Review Handout: some facts from M32000 Set Theory

(The following is a somewhat rough-and-ready summary of some definitions and facts shown in the level 3 course. It should be treated as a ready-reference: see those course notes for a proper exposition.)

Sect. 1

Theorem 1 (Russell) $\neg\exists z\forall x(x\in z)$. *That is, there is no universal set containing all sets.*

Proof: If z were a set so that $\forall x(x\in z)$ then by Ax.of Subsets, $r=\{x\in z\,|\,x\notin x\}$ is a set. However then $r=\{x\,|\,x\notin x\}$. But $r\notin r\Longleftrightarrow r\in r$! Contradiction! QED

Corollary 2 $R=_{\rm df}\{x\,|\,z\notin x\}$ is a proper class, *i.e.* is not a set.

Some basic theory

Definition 1 Ordered Pair $\langle x,y\rangle=_{\rm df}\{\{x\},\{x,y\}\}$.

Lemma 2 $\langle x,y\rangle=\langle u,v\rangle\longleftrightarrow x=u\wedge y=v$.

Definition 2 Ordered n-tuples, Cartesian products.

$\langle x_1\rangle=_{\rm df}x_1$; $\langle x_1,x_2\rangle$ is already defined; $\langle x_1,...x_n\rangle=_{\rm df}\langle\langle x_1,...,x_{n-1}\rangle,x_n\rangle$.

If $Y_1,...Y_n$ are classes, then $Y_1\times...\times Y_n=_{\rm df}\{\langle x_1,...x_n\rangle|x_i\in Y_i\,1\leq i\leq n\}$.

Definition 3 *An* $n-$ary relation *on* X is a class $R\subseteq X^n=_{\rm df}X\times...\times X$ (n *times*). *This is generalised to relations on* $X\times Y$ etc.

Definition 4 *If* R *is an* $n+1$*-relation* then

$\mathrm{dom}(R)=\{\langle x_1,...x_n\rangle\,|\,\exists x_{n+1}\,\langle x_1,...,x_{n+1}\rangle\in R\}$

$\mathrm{ran}(R)=_{\rm df}\{x_{n+1}|\exists x_1,...x_n(\langle x_1,...,x_{n+1}\rangle\in R\}$.

Definition 5 $R\restriction Z$ *is the restriction of* R *to* Z; $R``Z=\mathrm{ran}(R\restriction Z)$.

Definition 6 *For any function* f *we define* $f\colon X\longrightarrow Y$, f *is (1-1),* f *is onto as usual.*

Definition 7 *A strict total ordering (STO) is a relation* R *with a set* A *so that:*

(i) R *is transitive on* A*: i.e* ;$\forall x,y,z\in A(xRy\wedge yRx\longrightarrow xRz)$

(ii) R *is irreflexive; (iii) Trichotomy:* $\forall x,y,z\in A(xRy\vee yRx\vee x=y)$.

NB We do not assume $R\subseteq A\times A$. so that if $\langle A,R\rangle$ is a strict total order, and $B\subseteq A$, then so is $\langle B,R\rangle$.

Definition 8 *Let* $x\in A$*; then* $A_x=_{\rm df}\{y\in A\,|\,yRx\}$ (the R-initial segment determined by x.)

Definition 9 *If* A,B *are sets,* R,S relations*, we define the notion of isomorphism* $f\colon\langle A,R\rangle\cong\langle B,S\rangle$ *as that of* f *being a relation preserving bijection* $f(x)Sf(y)\Longleftrightarrow xRy$.

The crucial definition here is that of:

Definition 10 $\prec$ wellorders A *iff* $\langle A,R\rangle$ is a STO and

$$\forall x(x\subseteq A\wedge x\neq\varnothing\longrightarrow\exists y\in x(\forall z\in x(z\neq y\longrightarrow y\prec z).$$

We establish a number of basic lemmas about wellorders.

Lemma 4 *Let* $\langle A,\prec\rangle$ *be a wellorder. (We write "WO" for the class of wellorders.) Then*

$$\forall x \in A\ (\langle A, \prec\rangle \not\cong \langle A_x, \prec\rangle).$$

Lemma 5 *If* $\langle A, \prec_0\rangle, \langle B, \prec_1\rangle \in \mathrm{WO}$, *and* $f\colon \langle A, \prec_0\rangle \cong \langle B, \prec_1\rangle$ *then the isomorphism* f *between them is unique.*

Corollary 6 *If* $\langle A, \prec_0\rangle, \langle B, \prec_1\rangle \in \mathrm{WO}$, *and* $f\colon \langle A_x, \prec_0\rangle \cong \langle B_y, \prec_1\rangle$ *and* $g\colon \langle A, \prec_0\rangle \cong \langle B, \preceq_1\rangle$ *then* $g \upharpoonright A_x = f$.

Theorem 7 (Classification Theorem for WO's) *If* $\langle A, R\rangle, \langle B, S\rangle \in \mathrm{WO}$ *then exactly one of the following hold:*

(i) $\exists y \in B\ \langle A, R\rangle \cong \langle B_y, S\rangle$ *(ii)* $\exists x \in A\ \langle A_x, R\rangle \cong \langle B, S\rangle$; *(iii)* $\langle A, R\rangle \cong \langle B, S\rangle$.

Sect. 2 Ordinals

Definition 1 x *is transitive* ($\mathrm{Trans}(x)$) *if* $\forall z \in x (z \subseteq x)$.

Definition 2 x *is an* ordinal ("$x \in \mathrm{On}$") *if* $\mathrm{Trans}(x) \wedge \in\upharpoonright x \times x$ *wellorders* x.

Examples: $\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} \ldots \in \mathrm{On}$; $\{\{\varnothing\}\}, \{\varnothing, \{\{\varnothing\}\}\} \notin \mathrm{On}$.

The following gathers together some basic facts about ordinals:

Lemma 1 (1) $x \in \mathrm{On} \wedge y \in x \longrightarrow y \in \mathrm{On} \wedge y = \mathrm{pred}(x, y, \in) = x_y = \{z | z < y \wedge z \in x\}$;

(2) $x, y, \in \mathrm{On} \wedge x \cong y \longrightarrow x = y$;

(3) $x, y \in \mathrm{On} \rightarrow$ exactly on of (i)$x \in y$ (ii)$y \in x$ (iii) $x = y$ is true;

(4) $x \in y \in z \wedge x, y, z \in \mathrm{On} \rightarrow x \in z$;

(5) If C is a non-empty class (possibly proper) of ordinals then $\exists x \in C \forall y \in C\, [x \in y \vee x = y]$.

This implies:

Theorem 2 (Burali-Forti) $\neg \exists z\, (z = \mathrm{On})$.

Lemma 3 If A is a transitive set of ordinals (i.e. $A \subseteq \mathrm{On}$ and $\forall x \in A \forall y \in x (y \in A)$) then $A \in \mathrm{On}$.

The following theorem shows that we may use ordinals as canonical representatives of each equivalence class of isomorphic wellorderings.

Theorem 4 If $\langle A, \prec\rangle \in \mathrm{WO}$ then $\exists! x \in \mathrm{On}\, (\langle A, \prec\rangle \cong \langle x, \in\rangle)$.

By the above we may:

Definition 3 If $\langle A, \prec\rangle \in \mathrm{WO}$ then let $\mathrm{ot}(\langle A, \prec\rangle) =_{\mathrm{df}}$ that unique ordinal x from the last theorem with $\langle A, \prec\rangle \cong \langle x, \in\rangle$.

Nore: we usually use the greek alphabet for ordinals and we also write "$\alpha < \beta$" for $\alpha \in \beta$ (and similarly $\alpha \leq \beta$ for $(a \in \beta \vee \alpha = \beta)$ using the ordering symbol for the $\in$-ordering.

Definition 4 If x is a set of ordinals a) $\sup(x) =_{\mathrm{df}}$ least $\alpha \in \mathrm{On}\, \forall \beta \in x\, (\beta \leq \alpha)$;
b) $x \neq \varnothing \rightarrow \min(x) =_{\mathrm{df}} \bigcap x$.

Lemma 5 If x is a set of ordinals: $\sup(x) = \bigcup x$ (=$\max(x)$ if the latter exists).

Definition 5 $S(x) =_{\mathrm{df}} x \cup \{x\}$; "$\beta$ is a successor ordinal ($\mathrm{Succ}(\beta)$) $\Longleftrightarrow_{\mathrm{df}} \exists \gamma \in \mathrm{On}(\beta = S(\gamma))$;

$\underline{0} =_{\mathrm{df}} \varnothing$; $\underline{1} =_{\mathrm{df}} S(\underline{0})$; $\underline{2} =_{\mathrm{df}} S(\underline{1})$; in general $\underline{n+1} = S(\underline{n})$.

(These are the *von Neumann* naturals.)

Note that $\underline{n+1} = \{\underline{0}, \underline{1}, \underline{2}, \ldots, \underline{n}\}$.We may then define $\omega =_{\text{df}} \{\alpha \in \text{On} \mid \forall \beta \leq \alpha\, (\beta = 0 \vee \text{Succ}(\beta))\}$.

(As a set ω is the set of natural numbers $\mathbb{N}$.) ω is the first "limit ordinal" where we write

$$\text{“}\alpha \text{ is a limit ordinal” } (\text{Lim}(\alpha)) \Longleftrightarrow_{\text{df}} \alpha \neq 0 \wedge \neg\text{Succ}(\alpha).$$

• Note, the existence of ω requires the Ax.Infinity.

Sect. 3 Transfinite Induction and Recursion

We have already seen:

Theorem 1 (Transfinite Induction) If C is a non-empty class of ordinals then C has a least element.

Theorem 2 (Definition by transfinite recursion)

If $G: V \to V$ is a class function (so defined by some formula: $G(u) = v \longleftrightarrow \varphi_G(u, v, p)$ for some set parameter p), then there is a formula φ_F defining a unique function $F: V \to V$ satisfying

$(*)\ \forall \alpha\, F(\alpha) = G(F \restriction \alpha)$.

Proof: *Uniqueness*: Suppose two functions F_1, F_2 satisfied $(*)$, if they differed then by transfinite induction thee would be a least ordinal γ with $F_1(\gamma) \neq F_2(\gamma)$. However this is absurd as then $F_1(\gamma) = G(F_1 \restriction \gamma) = G(F_2 \restriction \gamma) = F_2(\gamma)$!

Existence: let us say that f is a δ-*approximation* if f is a function, $\text{dom}(f) = \delta$ and $\forall \gamma < \delta (f(\gamma) = G(f \restriction \gamma)$. Again by transfinite induction as in the uniqueness proof, if f, f' are δ- and δ'-approximations respectively, then f, f' must agree about their common domain of $\delta \cap \delta'$. We claim that for every δ there is a δ-approximation. If not then by Transfinite Induction there is a least δ_0 for which there is no δ_0-approximation. δ_0 is easily seen not to be zero, nor a successor, by appealing to $(*)$. So assume δ_0 is a limit ordinal. Let H be the set of all approximations. H is a set, since, we have already remarked any two such are identical on their common domain, it is linearly ordered by inclusion, *i.e.* if $f, g \in H$, either $f = g$, $f \subseteq g$, or $g \subseteq f$. Hence $\bigcup H$ is a δ_0-approximation. A contradiction.

We now define $F(\alpha)$ to be that unique value given to α by any δ-approximation with $\delta > \alpha$.
QED

There are many variations on definition by transfinite recursion. Defining ordinal functions often uses an appeal to:

Scheme of Tranfinite Ordinal recursion.

Let $G: \text{On} \times V \to V$, $H: \text{On} \times V \to V$, $x_0 \in V$ be given, with the former defined by formulae φ_G, φ_H respectively. Then there is a formula φ_F defining a function $F: \text{On} \to V$ where F satisfies:

$F(0) = x_0;$

$F(\alpha + 1) = G(\alpha, F(\alpha));$

$\text{Lim}(\alpha) \to F(\alpha) = H(\alpha, F \restriction \alpha).$

Examples: Ordinal Arithmetic. Fix $\gamma \in \text{On}$. We use the scheme above to define extensions of the arithmetic operations of $+$, $\times$ and exponentiation to the transfinite.

$\gamma+0=\gamma;\quad \gamma+(\alpha+1)=S(\gamma+\alpha);\ \mathrm{Lim}(\alpha)\to\gamma+\alpha=\sup\{\gamma+\xi\,|\,\xi<\alpha\}.$

$\gamma\cdot 0=0;\ \gamma\cdot(\alpha+1)=\gamma\cdot\alpha+\gamma;\ \mathrm{Lim}(\alpha)\to\gamma\cdot\alpha=\sup\{\gamma\cdot\xi\,|\,\xi<\alpha\}.$

$\gamma^0=1;\ \gamma^{\alpha+1}=\gamma^\alpha\cdot\gamma;\ \mathrm{Lim}(\alpha)\to\gamma^\alpha=\sup\{\gamma^\xi\,|\,\xi<\alpha\}.$

$\in$-Recursion

What we did for transfinite recursion "along the ordinals" can in fact be done along any *wellfounded relation*, which we define as a strict order where the trichotomy, or linearity condition is dropped from Def. 7 of 1.3, but where again if $\langle A,\ R\rangle$ is such a *partially* ordered set, then we require that for any $X\subseteq A$, if $X\neq\varnothing$, then there is an $x\in X$, which is *R-minimal*. We do not insist that x is R-least, only that for any $y\in A$, with yRx, that $y\notin X$. In particular we may do this for the set membership relation $\in$ itself, provided that it is wellfounded on sets.

The *Axiom of Foundation* states precisely this wellfoundedness requirement. If A is any set then $\in\restriction A\times A$ is wellfounded: given any non-empty $X\subseteq A$ there is $x\in X$ with x $\in$-minimal in X. (Equivalently we may say $x\cap X=\varnothing$.)

• Note. The AxFoundation is not a necessary axiom for doing mathematics; however it can be shown that assuming it is harmless for mathematical, or set theoretical practice. Notice that with a little bit of use of the axiom of choice, AxFoundation rules out the

• The Ax.Foundation rules out the existence of any infinite $\in$-descending chains: $\cdots x_{n+1}\in x_n\cdots\in x_1\in x_0$ (and hence for no set do we have $x\in x$). (*Exercise:* Assuming the Axiom of Choice, this is an equivalence: if there are no infinite $\in$-descending chains then Ax.Foundation holds.)

• Note that with the AxFoundation we can simplfy the definition of ordinal: it is simply a transitive set *linearly* ordered by $\in$ - since the AxFoundation guarantees for us that it is then *well*ordered. We assume this axiom in the next two theorems.

Theorem 3 (Principle, or Scheme, of $\in$-induction) *For any formula φ:*

$$\forall x\,[\,(\forall y\in x)\,\varphi(y)\to\varphi(x)]\ \to\ \forall x\varphi(x).$$

Theorem 4 (Transfinite Recursion along $\in$)

If $G\colon V\to V$ is defined by some formula $\varphi_G(u,v,p)$ then there is a unique function $F\colon V\to V$ (defined by some formula $\varphi_F(u,v,p)$) satisfying:

$$\forall x\ F(x)=G(F\restriction x).$$

Note that (i) in the above we can think of x as "pred$(V,x,\in)$" using our earlier notation; (ii) the proof of the above will need instances of Replacement and Comprehension schemes to collect together the approximating functions to F.

There are many variants of the above. The following is proved in the same way:

Theorem 5 (Transfinite Recursion along $\in$, second version)

If $G\colon V\times V\to V$ is defined by some formula $\varphi_G(u_0,u_1,v,p)$ then there is a unique function $F\colon V\to V$ (defined by some formula $\varphi_F(u,v,p)$) satisfying:

$$\forall x\ F(x)=G(x,F\restriction x).$$

The same argument works for any wellfounded partial order. A relation R is *set-like on A*, if for every $x\in A$, $\mathrm{pred}_{\langle A,R\rangle}(x)=_{\mathrm{df}}\{y|\ y\in A\wedge yRx\}$ is a set.

Theorem 6 (Generalized Transfinite Recursion Theorem)

Suppose $\langle A, R\rangle$ is a wellfounded relation, with R set-like on A. If $G: V \times V \to V$ is defined by some formula $\varphi_G(u_0, u_1, v, p)$ then there is a unique function $F: A \to V$ (defined by some formula $\varphi_F(u, v, p)$ depending on $\langle A, R\rangle$ and φ_G) satisfying:

$$\forall x\, F(x) = G(x, F \upharpoonright x).$$

Here if $\langle A, R\rangle$ is a wellfounded *partial order* then R is transitive, and this theorem can be proven like the last. Otherwise we have to replace R by R^* - the *transitive closure* of R in A, where for $x, y \in A$ we put

$$x\, R^* y \leftrightarrow_{\mathrm{df}} x\, R\, y \vee \exists n > 0\, \exists z_1 \in A, \ldots, z_n \in A (x\, R\, z_1 R z_2 \cdots R z_n R y).$$

We now define a hierarchy of sets that will include all sets of mathematical discourse.

Definition 1 (The V-hierarchy) *By ordinal recursion we define:*

$$V_0 = \varnothing;\ V_{\alpha+1} = \mathcal{P}(V_\alpha);\ \mathrm{Lim}(\alpha) \to V_\alpha = \bigcup_{\beta<\alpha} V_\beta\ .$$

We then define the wellfounded sets *to be*: $\mathrm{WF} =_{\mathrm{df}} \bigcup_{\alpha \in \mathrm{On}} V_\alpha$.

Is every set x in WF? That is what the Axiom of Foundation asserts.

Exercise: (i) Show that $\forall x (x \in \mathrm{WF} \longleftrightarrow x \subseteq \mathrm{WF})$. (ii) Assume Ax Foundation. Show that $V = \mathrm{WF}$.

Definition 2 (Rank function ρ) For $x \in \mathrm{WF}$ define $\mathrm{rk}(x) = \rho(x) =$ the least α such that $x \in V_{\alpha+1}$.

The following properties of the V_α-hierarchy hold true.

(i) V_α is transitive ;

(ii) $\alpha < \beta \longrightarrow V_\alpha \subseteq V_\beta$;

(iii) $V_\alpha = \{ x \in \mathrm{WF} \mid \rho(x) < \alpha \}$;

(iv) If $x \in \mathrm{WF}$ then $\forall y \in x (y \in \mathrm{WF} \wedge \rho(y) < \rho(x))$;

(v) If $x \in \mathrm{WF}$, then $\rho(x) = \sup \{ \rho(y) + 1 \mid y \in x \}$;

(vi) $\rho(\alpha) = \alpha$;

(vii) $\mathrm{On} \cap V_\alpha = \alpha$;

• By virtue of (ii) the V_α-hierarchy is often called the "cumulative hierarchy".

• (v) gives a way of defining the rank function ρ without using the power set operation - it can be defined by $\in$-recursion. This is sometimes useful in situations where we do not have the power set operation.

• It is easy to calculate the ranks of simple sets composed from objects in WF.

Exercise: Assume $x, y \in \mathrm{WF}$. Compute the ranks of the following sets in terms of $\rho(x), \rho(y)$: $\bigcup x$, $\mathcal{P}(x)$, $\{x\}$, $x \times y$, $x \cup y$, $\{x, y\}$, $\langle x, y\rangle$, ${}^y x$. [Note they are all less than $\max \{\rho(x), \rho(y)\} + \omega$.]

Once we have seen that these objects are all in WF, since $\omega = \mathbb{N} \in \mathrm{WF}$ the usual constructions of the rationals, reals etc shows that

- $\mathbb{Z}$, $\mathbb{Q}, \mathbb{R}, \mathbb{C} \in V_{\omega+3}$.

- (AC) Every group or topological space is isomorphic (homeomorphic) to a group (or top. space) in WF.

The latter is true of any mathematical structure: hence nothing is lost mathematically by assuming AxFound. which we shall do henceforth. Another important remark is that almost all of mathematics takes place within $V_{\omega+\omega}$: it is rare to find mathematical results that require more sets than we find in this initial segment of the V-hierarchy.

Definition 3 (Transitive Closure) *For x a set we define:*

$\bigcup^0 x = x;$ $\quad\bigcup^{k+1} x = \bigcup \{\bigcup^k x\};$ $\quad\mathrm{TC}(x) =_{\mathrm{df}} \bigcup \{\bigcup^k x | k \in \mathbb{N}\}$ or rewritten:

We can define $\mathrm{TC}: V \to V$ by $\in$-recursion (see (v) below):

$$\mathrm{TC}_0(x) =_{\mathrm{df}} x \cup \bigcup \{\mathrm{TC}(y) \,|\, y \in x\}.$$

The transitive closure of a set is the smallest transitive superset of the set (see (ii) and (iii) of the next Lemma). Note that $\{\langle x, y\rangle \,|\, x \in \mathrm{TC}(y)\}$ is a wellfounded relation. (Why?)

Lemma 5 (Lemma on TC) For any set x : (i) $x \subseteq \mathrm{TC}(x)$

(ii) If $\mathrm{Trans}(t) \wedge x \subseteq t \longrightarrow \mathrm{TC}(x) \subseteq t$. Hence $\mathrm{TC}(x)$ is the smallest transitive set t satisfying $x \subseteq t$.

(iii) Hence $\mathrm{Trans}(x) \longleftrightarrow \mathrm{TC}(x) = x$.

(iv) $x \in y \longrightarrow \mathrm{TC}(x) \subseteq \mathrm{TC}(y)$;

(v) $\mathrm{TC}(x) = \mathrm{TC}_0(x) =_{\mathrm{df}} x \cup \bigcup \{\mathrm{TC}(y) \,|\, y \in x\}$ (hence $\mathrm{TC}(\{x\}) = \{x\} \cup \mathrm{TC}(x)$

Sect 4 Cardinals

We do not immediately assume AC for this section.

Definition 1 $x \preccurlyeq y \Longleftrightarrow_{\mathrm{df}} \exists f [f : x \longrightarrow y \wedge f$ is (1-1)].

$x \approx y \Longleftrightarrow_{\mathrm{df}} \exists f [f : x \longrightarrow y \wedge f$ is (1-1) and onto].

$x \prec y \Longleftrightarrow_{\mathrm{df}} x \preccurlyeq y \wedge y \not\preccurlyeq x$.

Theorem 1 (Cantor-Schröder-Bernstein) If $x \preccurlyeq y \wedge y \preccurlyeq x$ then $x \approx y$.

Definition 2 If x can be wellordered then $|x| =_{\mathrm{df}}$the least α such that $x \approx \alpha$.

Of course, assuming AC (in the equivalent form of the Wellordering Principle (WP) -that every set can be wellordered) $|x|$ is always defined. Note that for any α $|\alpha|$ is always defined (without AC), and $|\alpha| \leq \alpha$.

Examples: $|2| = 2, |\omega| = \omega = |\omega + 1| = |\omega + \omega| = |\omega^\omega|$ (with the latter ordinal operations).

Definition 3 *If* $|\alpha| = \alpha$ *then* α *is a* cardinal. We write: $\mathrm{Card}(\alpha)$

We usually reserve the middle alphabet letters for cardinals: $\kappa, \lambda, \mu, \ldots$. If x is a set with $|x| \in \omega$ then we say that x is *finite.* If $x \approx \omega$ ($\Longleftrightarrow |x| = \omega$) then we call x *countably infinite.* It is immediate from the definition of $|x|$ that:

Lemma 2 $|\alpha| \leq \beta < \alpha \longrightarrow |\alpha| = |\beta|$.

Theorem 3 (Cantor) $\forall x\, (x \prec \mathcal{P}(x))$; *hence if AC holds* $\forall x\, (|x| \prec |\mathcal{P}(x)|)$ *and hence for every cardinal number there is a larger one.*

However AC is not needed to show there are arbitrarily large cardinals:

Theorem 4 (Hartogs Theorem) $\forall\alpha\exists\kappa(\kappa>\alpha\wedge\kappa$ a cardinal).

Definition 5 $\alpha^+ =_{\mathrm{df}}$ *the least cardinal greater than* α.

Definition 6 *a) Define* $\aleph_\alpha\,(=\omega_\alpha)$ *by recursion:* (i) $\aleph_0\,(=\omega_0)=\omega$; (ii) $\aleph_{\alpha+1}(=\omega_{\alpha+1})=\aleph_\alpha^+$; $\mathrm{Lim}(\alpha)\rightarrow\aleph_\alpha(=\omega_\alpha)=\sup\{\aleph_\beta\,|\,\beta<\alpha\}$.

b) A cardinal of the form $\aleph_{\alpha+1}$ *is called a* successor *cardinal, and one of the form* $\aleph_\alpha$ *with* $\mathrm{Lim}(\alpha)$ *a* limit *cardinal.*

ω_1 is the first *uncountable* cardinal number, and each of the ω_{n+1} are thus successor cardinals, with ω_ω being the first limit cardinal $>\omega$.

Lemma 5 (AC) *If* $\exists f\,[f\colon X\twoheadrightarrow Y\wedge f$ onto] *then* $|Y|\le|X|$.

Aside: Arguing as in Theorem 4 one can prove that $\exists g\colon\mathcal{P}(\omega)\twoheadrightarrow\omega_1$ (not using AC), but (it can be proven that) one cannot prove in ZF without AC that $\exists h\colon\omega_1\longrightarrow\mathcal{P}(\omega)$ which is (1-1). As is often done, we identify elements of $\mathbb{R}$ with subsets of ω *via* characteristic functions, $\mathbb{R}$ then is identified with $\mathcal{P}(\omega)$. (In particular $\mathbb{R}\approx\mathcal{P}(\omega)$ holds.) We may paraphrase the above remark by saying that without AC we cannot prove the existence of an uncountable sequence of distinct reals numbers.

Lemma 6 (AC) *If* $\kappa\ge\omega\wedge|x_\alpha|\le\kappa$ *for all* $\alpha<\kappa$, *then* $\kappa\ge|\bigcup\{x_\alpha\,|\alpha<\kappa\}|$.

Note: (1) If $\kappa=\omega$ this says that the union of countably many countable sets is countable. However we need some form of the axiom of choice to prove this.

(2) Levy has shown that it is consistent with ZF that both $\mathcal{P}(\omega)$ and ω_1 are countable unions of countable sets.

Definition 7 (Cardinal Arithmetic) $\kappa\oplus\lambda=_{\mathrm{df}}|\kappa\times\{0\}\cup\lambda\times\{1\}|$; $\kappa\otimes\lambda=|\kappa\times\lambda|$.

Note that these operations are commutative and so are completely different from those of ordinal arithmetic, as the following also makes clear.

Theorem 7 Let κ be an infinite cardinal. Then $\kappa=\kappa\otimes\kappa$.

Corollary 8 Let κ,λ be infinite cardinals, then $\kappa\otimes\lambda=\kappa\oplus\lambda=\max\{\kappa,\lambda\}$.

To define cardinal exponentiation we make use of:

Definition 8 (i) *Let* x,y *be sets. Then* ${}^y x=_{\mathrm{df}}\{f\,|\,f\colon y\rightarrow x\}$.

(ii) (AC) $\kappa^\mu=_{\mathrm{df}}|{}^\mu\kappa|$.

By considering characteristic functions of subsets of x it easily follows that

Exercise $|\mathcal{P}(x)|=2^{|x|}$, hence for cardinals $|\mathcal{P}(\kappa)|=2^\kappa$.

Lemma 9 a) If $\lambda\ge\omega\wedge2\le\kappa\le\lambda$ then ${}^\lambda\kappa\approx{}^\lambda2\approx\mathcal{P}(\lambda)$ (hence $\kappa^\lambda=2^\lambda$).

b) $\kappa^{\lambda\oplus\mu}=\kappa^\lambda\otimes\kappa^\mu$; $(\kappa^\lambda)^\mu=\kappa^{\lambda\otimes\mu}$.

The Continuum Hypothesis.

Cantor surmised that for any set of real numbers $X\subseteq\mathbb{R}$ either X was countable (and so $|X|=\aleph_0$) or else $X\approx\mathbb{R}$. In other words there could be no set of intermediate cardinality between that of $\mathbb{N}$ and that of $\mathbb{R}$. If so that would mean that the cardinality of $\mathbb{R}$ had to be that of the least cardinal number greater than $\aleph_0$: in our terms $\aleph_1$ (or ω_1). We shall write this as: The Continuum Hypothesis (CH): $2^{\aleph_0}=\aleph_1$.

The Generalised Continuum Hypothesis (GCH) $\forall\alpha\ 2^{\aleph_\alpha} = \aleph_{\alpha+1}$. The CH and GCH therefore assert that power sets do not grow too quickly: they take the least possible value.

In this course we shall prove Gödel's result that the GCH is consistent with the other axioms of ZF. This will be done by forming a special subclass of the cumulative hierarchy called L, the class of the *constructible sets.* In this hierarchy all the axioms of ZF together with the AC and GCH will hold.

The use of axioms

• Delivers a clear and explicit list of assumptions about the objects under study.

• Allows the use of tools from formal logic (notably the theorems mentioned below). We can speak about "provable in ZF set theory".

Recall that

(i) for any theory T we write $T \vdash \sigma$ if σ is derivable acording to the rules of first order predicate calculus from the set of axioms T. Recall that $T \vdash \sigma \Longleftrightarrow \exists T_0 \subseteq T$ (T_0 is finite and $T_0 \vdash \sigma$). Thus 'Φ is provable in ZF' is written 'ZF $\vdash \Phi$'.

(ii) T *is a consistent theory* iff for no σ do we have $T \vdash \sigma$ and $T \vdash \neg\sigma$ (iff we cannot deduce $T \vdash \sigma \wedge \neg\sigma$.)

Theorem (Completeness Theorem) *Let* $\Gamma \cup \{\sigma\}$ be a set of sentences in a first order language. Then:
- (Soundness Theorem) $\Gamma \vdash \sigma \Rightarrow \Gamma \vDash \sigma$
- (Completeness) $\Gamma \vdash \sigma \Leftarrow \Gamma \vDash \sigma$.

Theorem (AC)((Downward) Löwenheim-Skolem Theorem) *Let* $\mathfrak{A}$ *be a model with domain* $\mathrm{dom}(\mathfrak{A})$. Let $X \subseteq \mathrm{dom}(\mathfrak{A})$. *Then there is an elementary substructure* $\mathfrak{B} \prec \mathfrak{A}$ *with* $X \subseteq \mathrm{dom}(\mathfrak{B}) \wedge |\mathrm{dom}(\mathfrak{B})| = \max\{|X|, \omega\}$.

[Here to say $\mathfrak{B} \prec \mathfrak{A}$ is to say for any formula $\varphi(v_0, ..., v_{n-1})$ of the language approproate for $\mathfrak{A}$ and any $\vec{b} = b_0, ..., b_{n-1}$ from $\mathfrak{B}$ we have $\mathfrak{A} \vDash \varphi(\vec{b}) \leftrightarrow \mathfrak{B} \vDash \varphi(\vec{b})$. If $\mathfrak{B} \prec \mathfrak{A}$ then we say $\mathfrak{B}$ is an *elementary substructure* of $\mathfrak{A}$]

We shall have occasion to speak about Incompleteness results:

Theorem (Gödel's Second Incompleteness Theorem) *Let T be any recursively given theory expressible in a formal language $\mathcal{L}_T$ from which can be proven all the axioms of Peano Arithmetic (PA). Let* Con_T *be a sentence of $\mathcal{L}_T$ which formally expresses (via some reasonable coding) that there is no proof of "$0 = 1$" from T. Then* $T \nvdash \mathrm{Con}_T$, *unless T is inconsistent.*