin's first lecture we can see that

$$N((a + b\sqrt{-D})) = a^2 + Db^2.$$

So

$$\zeta_K(s) = \sum_{\substack{a\in\mathbb{N}\\ b\in\mathbb{Z}}} \frac{1}{(a^2 + Db^2)^s} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where a_n is the number of solutions to $a^2 + Db^2 = n$, with $a \in \mathbb{N}$ and $b \in \mathbb{Z}$. By lemma 1 we have

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s \int_1^{\infty} \frac{A(x)}{x^{s+1}}\, dx$$

with $A(x) = \sum_{n \leqslant x} a_n$. By lemma 2 we know

$$A(x) = \frac{\pi x}{2\sqrt{D}} + \mathcal{O}(\sqrt{x}).$$

So

$$\zeta_K(s) = s\int_1^\infty \left(\frac{\pi x}{2\sqrt{D} x^{s+1}} + \frac{E(x)}{x^{s+1}} \right) dx$$

$$= \frac{\pi s}{2\sqrt{D}(s-1)} + s\int_1^\infty \frac{E(x)}{x^{s+1}}\, dx,$$

where $E(x) = \mathcal{O}(\sqrt{x})$. This integral converges for $\Re(s) > \frac{1}{2}$, giving us our analytic continuation:

$$(s-1)\zeta_K(s) = \frac{\pi s}{2\sqrt{D}} + s(s-1)\int_1^\infty \frac{E(x)}{x^{s+1}}\, dx.$$

From this we get

$$\lim_{s \to 1}(s-1)\zeta_K(s) = \frac{\pi}{2\sqrt{D}}.$$

From Sandro's lecture, $d_K = -4D$, so $2\sqrt{D} = \sqrt{|d_K|}$, as required. □

Assuming the class number formula, and since we know that $r_1 = 0, r_2 = 1$, and $h_K = 1$, we get that $w = 2R_K$. Thus to calculate R_K, which is a rather unpleasant determinant, we need only check if $i \in K$.