Revision Topics: Axiomatic Set Theory

1 The ZFC axioms themselves in the form given using *terms* as at 1.2.2.

(2-5 below are essentially revision topics from 3rd Year Set Theory Course and are really *assumed* for this course.)

2 Basic definitions of: ordered pair n-tuple etc, *wellorderings*; the *Classification Theorem* for wellorderings (that any two such are comparable: either of the same *order-type*, or one isomorphic to an initial segment of the other); Zermelo's theorem that any wellorder is isomorphic to an ordinal; *transfinite recursion* along $\langle \mathrm{ON}, < \rangle$ or along $\in$.

3 Definitions and basic properties of ordinal arithmetic.

4 Wellfounded sets, basic definitions and properties of the V_α hierarchy (Def 1.17,1.18,Ex 1.3,1.9). Be able to compute *ranks* of various sets obtained from arbitrary $x, y \in \mathrm{WF}$. Be able to work with at least one definition of *transitive closure* TC say and prove basic facts thereto.

5 Basic theory of cardinals and their arithmetic; (statement of) *Cantor-Schröder-Bernstein Theorem*; statement and proof of *Cantor's Theorem*.

6 Relativisation of terms and formulae to class terms W; *downwards and upwards absoluteness* between class terms $W \subseteq Z$. The absoluteness of, in particular, Δ_0-formulae.

7 Definition of *cofinality*; *regular* and *singular* cardinals; that $\mathrm{cf}(\lambda)$ is always a regular cardinal; that under AC κ^+ is always regular. The *aleph* and *beth* functions. CH and GCH. *Weakly and strongly inaccessible* cardinals. *Normal* (or "*Continuous*") *functions* and their *fixed points.*

8 *C.u.b.* and *stationary* sets below a regular cardinal κ, definitions and examples. The closure of the field of subsets of κ which are c.u.b., under $<\kappa$ many intersections; the closure of a κ-sequence of c.u.b. sets under *diagonal intersection*; *Fodor's lemma*. (**Omit:** Def 2.17, Ex 2.11, Thm. 2.20, Sections 2.2, 2.6.2.)

9 *Mostowski-Shepherdson Collapsing Lemma* on extensional structures $\langle X, \in \rangle$.

10 The *Montague-Levy Reflection Theorem* (Lemma 2.42, Thm 2.43, Cors. 2.44. Cor 2.45 we did not really cover so can be **omitted**, although this argument appeared independently in the GCH proof in L).

11 The class of sets *hereditarily of cardinality* $<\kappa$, H_κ. Know that $(\mathrm{ZF}^-)^{H_\kappa}$ for κ regular, and full $(\mathrm{ZFC})^{H_\kappa} \Longleftrightarrow \kappa$ strongly inaccessible and be able to prove them. **Omit** Subsection 2.6.2.

12 *Definite* terms and formulae, (you will not be asked to give the definitions of these), Be able to show that certain simple terms and expressions are capable of being expressed in a Δ_0 fashion (as in Lemma 3.7) or definite (as in 3.8,3.9). Section 3.5: Understand how the statement of Correctness Theorem is used. In 3.5.1 understand that Thm 3.25 is a version of Godel Incompleteness.

13 *The Construction of* L. You should have an understanding of, and be able to give a descriptive account of how L is built. However you should be able to give a explanations of the Sat definition & the Def function, of $\iota(x, u, h)$ *etc.*, and explain what these mean; absoluteness and properties of Def. (See Ex. 3.4,3.5, Lemmas 3.17,3.18.)

14 *Properties of* L*:* that each axiom of ZFC holds in L; proof that $\mathrm{ZF} \vdash (V = L)^L$; how a *consistency proof* such as *e.g.* $\mathrm{Con(ZF)} \Longrightarrow \mathrm{Con}(\mathrm{ZF} + V = L)$ works. Be able to give a description of the main outline of the proofs of: $(\mathrm{AC})^L$, $(\mathrm{GCH})^L$.

15 In Section 4.5: Know that $x \in \mathrm{OD} \longleftrightarrow x \in \mathrm{OD}^*$ (the latter Def.3.19). Understand 'outright definable' and $\mathrm{Def}_0(\langle x, \in \rangle)$ function. Thm 4.21 gives φ_{OD} as a first order definition within $\mathcal{L}$ of the class OD. Thm 4.22 that the class OD (and hence the class HOD) has a definable wellorder. Def. of HOD and that $(\mathrm{ZFC})^{\mathrm{HOD}}$.

16 **Omit** Sections 4.6, 4.7.

Note: The exam will be 2 and 1/2 hours, all 4 questions to be attempted. 1 (one) A4 page (= 2 sides) of handwritteen notes may taken to the Exam and inserted into the Exam script afterwards. Past papers: see Blackboard (–>Organisations -> Resources for Students –> Examinations –> Past Examinations).