PROOF THAT $\sqrt{2}$ IS IRRATIONAL #768

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Dedicated to Professor Trevor D. Wooley on the occasion of his nth birthday

Abstract. Using the same methods as a new proof that $\zeta(3)$ is irrational we show that $\sqrt{2}$ is irrational. Again.

1. Prolegomenous lemmata

Theorem 1 (Brun's irrationality criterion, [Bru10])**.** *Let (x_n) and (y_n) be strictly increasing sequences of natural numbers such that (x_n/y_n) is a strictly increasing sequence and tends to some limit L. If the sequence (δ_n) given by*

$$\delta_n = \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$$

is strictly decreasing then L is irrational.

Let

$$x_n = \sum_{k=0}^{\infty} \binom{2n-1}{2k} 2^k,$$

and

$$y_n = \sum_{k=0}^{\infty} \binom{2n-1}{2k+1} 2^k.$$

Lemma 2. *For all $n \geq 1$,*

$$x_{n+1} > x_n, \quad y_{n+1} > y_n.$$

Proof. Exercise. □

Lemma 3. *Both sequences (x_n) and (y_n) satisfy the recurrence*

$$u_{n+2} = 6u_{n+1} - u_n$$

for $n \geq 1$.

Proof. First define

$$\lambda_{n,k} = \binom{2n-1}{2k} 2^k,$$

so that

$$x_n = \sum_{k=0}^{n} \lambda_{n,k}.$$

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Now let

$$L_{n,k} = \frac{2(2k^2 - k - 2 + 6n - 4n^2)}{(n-1)(2n-1)}\lambda_{n,k}.$$

It can be verified that

$$\lambda_{n+1,k} - 6\lambda_{n,k} + \lambda_{n-1,k} = L_{n,k} - L_{n,k-1}.$$

Summing over k we thus have

$$\sum_{k=0}^{n+1}(\lambda_{n+1,k} - 6\lambda_{n,k} + \lambda_{n-1,k}) = L_{n,n+1} - L_{n,-1} = 0.$$

Hence $x_{n+1} - 6x_n + x_{n-1} = 0$ as required.

Exercise. Let

$$\mu_{n,k} = \binom{2n-1}{2k+1}2^k$$

and pull out of your hat a function $M_{n,k}$ with the properties that $M_{n,n+1} = M_{n,-1} = 0$ and

$$\mu_{n+1,k} - 6\mu_{n,k} + \mu_{n-1,k} = M_{n,k} - M_{n,k-1},$$

and thus complete the proof. □

Lemma 4. *For all $n \geq 1$,*

$$x_{n+1}y_n - x_n y_{n+1} = 2.$$

Proof. By the previous lemma we have the identities

$$x_{n+1} - 6x_n + x_{n-1} = 0$$
$$y_{n+1} - 6y_n + y_{n-1} = 0.$$

Multiplying the first by y_n, the second by x_n, and then taking the difference we deduce

$$x_{n+1}y_n - x_n y_{n+1} = x_n y_{n-1} - x_{n-1}y_n.$$

And so, by induction,

$$x_{n+1}y_n - x_n y_{n+1} = x_2 y_1 - x_1 y_2 = 7 - 5 = 2.$$

□

Corollary 5. *For all $n \geq 1$,*

$$\frac{x_{n+1}}{y_{n+1}} > \frac{x_n}{y_n}.$$

Lemma 6.

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \sqrt{2}.$$

Proof. We prove by staggered inductions the claims:

$$P(n) : 2y_n^2 - x_n^2 = 1$$

and

$$Q(n) : 2y_{n+1}y_n - x_{n+1}x_n = 3.$$

The basis cases $P(1), P(2), Q(1)$ are easily checked numerically. We have the following implications,

$$P(n) \wedge P(n-1) \wedge Q(n-1) \Rightarrow P(n+1)$$
$$P(n) \wedge Q(n-1) \Rightarrow Q(n).$$

And so the above cases together with the upcoming inductive steps will prove the claims for all n.

First assume we know $P(n-1), P(n)$, and $Q(n-1)$. By lemma 3 we know

$$2y_{n+1}^2 - x_{n+1}^2 = 2(6y_n - y_{n-1})^2 - (6x_n - x_{n-1})^2.$$

Multiplying this out gives us

$$2y_{n+1}^2 - x_{n+1}^2 = 36(2y_n^2 - x_n^2) + (2y_{n-1}^2 - x_{n-1}^2) - 12(2y_n y_{n-1} - x_n x_{n-1}).$$

Whence, by the inductive hypotheses,

$$2y_{n+1}^2 - x_{n+1}^2 = 36 + 1 - 36 = 1.$$

Now suppose we know that $P(n)$ and $Q(n-1)$ are true. To prove $Q(n)$ we take the two recurrence relations

$$x_{n+1} - 6x_n + x_{n-1} = 0$$
$$y_{n+1} - 6y_n + y_{n-1} = 0$$

and multiply the first by x_n, the second by y_n, then take the difference. This gives us

$$2y_{n+1}y_n - x_{n+1}x_n = 6(2y_n^2 - x_n^2) - (2y_n y_{n-1} - x_n x_{n-1})$$

and by the inductive hypotheses this is

$$2y_{n+1}y_n - x_{n+1}x_n = 6 - 3 = 3.$$

To prove the lemma we now divide the identity $P(n)$ by y_n^2 to get

$$\frac{x_n^2}{y_n^2} = 2 - \frac{1}{y_n^2}.$$

But $y_n \to \infty$, so we have the result. □

2. Finding a subsequence

Let

$$\delta(m, n) := \frac{x_n - x_m}{y_n - y_m}$$

for $1 \leq m < n$.

Think of δ as a function from the following array to $\mathbb{Q}$:

1,2	1,3	1,4	1,5	1,6	…
	2,3	2,4	2,5	2,6	…
		3,4	3,5	3,6	…
			4,5	4,6	…
					⋱

We want an increasing sequence n_k such that $\delta(n_k, n_{k+1}) > \delta(n_{k+1}, n_{k+2})$. Do this in two steps;
Step 1 $\lim_{n\to\infty} \delta(m, n) = \sqrt{2}$, so after applying δ to the table each row tends to $\sqrt{2}$.
Step 2 For each m, eventually $\delta(m, n) > \sqrt{2}$.

Together these imply a way of stepping down the table finding the requisite sequence (n_k).

Lemma 7. *For any $m \in \mathbb{N}^+$,*

$$\lim_{n\to\infty} \delta(m, n) = \sqrt{2}.$$

Proof. Exercise. □

Lemma 8. *For all $m \in \mathbb{N}^+$ there is some $N_m \in \mathbb{N}^+$ such that if $n > N_m$ then $\delta(m, n) > \sqrt{2}$.*

Proof. After some rearrangement the lemma is equivalent to showing that for any m there is N_m such that for $n \geq N_m$ we have

$$0 < y_n\sqrt{2} - x_n < y_m\sqrt{2} - x_m.$$

In particular, if the sequence $y_n\sqrt{2} - x_n \to 0$ then we will have the result.

During the proof of lemma 6 we proved the identity

$$2y_n^2 - x_n^2 = 1.$$

We can factor the left hand side and divide by one of the factors to uncover the identity

$$\sqrt{2}y_n - x_n = \frac{1}{\sqrt{2}y_n + x_n} \to 0.$$

And so we're done. □

Theorem 9. *$\sqrt{2}$ is irrational.*

Proof. Start by picking values $n_1 < n_2$ such that $\delta(n_1, n_2) > \sqrt{2}$. The values $n_1 = 1$, $n_2 = 2$ will suffice since then $\delta(n_1, n_2) = 3/2 > \sqrt{2}$. We now need to find $n_3 > n_2$ such that $\delta(n_2, n_3) < \delta(n_1, n_2)$. By lemma 7 we know

$$\lim_{n\to\infty} \delta(n_2, n) = \sqrt{2}$$

and by lemma 8, for all sufficiently large n we also have

$$\delta(n_2, n) > \sqrt{2}.$$

These two facts mean that for all sufficiently large n we will have

$$\sqrt{2} < \delta(n_2, n) < \delta(n_1, n_2),$$

and so we can take one of these values of n as n_3 and then repeat the whole process to find n_4 and so on. □

References

[Bru10] V. Brun, 'Ein Satz über Irrationalität,' *Archiv for Mathematik og Naturvidenskab (Kristiania)*, **31**, (1910), p. 6.