Being a treatise on some pfaffian functions of degree two

OR

How I learned to stop worrying and love the quotient of Bessel functions

This note establishes the pfaffianicity of two unlikely looking functions. To wit, we have the following.

Lemma 1. *Let $J_n(x)$ be the Bessel function of the first kind, that is the solution to*

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0,$$

that is nonsingular at the origin, and let $Y_n(x)$ be the Bessel function of the second kind, that is the solution to the above equation that is singular at the origin. Then the following functions are both pfaffian:

$$f(x)=\frac{J_1(x)}{J_0(x)},\quad g(x)=\frac{Y_1(x)}{Y_0(x)}.$$

Proof. Consider the system of ODEs:

$$\frac{df_1}{dx}=f_1 \tag{1}$$

$$\frac{df_2}{dx}=(f_1+f_2)^2. \tag{2}$$

The solutions f_1, f_2 for any initial conditions will be a pfaffian chain by definition. Equation (1) is clearly satisfied by $f_1(x)=e^x$ which leaves us to solve

$$\frac{df_2}{dx}=e^{2x}+f_2(x)(f_2(x)+2e^x).$$

Setting $f_2(0)=a$ the solution can be verified as

$$f_2(x)=\frac{c_aY_1(2e^{x/2})+d_ae^{x/2}J_1(2e^{x/2})-d_ae^xJ_0(2e^{x/2})-c_ae^xY_0(2e^{x/2})}{c_aY_0(2e^{x/2})+d_aJ_0(2e^{x/2})}.$$

where

$$c_a=(1+a)J_0(2)-J_1(2),\quad d_a=Y_1(2)-(1+a)Y_0(2),$$

The composition of pfaffian functions preserves their state of being pfaffian, so we may let $u=2e^{x/2}$ to simplify the above. Then, setting $a=J_1(2)/J_0(2)-1\approx 1.5759$ we get $c_a=0$ and the solution simplifies to

$$f_2(x)=\frac{uJ_1(u)}{2J_0(u)}-\frac{u^2}{4}.$$

As $u^2/4$ and $2/u$ are both pfaffian we may add the first and multiply by the second to get the pfaffian function

$$f(x)=\frac{J_1(x)}{J_0(x)}.$$

Choosing $a=Y_1(2)-(1+a)Y_0(2)\approx -1.2097$ similarly proffers $d_a=0$, and, when dressed down, gives us the pfaffian function

$$g(x)=\frac{Y_1(x)}{Y_0(x)}.$$

□

The above can be viewed as a sort-of 'top down' approach to deriving these functions' pfaffianicity. That is, we define an interesting ODE and happen across the quotient of Bessel functions as a solution. Quicker, neater, and perhaps more handy, is the bottom up approach. That is, we simply look at the function and see if its derivative is anything nice. And in this case, it is.

Lemma 2. *Let $f(x) = -x^{-1}, g(x) = Z_1(x)/Z_0(x)$, and $h_n(x) = Z_n(x)$ for $0 \leqslant n \leqslant t$, where Z is either J or Y, i.e. Z_n is the nth Bessel function of either the first or second kind. Then $f, g, h_0, h_1, \ldots, h_t$ is a pfaffian chain of order $t+3$ and degree 2 on any interval $I \subset \mathbb{R}$ not containing 0 or any zeros of $Z_0(x)$.*

Proof. The proof is by simple computation, videlicet we have:

$$\begin{aligned}
\frac{df}{dx} &= f^2, \\
\frac{dg}{dx} &= 1 + fg + g^2, \\
\frac{dh_0}{dx} &= -gh_0, \\
\frac{dh_n}{dx} &= h_{n-1} - nfh_n, \quad \text{for } 1 \leqslant n \leqslant t.
\end{aligned}$$

The last two equations follow from the differential identity

$$\frac{d}{dx}(x^n Z_n(x)) = x^n Z_{n-1}(x).$$

Using the product rule on the left-hand side and equating the two expressions gives the desired result. The second equation then follows using the quotient rule. □

Corollary. *The Bessel functions of the first and second kind, J_n and Y_n respectively, are pfaffian for all $n \in \mathbb{Z}$ on any interval $I \subset \mathbb{R}$ containing neither 0 nor a zero of $J_0(x)$ or $Y_0(x)$ respectively.*