IRRATIONAL POLYGONS AND CONVERGENTS

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Abstract. We show how anyone with patience, a protractor, and a calculator that's almost out of batteries can find the best rational approximations to any real number.

1. Introduction

It is an important moment in every person's life when he first sits down with his pencil and his paper, his ruler and protractor, and of course his scientific calculator, and decides that today is the day he is going to draw a regular n-sided polygon. The process is utterly intuitive. First one determines the interior angle of the shape, which thanks to a nifty proof by induction one knows to be $\theta = \pi(1 - 2/n)$. Next one draws a line of unit length, then turns left until the next line will make an angle of θ with the previous one, and repeats the process. After n steps one returns whence one started, and the polygon is complete.

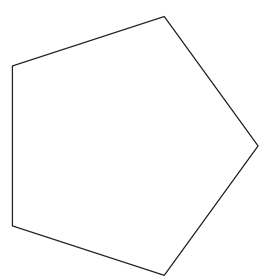

Figure 1. A regular 5-sided polygon.

A braver person might ask what happens if one picks a value of n that is not a natural number, but a *rational* one, say $n = 5/2$. Nothing in the above algorithm seems to forbid this, so one finds that the internal angle for such a shape is $\theta = \pi/5$, and one starts drawing. And voila! After five lines one gets back to one's starting point, and one has entered the world of star polygons.

Ostensibly the natural setting for star polygons is the complex plane. Given integers a and b, an (a/b)-gon is realized by plotting the a^{th} roots of unity, $\exp(2\pi ik/a)$, $0 \leq k < n$, and then joining each root to the one b after, i.e. joining $\exp(2\pi ik/a)$ to $\exp(2\pi i(k+b)/a)$ for each k. This method removes all restrictions on the integers a and b, allowing such shenanigans as $(1/0)$-gons, or proffering an $(8/2)$-gon that isn't just a square.

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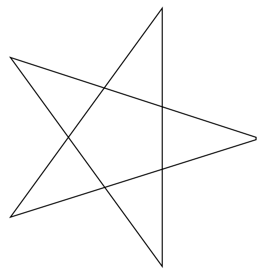


FIGURE 2. A (5/2)-gon, yesterday.

As with many things in life, though, in passing to the complex plane we have made it less clear how to extend the set-up still further. We've have seen n-gons when n is an integer and n-gons when n is rational. What if n is irrational? The joining-roots-of-unity interpretation is less intuitive here. Following the above schema we should plot the points $\exp(\sqrt{2}\pi ik)$ for $k \in \mathbb{N}$, and join $\exp(\sqrt{2}\pi ik)$ and $\exp(\sqrt{2}\pi i(k+1))$ for each k. The equidistribution theorem in ergodic theory tells us that these "$\sqrt{2}^{\text{th}}$ roots of unity" are dense on the circle $|z| = 1$, thus giving us a countable set of lines that are dense in the annulus $R \leq |z| \leq 1$, where we can easily check that $R = \sin((\sqrt{2}-1)\pi/2) \approx 0.606\ldots$.

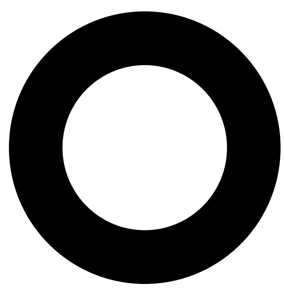


FIGURE 3. A $\sqrt{2}$-gon.

This is certainly a valid way of obtaining a $\sqrt{2}$-gon, but it somehow feels less fulfilling. And so we pick up our pen and paper, our ruler and protractor, and of course our scientific calculator, and we go back to the drawing board.

Let's take a different irrational number this time, say π. The internal angle of a π-gon should be $\theta = \pi - 2$. And so we draw a line of unit length, turn left so our next line makes an angle of $\pi - 2$ with the previous one, and then repeat. We wouldn't expect to ever return to the starting point, but every now and then we do get rather close. If we were very patient and very careful and we kept track of the number of lines we'd drawn when we made our closest approaches thus far to the starting point then we'd form the sequence $(3, 22, 333, 355, 103993, \ldots)$. These numbers should be familiar as the numerators in the best rational approximations to π, the rational numbers that give a better approximation to π than any other rational with smaller denominator. Since there is some variety in the meaning of the term "best rational approximations" we had better define what is meant in this note.

Definition 1. Given a real number α, a *best rational approximation* to α is a rational number p/q with $q > 0$ and $\mathrm{hcf}(p, q) = 1$ such that for any rational number s/t with $1 \leq t \leq q$ and $p/q \neq s/t$ we have

$$|q\alpha - p| < |t\alpha - s|.$$

This definition gives what some of the literature refers to as "best rational approximations of the second kind", and gives the same numbers as the convergents of α, those rational numbers obtained by truncating the continued fraction of α after finitely many terms. We will need a small result connecting the best rational approximations of α and those of its reciprocal.

Lemma 2. *Let $\alpha \in \mathbb{R}^\times$ and suppose the best rational approximations to α are p_i/q_i, $i \in \mathbb{N}$. Then the best rational approximations to $1/\alpha$ are q_i/p_i, $i \in \mathbb{N}$.*

Proof. We assume without loss of generality that $\alpha > 1$; if $\alpha = 1$ then the proof would be shorter than this sentence is. Suppose that in simple continued fraction form the number α is equal to

$$\alpha = [a_0; a_1, a_2, a_3, \ldots].$$

Then $p_i/q_i = [a_0; a_1, \ldots, a_i]$. Meanwhile, the continued fraction for $1/\alpha$ is

$$\frac{1}{\alpha} = [0; a_0, a_1, a_2, \ldots].$$

And so if r_i/s_i are the best rational approximations to $1/\alpha$ then

$$\begin{aligned}\frac{r_i}{s_i} &= [0; a_0, \ldots, a_i] \\ &= \frac{1}{[a_0; a_1, \ldots, a_i]} \\ &= \frac{q_i}{p_i}.\end{aligned}$$

And since best rational approximations are necessarily in lowest terms, we must have $r_i = q_i$ and $s_i = p_i$. □

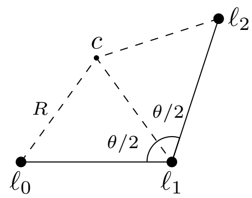
We can now state our theorem.

Theorem 3. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\theta = \pi(1 - 2/\alpha)$. Let ℓ_0 be a point on a piece of paper, and draw a line of length L to reach a point ℓ_1. Given a line from ℓ_i to ℓ_{i+1}, ℓ_{i+2} is formed by drawing a line of length L from ℓ_{i+1} at an angle of θ to the line from ℓ_i to ℓ_{i+1}, always with the same orientation. Form a sequence $(i_k)_{k\in\mathbb{N}^+}$ where i_k is such that the distance from ℓ_0 to ℓ_{i_k} is smaller than the distance from ℓ_0 to ℓ_i for any $i < i_k$. Then the numbers $i_1 < i_2 < i_3 < \ldots$ are the numerators in the best rational approximations to α.*

2. An elementary proof

Since the above theorem is elementary by nature we strive here to give an elementary proof. A snappier proof will be given afterwards that sacrifices geometric charm for brevity.

Clearly the length L mentioned in the statement of the theorem is irrelevant: the whole construction scales linearly with L so we will always obtain the same sequence $i_1, i_2, \ldots$ regardless of our choice of L. So we may as well take $L = 1$. As a preliminary result we need the circumradius of the shape we are constructing, that is the radius of the circle that circumscribes our "α-gon". Drawing the first two lines of the above construction gives something like the picture below.

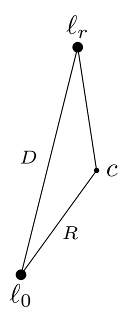


The picture implicitly suggests that $\theta > \pi/2$, but the geometry works in the acute case too. We want to calculate R. First we note that the triangle $\triangle \ell_0\ell_1 c$ is isosceles, and so the angle $\angle \ell_0 c \ell_1$ is just $\pi - 2\theta/2 = 2\pi/\alpha$. So, by the sine rule,

$$R = \frac{\sin(\theta/2)}{\sin(2\pi/\alpha)}.$$

Substituting $\pi(1-2/\alpha)$ for θ and simplifying we deduce that $R = \operatorname{cosec}(\pi/\alpha)/2$.

Now we ask the question: how far are we from ℓ_0 after r lines? We now have the following situation:



The angle $\angle \ell_0 c \ell_r$ increases by $2\pi/\alpha$ with each extra line, so the angle $\angle \ell_0 c \ell_r$ is either $2\pi r/\alpha$ or $2\pi - 2\pi r/\alpha$, but since we are dealing with sines and cosines a modulus sign at an opportune moment will cancel out the difference between these two possibilities. The triangle $\triangle \ell_0 c \ell_r$ is again isosceles so the two equal angles are each

$$\angle c\ell_r\ell_0 = \frac{1}{2}\left(\pi - \frac{2\pi r}{\alpha}\right) = \frac{\pi}{2} - \frac{\pi r}{\alpha}.$$

The quantity we are interested in is D. Using the sine rule again we have

$$\begin{aligned} D &= \left| \frac{R\sin\left(\frac{2\pi r}{\alpha}\right)}{\sin\left(\frac{\pi}{2} - \frac{\pi r}{\alpha}\right)} \right| \\ &= \left| \frac{2\sin\left(\frac{\pi r}{\alpha}\right)\cos\left(\frac{\pi r}{\alpha}\right)}{2\cos\left(\frac{\pi r}{\alpha}\right)\sin\left(\frac{\pi}{\alpha}\right)} \right| \\ &= \left| \frac{\sin\left(\frac{\pi r}{\alpha}\right)}{\sin\left(\frac{\pi}{\alpha}\right)} \right|. \end{aligned}$$

Now we have a formula for how far the end of each line is from the starting point, so we just need to find the successive minima for this sequence of distances, that is the values r for which this distance is smaller than for any previous value. We have the following result.

Lemma 4. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $(D(r))_{r\in\mathbb{N}^+}$ be the sequence given by*

$$D(r) = \left| \frac{\sin\left(\frac{\pi r}{\alpha}\right)}{\sin\left(\frac{\pi}{\alpha}\right)} \right|.$$

Then for all $r > 1$, $\bigwedge_{i=1}^{r-1} D(r) < D(i)$ if and only if there is an integer s such that r/s is a best rational approximation to α.

Proof. First we should note that the term $|\sin(\pi/\alpha)|$ has no effect on the sequence of successive minima we get. If we wish to compare $D(r_1)$ and $D(r_2)$ then, since both contain the same denominator, we are just comparing their numerators. We have thus reduced the problem to showing that the sequence $(|\sin(\pi r/\alpha)|)_{r\in\mathbb{N}^+}$ takes its successive minima at the numerators of the best rational approximations to α. Bearing in mind lemma 2, this is the same as saying that the successive minima of the sequence occur at the denominators of the best rational approximations to $1/\alpha$. And so if we define

$$\delta(r) := \left|\sin\left(\frac{\pi r}{\alpha}\right)\right|$$

for $r \in \mathbb{N}^+$ then we claim that for a given r we have that $\delta(r) < \delta(R)$ for every $R < r$ if and only if there is a $p \in \mathbb{N}$ such that for every $R < r$ and every $P \in \mathbb{N}$,

$$|r\alpha^{-1} - p| < |R\alpha^{-1} - P|.$$

We note that in this case we must have $|r\alpha^{-1} - p| < 1/2$, for else we could take $R = 1$ and P being the nearest integer to $1/\alpha$ to get the right hand side smaller than $1/2$.

We first prove that the best rational approximations minimize the sequence. So suppose p/r is a best rational approximation to $1/\alpha$, but suppose that there is some $R < r$ such that $\delta(R) \leq \delta(r)$. Since $|\sin(x)|$ is periodic with period π and an even function we have

$$\delta(r) = \left|\sin\left(\pi\left|\frac{r}{\alpha} - p\right|\right)\right|$$

and, for any P we choose,

$$\delta(R) = \left|\sin\left(\pi\left|\frac{R}{\alpha} - P\right|\right)\right|.$$

As mentioned above, we must have $|r\alpha^{-1} - p| < 1/2$, and we choose P so that $|R\alpha^{-1} - P| \leq 1/2$. Hence

$$\pi\left|\frac{r}{\alpha} - p\right|, \pi\left|\frac{R}{\alpha} - P\right| \in (0, \pi/2].$$

But $|\sin(x)| = \sin(x)$ is monotonically increasing on this interval, and so if $\delta(R) \leq \delta(r)$ then $|R\alpha^{-1} - P| \leq |r\alpha^{-1} - p|$. That is, p/r would not be a best rational approximation to $1/\alpha$, a contradiction.

The other implication is proved in much the same way. Suppose that $\delta(r) < \delta(R)$ for every $R < r$ but that there is some P such that for every $p \in \mathbb{N}$,

$$\left|\frac{R}{\alpha} - P\right| \leq \left|\frac{r}{\alpha} - p\right|.$$

Again, we must have the left hand side being at most $1/2$, and we can pick p so that the right hand side also falls in this interval. But then we should have

$$\sin\left(\pi\left|\frac{R}{\alpha}-P\right|\right) \leq \sin\left(\pi\left|\frac{r}{\alpha}-p\right|\right),$$

contradicting the fact that $\delta(r) < \delta(R)$. Hence for every $R < r$ and any P we must have $|r\alpha^{-1} - p| < |R\alpha^{-1} - P|$, so that p/r is a best rational approximation to $1/\alpha$. $\square$

3. An alternative proof

The above proof is a nicely geometric justification of a fact that seems to lie somewhere between geometry and number theory. A shorter proof can be found by looking in the complex plane where the problem is neater to state. Again, we note that we may as well set the length L of our lines to be 1. We can thus consider our first line as going from 0 to 1 in $\mathbb{C}$. The next line should make an angle of θ with the previous one, which it is easy to check means that as a complex number it should have argument $\pi - \theta = 2\pi/\alpha$. So after two lines we should be at $1 + \exp(2\pi i/\alpha)$. And so we continue, adding $\pi - \theta$ to the argument of our exponential and adding this to the previous total. So after r lines we are at

$$\sum_{k=0}^{r-1} \exp\left(\frac{2\pi i k}{\alpha}\right).$$

From here we again define a function $d(\alpha, r)$ which tells us how far we are from the starting point, i.e. from zero, after drawing r lines. This is simply the absolute value of the above series,

$$d(\alpha, r) = \left|\sum_{k=0}^{r-1} \exp\left(\frac{2\pi i k}{\alpha}\right)\right|.$$

And now our theorem is a little more obfuscated.

Theorem 5. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Set $\lambda_0 = 1$ and let $r_1 = \min\{r \in \mathbb{N}^+ : d(\alpha, r) < \lambda_0\}$ and $\lambda_1 = d(\alpha, r_1)$. Recursively define $r_n = \min\{r \in \mathbb{N}^+ : d(\alpha, r) < \lambda_{n-1}\}$ and $\lambda_n = d(\alpha, r_n)$. Then $r_1, r_2, r_3, \ldots$ are the numerators in the best rational approximations to α.*

Proof. The first thing to do is to simplify the function $d(\alpha, r)$. The series it deals with is just a geometric one, so we can immediately arrive at

$$d(\alpha, r) = \left|\frac{\exp\left(\frac{2\pi i r}{\alpha}\right) - 1}{\exp\left(\frac{2\pi i}{\alpha}\right) - 1}\right|.$$

Replacing the exponentials with trigonometric functions and using the double angle formula for cosine this drastically simplifies to

$$d(\alpha, r) = \left|\frac{\sin\left(\frac{\pi r}{\alpha}\right)}{\sin\left(\frac{\pi}{\alpha}\right)}\right|.$$

Unsurprisingly, then, we have the same distance-from-the-starting-point function. The proof from here is identical to that given in the previous section. $\square$

The theorems above specify irrational α, but in fact nothing in the proof uses the fact that α is irrational, and the claims hold true for most rational α too. The only problematic cases are when $\alpha = 0$ for the obvious reason that $\theta = \pi(1 - 2/\alpha)$ is undefined when $\alpha = 0$, and the other problems are the Egyptian fractions, those numbers of the form $1/n$ for some integer n. The problem with these is that the best rational approximations to these numbers are just the numbers themselves, and so in particular the only numerator in their best rational approximations is 1. In some sense the algorithm still works, since a "$(1/n)$-gon" will just be a straight line forever moving away from the starting point, and so the closest we ever get is after one line. But this isn't exactly in the spirit of the algorithm, and so we discount that possibility too.