Conceptualism: sets and classes

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Abstract

We outline an extension of Martin's view of a conceptual realism, to a Cantorian realm of absolute infinities. We then formulate a strong reflection principle within this framework to obtain extra-constructible large cardinals.

1 Introduction

This paper aims to address the topic of the foundations of mathematics[1] by considering a question in the foundations of set theory. I shall take it as a starting point that ZFC set theory is *a* foundations for mathematics, even if there are alternative foundational systems with probable benefits in terms of application, such as category theory for algebraic concepts *etc.* In general my view point is that any such system can receive its interpretation in terms of set theory, or with some exceptions variants thereof. However I am ecumenical: if category theory can solve the Continuum problem, then I am a category theorist.

I shall focus on conceptual issues, which I take it we'll agree are relevant to foundations. Firstly, to continue stating my personal viewpoint, is that there is not now, nor has there ever been the number 6 sitting in my fridge, and that $\aleph_{23}$ is not located, or stuck, in some quarter of the universe quarantined off in Benacerraffian isolation along with other mathematical *objects*. What we have are mathematical *concepts*. We have no need to locate the object $\aleph_1$ whatever that might be, in order to formulate the Continuum Problem, nor even have to be able to perceive in some Gödelian fashion the set $\mathbb{R}$ of real numbers in order to do analysis. Gödel's emphasising the role of perception of mathematical objects (*cf* [2] p.128), and the similarity to a kind of quasi-empirical research about 'objects' does not seem to advance us further.

My viewpoint today I think is closest to what Martin has identified in [8] when discussing Gödel's Conceptual Realism. The main tenor of [8] is that much, but not all, of Gödel's realism can be construed as being about concepts rather than objects. (For Gödel both concepts and objects have some real existence in some

[1] A paper given at the *"Foundations of Mathematics: what are thay and what they for?"* conference, Cambridge July 2012

form.) Much of Martin's paper is taken up with discussing various aspects of Gödel's writings on this at various periods, that we shall not repeat here.

2 The concept of 'set of'

To summarise our paper in one sentence, we aim to extend Martin's notion of " 'concept of set' in the indirect sense" [8] to a similarly indirect sense of 'concept of absolute infinity' (which I may abbreviate to 'concept of (proper) class' or perhaps to be more neutral to 'concept of a part (of V)').

Martin distinguishes the two senses of 'concept of set' ([8] p.212):

> My sense differs from the straightforward sense in that instances of a concept of set in the straightforward sense - the objects that fall under the concept - are sets (or, at least, what the concepts are count as sets). The instances of a concept of set in my sense are not sets. There are two versions of my sense. In one version the instances are concepts: straightforward-sense concepts of set. In the other version the instances might be described as set structures or universes of sets.

It is this final 'other version' that I shall want to mostly take here. However first (p.213, *ib.*):

> A concept of set expressed by axioms such as comprehension axioms cannot put any constraint on which objects count as sets and which do not. Such axioms put constraints on the isomorphism type of set theoretic structure ... a concept of set could count as concept of set in my [indirect] sense even if it determined completely what objects count as sets and what counts as the membership relation. A concept of this sort would have at most one instance: it would allow at most one structure to count as a set-theoretic universe ...

What is ultimately at play here is the point Martin wishes to make that *instantiation* of a concept for mathematics (or set theory) is not needed: what we require is *uniqueness* (up to isomorphism) in order to make sense and understand concepts. He reads Gödel as primarily not needing instantiation in many crucial places: for example, he notes that neither it nor perception of objects plays any significant role in Gödel's justifications of strong axioms of infinity,

His primary point is perhaps plainly put ([8] p.215): the example of Axiom of Extensionality: it does not say what a set is, it only prescribes what it means for any two sets to be equal. The concept of set does not determine what it is for an object to be a set (as he states in [9]). The notion is objective: we understand it, talk about it, as no doubt they do on some other planet with discretely individuated intelligences. (It is not for nothing that we engrave on steel plates pictures of Pythagoras' theorem and place them on the moon, or send them out on Voyager 23.)

In short we understand the concept 'set of' without having to *perceive* it in some Gödelian manner. Hence:

• Instantiation is not needed either in mathematics or in set theory; thus
• This is closer to a structuralist viewpoint.

In his paper for the Exploring the Frontiers of the Infinite Series [9] he considers two basic concepts, that of 'natural numbers' (or rather 'ω-sequence'), and 'set'. He identifies three properties a basic concept may have:

(i) First order completeness: the concept determines truth values for all first order statements.

(ii) Full determinateness: the concept fully determines what any instantiation would be like.

(iii) Categoricity.

The concept of natural number yields IPA: Informal Peano Axioms, (not in the usual first order sense) which in turn yields categoricity of $\mathbb{N}$. However categoricity alone does not imply 1st order completeness: but he believes in full determinateness for $\mathbb{N}$ and ([9], p13):

> I believe that full determinateness of the concept is the only legitimate justification for the assertion that the concept is instantiate or that natural numbers exist.

Whilst neither endorsing or denying the last quotation, I'll go along with it for the present purposes. I shall ignore in any case discussion of the natural numbers for this paper. He then applies a similar sequence of considerations for the concept of sets. For him the modern, iterative concept has four important components:

(1) the concept of natural number
(2) concept of 'set of x's'
(3) concept of transfinite iteration
(4) concept of absolute infinity.

He remarks that (1) can be subsumed under (2) and (3). My remark is that (4) is perhaps not on everyone's list of components. He is thinking of the concept of sets as the concept of 'structuralist's structure' and thus does not have to add anything as to what kind of things sets are. We adopt this view here. (Martin remains silent as to which flavour of structuralism's structure might be at play here, and we comment on this at the end of the section.) A set structure is then what is obtained by iterating the concept 'set of x's' absolutely infinitely many times.

We have only glimmerings of what goes on when considering subsets of $V_{\omega+1}$: is the Continuum Hypothesis true? Is every definable subset of the plane definably uniformisable? So we are hopelessly far from first order completeness. However, when considering subsets of V_ω we are, somewhat recently, in a better position. We now know that adding the assumption of Projective Determinacy to analysis, or to the theory of hereditarily countable sets give us as complete a picture of HC as PA does for V_ω = HF. Martin asks:

Question: Which informal axioms are implied by the concept of set?

He lists two (p.14).

(I) If a and b have the same members, then $a = b$.

(II) For any property P, there is a set whose members are those x's that have P.

The first is Extensionality, and the second is an Informal Comprehension scheme: informal since "property" is not specified in generality. However any worries can be dispelled since it will be clear that the few instances we shall use are clear cases of properties.

Martin seeks to further soothe any worries that we need to specify what objects sets are in order to 'fully understand' the concept. He will ignore whatever structural constraints one may put on what sets actually are, other than the structural constraints of (I) and (II), and continues as follows:

Theorem 1 *(Essentially Zermelo) Axioms (I) and (II) are categorical: if* $(\mathfrak{V}_1, \in_1)$ *and* $(\mathfrak{V}_2, \in_2)$ *are two structures satisfying (I) and (II) with the same x's, then with each set* $b \in_1 \mathfrak{V}_1$ *we associate a* $\pi(b) \in_2 \mathfrak{V}_2$.

Proof: Let P be the property of being an x such that $x \in_1 b$. By the Informal Comprehension Scheme there is a $c \in_2 \mathfrak{V}_2$ such that

$$\forall x[x \in_2 c \leftrightarrow P(x)].$$

Q.E.D.

• This is the basis of Zermelo's proof that any two models of ZFC (without *urelemente)* of the same ordinal height are isomorphic.

• The notion (3) of transfinite iteration is just that of ordinals or even wellorderings. Martin points out that this makes one have confidence in the full determinateness of small transfinite ordinals or the associated levels of the L_α-hiearchy, and he further remarks that an Informal Wellfoundedness Axiom would play the role of Informal Comprehension Axiom here.

• Indeed the same argument shows that if the $\alpha \to V_\alpha$ operation is iterated along the absolute infinity of *all* the ordinals, the universes obtained are categorical, and so unique up to isomorphism. As Martin has remarked elsewhere [7] the isomorphism argument following Zermelo works here too.

In short, what is unfolded from the iterative concept of set for Martin is the above fact. We did not need instantiation for the above argument, or indeed to know what *objects* $\{\varnothing\}$, or $\aleph_{23}$ *are*.

Nor do we, I take it to mean, actually need to assert that any *structure* such as $(\mathfrak{V}, \in)$ actually exists. This latter existence need not follow from the concept alone. Burgess in [1] analyses potential kinds of structuralism into three sorts, of which the first two, the 'eliminating objects' or *in re*, and 'natureless objects' or *ante rem*, (he calls them "hard-headed" and "mystical") are the most prevalent. He also identifies a third possible meaning, the 'arbitrary structure' (picked out by a use of the Hilbertian ϵ-symbol). His discussion centers around the idea introduced by Pettigrew [11] and also Shapiro [12] of using "an introduced parameter" as means of referring to mathematical concepts not only such as i or $\sqrt{2}$ but also the "the (algebraic) structure of the natural numbers" or the "real closed field" etc. The difficulties of extending structuralism to set theory to deal with all of V he says are well-known. Of the two (or three) kinds the 'mystical' option seems closest to what one might want (I hesitate to claim anything for Martin here) in that we are *talking about a special model whose distinctive metaproperty is to have no distinctive properties* in Burgess's words. Well, I said 'closest', but perhaps for many set theorists, this does not ring very close. Set theorists are probably either more 'formalist', and think of 'constructing' formally very distinctive models (probably by forcing), or else more 'realist' in attempting to ascertain 'V' 's distinctive features. The latter's use of 'V' (as being the 'set of'/structure concept obtained by iterating power set along the ordinals) sets up 'V' as one of Pettigrew's 'distinctive free variables', albeit not within a strictly mathematical discussion, since we wish to restrict the domain of mathematics to sets, and not to include proper class entities such as V. A set theorist of the latter kind may well say "let 'V' be the universe of sets" and mean the one obtained by iterating power set along the ordinals, just as in the phrase "Let $\mathbb{N}$ be the natural number system": here $\mathbb{N}$ is then an example of one of Pettigrew's dedicated free variable.

However Burgess has other reasons for doubting that this form of structuralism can be deployed in the case of set theory. He continues (his emphasis):

> But if that is how set theory is conceived, then there seems to be no room for the activity, important to many set theorists, of going back to an intuitive notion of set motivating the axioms in order to motivate more axioms to settle questions not settled by the existing axioms. Structuralism here ties set theory to a particular axiom system in a way that seems to *block the road of inquiry.*

The difficulty about there being 'no room' seems to be alleviated if one allows for the fact that we are currently at a stage of enquiry where we have no definite knowledge about this 'special model' (or the equivalence class up to isomorphism of this special model). Structuralists' arguments as applied to the natural number structure or real continuum structure are being applied to concepts that are well-trodden and enjoy virtual unanimity of conception amongst mathematicians. Set Theory does not (yet) have that status. We have an intuitive notion of ordinal, and of informal recursion along On. If we allow ourselves to apply the latter to the power set operation then this gives us our 'set-of' concept, our 'structure'. With that pre-formal perspective, we then formalise the subject and then afterwards our view of V evolves as we discover more about its properties and potential embedding spectra (a.k.a. potentially new axioms of infinity.)

3 Stepping up to other absolute infinities.

To set the record straight Martin states that he is dubious about the notion of absolute infinities (p19, [9]). This is precisely the point where we want step up to and beyond. Yet it would seem that he might accept the following argument concerning mappings between the ordinal classes without difficulty.

Just as the argument that for any two $\mathfrak{V}_1 = (V_1, \in_1)$, $\mathfrak{V}_2 = (V_2, \in_2)$ obtained by iterating the V_α function throughout all the absolute infinity of ordinals, we have an isomorphism $\pi : (V_1, \in_1) \to (V_2, \in_2)$ (Thm 1), then we see that $\pi \restriction \mathrm{On}^{\mathfrak{V}_1} : \mathrm{On}^{\mathfrak{V}_1} \cong$

$\mathrm{On}^{\mathfrak{V}_2}$ where $\mathrm{On}^{\mathfrak{V}_i}$ is the absolute infinity of von Neumann ordinals in the model $\mathfrak{V}_i$.

We want to take a Cantorian view, perhaps even a naive view, about absolute infinities. We recognise the *logical* necessity of such: the Russell, Burali-Forti, Cantor arguments. If we wish to see what follows as a *logical necessity* from the concept of set (1)-(4) then a consequence of this is acknowledging these arguments. Purloining some terminology from mereology, we may view absolute infinities as the *parts of* V, or rather what is left after we have identified the 'set-sized' parts of V with the corresponding set of V. We continue to use the word 'part' or '(class-sized) part' or 'absolute infinity' but these would seem little different from 'proper class'.

We should like to take a view-point that sees the universe V of sets identified as the realm of all mathematical discourse. Like Cantor we could restrict mathematics to the world of sets, and so elements of V. We don't regard the absolute infinities, such as V itself for example, as strictly mathematical objects or even structures within mathematics. (Very little of mathematics seems to be restricted with this view *pace* a few 'large categories'.)

However, of the parts of V the ordinals occupy a special place. [2] Cantor one assumes would have thought so, and we too see the ordinals as the quintessentially transfinite objects that give set theory (beyond the hereditarily finite sets) its character. Without ω and at least the countable ordinals there is little set theory. We should like to list the concept of ordinal number amongst the 'fundamental concepts' that Martin mentions as named by Fefermann, and that he himself calls 'basic.' This might seem controversial, since Martin only wants to allow concepts that are to some extent atomic, that is not built out of other concepts, and for this he mentions only natural number and the set concept, but would not, presumably, include the concept of von Neumann ordinal which requires the notion of 'transitive set.' However I note that when Martin comes to consider the concept of *ω-sequence* (as opposed to just simply natural numbers), he remarks that although one can define such from sets, he will take such as basic and consists of some objects coming equipped with a successor function *etc.* or alternatively a successor *relation.* For us we should have to take an ordinal as some objects, together with a predecessor relation, with the additional well ordering requirement.

Whether much turns on our selecting the ordinal concept as basic, I am not sure, but from the ordinals much can be derived when we consider the addition of power set operations and replacement: the V_α hierarchy itself is obtained by iterating the power set operation along the ordinals. In our Cantorian, pre-theoretic thinking, the ordinals, like the natural numbers, are determinate. Before Cantor the natural numbers would have constitued an 'absolute infinity' - he showed us otherwise. Later we come to formalise our set theory and eventually contemplate strong axioms of infinity within the language of that theory, but these do not affect ordinals - they are not 'longer' because we discover/posit/assert that there are inaccessible or measurable cardinals (which are in any case cardinal-theoretic properties, not ordinal-theoretic ones) any more than the natural numbers are 'longer than we thought' because of the Skewes number.

[2] The centrality of the ordinals to Cantor, and to modern set theory is emphasised in [4].

One additional caveat in the above discussion is that our phrasing "the ordinals are determinate", cannot be meant in the strong sense of Martin: *"a concept is fully determinate if it is determined, in full detail, what a structure instantiating it would be like.*([9]p5) since he only seems to accept the determinateness of small countable ordinals. Martin does not mind if someone takes determinateness of a concept to imply instantiations of it exist. His objection is that the concept of set, and presumably the concept of ordinal in generality is so determinate.

We take in this paper the view that we do have *sufficient determinateness* of von Neumann ordinals: these are the transitive sets wellordered by $\in$. The fact that we use the concept of set to state this definition, should not mean that we do not fully understand this. There may be uncountable ordinals, inaccessible initial ordinals, *etc.* and these varying 'details' beyond the purely ordinal-theoretic, may be what Martin views as insufficiently determining the concept. However the base concept of the von Neumann ordinal as just defined allows one *given any putative instantiation of it,* to tell, figuratively speaking, whether it is, or is not, an ordinal. There is a world of difference between asserting this sufficient determinateness of the von Neumann ordinal concept, and, say, that of the concept of power set of $V_{\omega+1}$. We are perhaps cutting the division between instantiations and determinateness in a different way to Martin: whereas he does not mind if determinateness is taken to imply instantiation, we are not saying this for the sufficient determinateness, or whatever amount of determinateness one wants to call it, that determines our description of von Neumann ordinal (again modulo understanding the 'set of' concept).

We therefore let $\mathscr{C}$ denote the collection of the parts of the domain of the universe $\mathfrak{V}$. When talking about a structure with its parts as a predicate such as $\mathfrak{V} = (V, \mathscr{C}, \in)$ we are thinking of a two sorted language with variables $x, y, z, \ldots$ for sets in V, and $X, Y, Z, \ldots$ for the parts in $\mathscr{C}$.

Theorem 2 *If we have two structures of sets* $\mathfrak{V}_i = (V_i, \in_i)$ $(i = 1, 2)$ *satisfying Martin's (1) and (2) above, with collections of parts* $\mathscr{C}_i$, *we may define an isomorphism* $\pi : (V_1, \in_1) \to (V_2, \in_2)$ *as before.* π *then extends to an isomorphism:*

$$\pi : (V_1, \mathscr{C}_1, \in_1) \cong (V_2, \mathscr{C}_2, \in_2).$$

Proof: Let $(V_1)_\alpha$ denote the set of $\mathfrak{V}_1$-sets of rank α in the sense of $\mathfrak{V}_1$ (and similarly $(V_2)_\beta$ etc). It suffices to show for every part $X \subseteq V_1$ (thus $X \in \mathscr{C}_1$) there is a $Y \subseteq V_2$ with $\pi(X \cap (V_1)_\alpha) = Y \cap (V_2)_\beta$ where $\alpha \in_1 \mathrm{On}^{\mathfrak{V}_1}$ and $\beta \in_2 \mathrm{On}^{\mathfrak{V}_2}$ with $\pi(\alpha_1) = \beta$, and conversely - since then we may define $\pi(X) = \bigcup_{\alpha \in_1 \mathrm{On}^{\mathfrak{V}_1}} \pi(X \cap (V_1)_\alpha)$. *etc.*, thereby yielding $\pi(X) \in \mathscr{C}_\in$. Q.E.D.

Here we are taking the 'informal union' of the sets of the form $\pi(X \cap (V_1)_\alpha)$. However we are not declaring this union to be a 'set' or any such, so no formal axiom is needed. This is unproblematic as it is simply taking a union (or fusion if you will) of the parts $\pi(X \cap (V_1)_\alpha)$ and thus is a part of V_2. A point to be mentioned is that we obtain the map π from Martin's argument at Theorem 1 above which turned on a use of his Informal Comprehension Scheme: nothing further is needed to extend the map to the parts of each universe.

Much of the above could be given a simple, and natural, explanation in a formal second order logical framework, but we are intentionally restricting our appeal to second order formal methods, and giving an account of informal reasoning that leads to the formalisations that we now have.

4 What is the character of $\mathscr{C}$?

I shall write from now on $(\mathfrak{V}, \mathscr{C}, \in)$ since we have argued that this is a conceptual structure unique up to isomorphism. We think of elements of $\mathscr{C}$ as the absolutely infinite parts of V. *Prima facie* there may seem not much that can be said. But there is more to the unfolding of the concept set of/part of.

As we adopted a non-instantiative approach to V we need not feel queasy that we are positing new, instantiated (and large) entities: just as we have adopted a view of $(V, \in)$ as a structure unique up to isomorphism, and seen how we can extend that to a view of V together with its parts, we do not have to say anything further about ontological committment.

Question: Which informal axioms follow from the concepts of 'set of/part of' or 'set of/absolute infinity of'?

We ask this question deliberately to mirror the same question of Martin's above concerning the concept of 'set of' alone. Should we be adopting some kind of Informal Comprehension Scheme involving a properties scheme with both sets and parts of V? Well we could, however we should like to hold back from too much overtly informal second order reasoning. Thus we might make the following observations about sets and absolute infinities directly: clearly

$$\{(x,x) \mid x \in V\} \text{ and } \{(y,x) \mid y \in x \in V\}$$

are both absolute infinities (here "(y,x)" denotes the usual ordered pair of y and x and later (z,y,x) for ordered triple). Continuing with this idea, and allowing sets to reappear also as parts of V we might be tempted to argue that if X and Y are absolute infinities, then there is some part of V that is their intersection: some Z so that $Z = X \cap Y$. This is informal reasoning, rather than a formalised axiom. Similarly one could claim that a finite number of instances of informal arguments establishes the following informal, but more, or less, intuitive, principles:

(i) For any two parts $X, Y \in \mathscr{C}$ there is a part of the universe $Z \in \mathscr{C}$ which is the collection of all those t which are both in X and in Y.

We have expressed this in English to emphasise the informal nature of the reasoning leading to this conclusion. Similarly:

(ii) For any $X, Y \in \mathscr{C}$ the collection of those t in X but not Y forms a part of V.

Still intending informality, but less ponderously expressed:

(iii) $\forall X \exists Y (Y = V \setminus X)$
(iv)$\forall X, Y \exists Z (Z = X \times Y)$
(v) $\forall X \exists Y (Y = \mathrm{dom}(X))$
(vi) $\forall X \exists Y \forall xyz((x, y, z) \in X \leftrightarrow (z, x, y) \in Y)$
(vii) $\forall X \exists Y \forall xyz((x, y, z) \in X \leftrightarrow (x, z, y) \in Y)$.

Just as Martin would invoke a small number of instances of the informal notion of 'Property' in his Informal Comprehension Scheme (and those properties that he does invoke are defined from the structures involved, which he claims legitimates their use, *ib.* p.16), so we are using a small number of instances of rudimentary reasoning about parts. What we have done is to show that whatever the collection of parts $\mathscr{C}$ is, a small number of instances of informal reasoning leads from simply given parts to other parts, and in particular from absolute infinities to parts (that in some cases are also absolute infinities - but may not be). Of course whatever $\mathscr{C}$ is, if we accept the above we have shown:

Proposition 1
$\mathfrak{V} = (V, \mathscr{C}, \in)$ *satisfies the formal von Neumann-Bernays-Gödel axioms.*

since (i)-(vi) capture Bernays' finite axiomatisation of NBG. If the reader does not wish to accept this last move, then this will not harm what follows.

5 Global Reflection principles

On its own the iterative concept of set says nothing about Reflection, but it is perhaps remarkable that first order reflection is a theorem of ZF due independently to Montague and Levy.

Gödel again:

> "All the principles for setting up the axioms of set theory should be reducible to Ackermann's principle: The Absolute is unknowable. The strength of this principle increases as we get stronger and stronger systems of set theory. The other principles are only heuristic principles. Hence, the central principle is the reflection principle, which presumably will be understood better as our experience increases. Meanwhile, it helps to separate out more specific principles which either give some additional information or are not yet seen clearly to be derivable from the reflection principle as we understand it now." (Wang [14].)

Peter Koellner in [6] suggests that *intrinsic reflection theorems* are those that derive from the iterative concept of set and moreover these are bound in strength by that of an ω-Erdős cardinal. Such cardinals are consistent with $V = L$ and hence are *intra-constructible.* Koellner in this paper seeks to analyse some suggestions for reflection principles of Tait [13] who proposed some as giving large cardinal strength of measurable cardinals. However Koellner shows that Tait's principles are either inconsistent or intra-constructible. Koellner gives a heuristic argument as to why all intrinsic reflection theorems are intra-constructible.

I find it difficult to see how higher order reflection principles such as those of Bernays which deal with Π^1_n or even Π^m_n reflection schemes, follow from the iterative concept of set. If one takes a Zermelian approach [17] which involves a never-ending tower of normal domains indexed by inaccessible cardinals then this potentialist never-to-be-completed universe of sets and domains hardly leaves scope for higher type quantification over 'everything'. Hence it is better to adopt, as Koellner does, an 'actualist' stance where the universe of V is built by iterating the rank function along the absolute infinity of On and that is it: we have the concept of a set structure, that is a universe, and over this we may consider higher type quantifications leading to the satisfaction of some higher type sentence Ψ say. However it is hard to see how we can properly formulate the truth conditions for such a formula Ψ with the tools at hand. The second (or higher) quantifiers have to range over something. One can perhaps do something with the iterative concept plus plural quantification plus reflection thereof, but the higher order reflection needed to get Π^m_n reflection and thence Π^m_n-indescribable cardinals (still intra-constructible) needs further concepts.

Reflection from the iterative concept of sets with classes.

We consider $\mathfrak{V} = (V, \mathscr{C}, \in)$ together with its parts. We let $\mathscr{L}^+$ be the usual first order language, augmented with second order variables $X_1, X_2, \ldots$ but without second order quantification. The interpretation of the second order variables from a formula φ in $\mathscr{L}^+$ is that the X_i range over the parts in $\mathscr{C}$. We may further stratify the language in the usual manner with φ being from $\Sigma_1, \Sigma_2, \ldots \Sigma_\omega = \mathscr{L}^+$.

Formula-by-formula reflection now is unexceptional: fix an $i \leq \omega$, then for any $\varphi \in \Sigma_i$:

$$\forall \alpha \exists \beta > \alpha : \forall \vec{x}_i \in V_\beta \forall \overrightarrow{X}_j \in \mathscr{C} : \varphi(\vec{x}_i, \vec{X}_j)^{(V,\mathscr{C},\in)} \leftrightarrow \varphi(\vec{x}_i, \overrightarrow{X_j \cap V_\beta}))^{(V_\beta, V_{\beta+1},\in)}.$$

Here we have identified the parts of V_β with $\mathscr{P}(V_\beta) = V_{\beta+1}$. This is consonant with what we have done: $V_{\beta+1} = \{X \cap V_\beta \mid X \in \mathscr{C}\}$. Here the strength is rather weak, even if we invert some quantifiers and informally reflect the whole language at once:

$$\forall \alpha \exists \beta > \alpha : \forall \varphi \in \Sigma_k \; \forall \vec{x}_i \in V_\beta \forall \overrightarrow{X}_j \in \mathscr{C} : \varphi(\vec{x}_i, \vec{X}_j)^{(V,\mathscr{C},\in)} \leftrightarrow \varphi(\vec{x}_i, \overrightarrow{X_j \cap V_\beta}))^{(V_\beta, V_{\beta+1},\in)}$$

we have something less than Π^1_1-indescribability, and so are firmly intra-constructible.

However now let us express the ineffability of V together with its parts $\mathscr{C}$ by asking that we have a form of reflection that takes the whole of $(V, \mathscr{C}, \in)$. down to some $(V_\beta, V_{\beta+1}, \in)$ in some very uniform way. We express this by asserting the explicit existence of a *connection*, or *reflecting map j* as follows:

$$\forall \alpha \exists \beta > \alpha \exists j_\beta : (V_\beta, V_{\beta+1}, \in) \longrightarrow_{\Sigma_1} (V, \mathscr{C}, \in) \qquad \text{(GRP)}$$

where $j_\beta \restriction V_\beta = \mathrm{id} \restriction V_\beta$, and the elementarity is Σ_1 in the language $\mathscr{L}^+$.

1) Notice that $j_\beta(\beta) = \mathrm{On}$ where β is a 'part' of V_β and so is in $V_{\beta+1}$ and similarly $\mathrm{On} \in \mathscr{C}$.

(This is because

$$\forall\tau(\tau \text{ is an ordinal } \leftrightarrow \tau \in \beta)^{(V_\beta, V_{\beta+1}, \in)}$$

is a Π_1 formula about the class β and and so goes up to $(V, \mathscr{C}, \in)$ about $\pi(\beta) = \mathrm{On}$.)

2) More generally for $X \in V_{\beta+1}$ $j_\beta(X) \cap V_\beta = X$.

3) The assumed elementarity ensures that β is an inaccessible cardinal, however there is more to come.

Whilst the assertion of j_β's existence is an assertion that there is a $\mathscr{P}(V_\beta)$-sized collection of ordered pairs $(X, j_\beta(X))$ of classes these can be thought of as a single $Z = \{(y, X) \mid y \in j_\beta(X)\}$. We may thus view j_β either as a plurality of a small number of parts of V of a particular kind[3], or else the result of a single Σ^1_1-assertion about the existence of such a Z.

Let a slightly weaker principle be defined:

$$\exists j : (V_\beta, V_{\beta+1}, \in) \longrightarrow_{\Sigma_1} (V, \mathscr{C}, \in) \qquad (\mathrm{GRP}_0).$$

Proposition 2 $\mathrm{GRP}_0 \Longrightarrow$ *There is an absolute infinity of measurable Woodin cardinals.*

Proof: We prove that there is a proper class of measurable cardinals. Arguments that can be found, for example, in [5] establish also the Woodiness of κ. Suppose GRP_0 holds as witnessed by a j with critical point κ. Define a field of classes U on $\mathscr{P}(\kappa)$ by

$$X \in U \leftrightarrow \kappa \in j(X).$$

As $\mathscr{P}(\kappa) \subseteq V_{\kappa+1} \subseteq \mathrm{dom}(j)$ by Σ_1 -elementarity, this is an ultrafilter (and this is the point of ensuring that j acts on all classes of V_β). The strong inaccessibility of κ is easy to establish from our assumptions, and this yields the δ-additivity of U for any $\delta < \kappa$, and non-principality of U trivially follows from $j \restriction \kappa = \mathrm{id} \restriction \kappa$. Thus U establishes κ is a 'measurable cardinal' (and thus we have a strongly *extra-constructible principle*). However then:

For any $\alpha < \kappa$:

$$\text{“}\exists\kappa > \alpha(\kappa \textit{ a measurable cardinal}\text{”}^{\langle V, \in\rangle} \Longrightarrow$$
$$\text{“}\forall\alpha\exists\lambda > \alpha(\lambda \textit{ a measurable cardinal})\text{”}^{\langle V_\kappa, \in\rangle} \Longrightarrow$$
$$\text{“}\textit{There is a proper class of measurable cardinals}\text{”}^{\langle V, \in\rangle}.$$

Q.E.D.

By the work of Martin & Steel [10], and Woodin (see [16]), the consequent of the above proves the following:

Corollary 1 $\mathrm{GRP}_0 \Longrightarrow$ Projective Determinacy, $\mathrm{AD}^{L(\mathbb{R})}$, *and no statement of analysis can be forced to change its truth value by Cohen style set forcing.*

[3] This plural view is discussed further in[3]

We finally remark that both GRP and GRP_0 do not imply further large cardinals: the principles are consistent relative to that of ZFC and the assertion of the existence of 'weakly sub-compact cardinals' (from which they are derived) but they do not imply any form of sub- or supercompact cardinal. They thus seem to sit at a watershed between those weaker large cardinals and those that imply there are $\mathcal{L}$-elementary embeddings $j : V \longrightarrow M$ with critical point some κ so that $j(\kappa^+) > \sup j``\kappa^+$. (All weaker large cardinals have equality here.) This may look like an arcane technicality, but this 'jump' discontinuity is at the base of many arguments involving, for example, supercompact cardinals and in particular forcing arguments. It is in some sense a natural threshold, but it is somewhat hard to assess exactly its significance.

6 Conclusions

We have argued that the natural extension of the concept 'set of' (in the Martinian fashion) to include the logically necessary 'absolute infinities' following on from a Cantorian or a ZF viewpoint, yields a conceptual framework which in turn entails, it can be argued, an informal axiom scheme of comprehension in the form of the Bernays finite axiomatisation of NBG. We have done this in order to avoid requiring the existence of either sets or of classes as instantiated mathematical objects.

A strong reflection principle, the Global Reflection Principle is then introduced, which does require the assertion of the existence of a connection or map exemplifying the reflection of simple existential assertions between the universe V together with its absolutely infinite parts, and those of some one V_β together with its collection of parts which we identify with $V_{\beta+1}$. GRP_0 then yields proper classes of sufficiently large cardinals to use Martin & Steel's result that Projective determinacy holds, Woodin's results that $\text{AD}^{L(\mathbb{R})}$ and that both these statements as well as any other statements of analysis cannot be changed by Cohen style set-forcing techniques.

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