Minimality Arguments for Infinite Time Turing Degrees

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Abstract

We show that the length of the naturally occurring jump hierarchy of the infinite time Turing degrees is precisely ω, and construct continuum many incomparable such degrees which are minimla over $\mathbf{0}$. We show that we can apply an argument going back to that of H. Friedman to prove that the set ∞-degrees of certain Σ^1_2-correct KP-models of the form L_σ ($\sigma < \omega_1^L$) have minimal upper bounds.

1 Introduction

Obtaining minimality results in degree theory has a long history: the methods go back to those of Spector when he constructed minimal Turing degrees, and to Gandy-Sacks, [2], for minimal hyperdegrees. The perfect set construction is the common thread to these proofs. A further feature, which is shared, either directly or indirectly, by such arguments, is the use of a selection principle in order to typically, directly shrink a perfect set $T \subseteq {}^\omega\omega$ to a T' so that a particular function is either continuous one-to-one, or constant on the branches of T'. For example in Sack's construction [8], using uniformly hyperarithmetically pointed perfect trees, Kreisel Selection is appealed to in order to directly shrink hyperarithmetically pointed trees to uniformly pointed ones, and ultimately, so that relevant functions fall into this dichotomy on some tree. In Kechris' results ([6],[5]) category theoretic arguments are used to find a suitable T' on which, for example in [6], a Δ^1_{2n} function $F : {}^\omega\omega \to {}^\omega\omega$ can be seen to be (1-1) and continuous, or just constant. For a set $D = \{\boldsymbol{d_n}\}_{n<\omega}$ of Δ^1_2 degrees H. Friedman used a slightly different tack: he wished to construct a shrinking sequence of perfect trees with certain pointedness properties, whose final intersection would not necessarily have to be so rigourously pointed (the source of difficulties for Sacks in [8]), but such that branches f "remote" from $\{\boldsymbol{d_n}\}_{n<\omega}$, that is $f \notin \cup_n L[d_n]$, would be Δ^1_2-minimal over D. The trick was to use a notion of pointedness that was Π^1_2 and appeal to Schoenfield's Σ^1_2 absoluteness theorem, and, *via* Uniformisation, to the Basis Theorem for Σ^1_2. This was done in order to indirectly shrink perfect T's which might have a branch f lying in a particular $\Sigma^1_2(T)$ set, with f itself not in $L[T]$, by using Mansfield's Perfect Set argument (or its hyperarithmetic analogue.) We lifted this trick to obtain minimal upper bounds for Δ^1_3-degrees, assuming sharps for reals, and $\neg O^\dagger$ ([10]) or the failure of some $\Delta^1_2(z)$-Determinacy ([11]). Something of the Friedman argument remains also in [6] where Friedman's key lemma has as analogue a shrinking construction on trees to see that if a branch in a certain Σ^1_{2n} set is remote, (now this means not in the smallest $\Sigma^1_{2n}(T)$ set $\mathcal{C}_{2n}(T)$), then by Π^1_{2n-1}-Uniformization (we are assuming Δ^1_{2n}-Determinacy here) we may shrink T to T' so that now all branches fall within the set and have the same ordinal value arising from the Π^1_{2n-1} norm.

We exploit this trick one more time in constructing minimal upper bounds for degrees of computability abstracted from the Infinite Time Turing machines of [3]. The notion of relative computability here is more akin to hyperdegrees (even Δ^1_2-degrees) than Turing degrees. We use "∞"-pointed trees (a Π^1_2-notion) to construct recursively a splitting system of trees all of whose paths dominate the degrees of a countable Σ^1_2-correct model. As in [1] the branches f of the final fusion tree, if remote from M, are seen to be minimal upper bounds for the degrees of M. (Here "remoteness" means that f is not itself computable from any real of M, in any time that is less than any ordinal which is itself computable from f.) The use of Σ^1_2-absoluteness is as follows: it is used to contract trees of M which have branches convergent in V with respect to a certain computation, to subtrees, still in M, with a uniform M-bound on the number of steps for convergence for all the branches (whether lying in V or M). This use of the Σ^1_2-absoluteness (or, eventually, of Σ_1-reflection) is in one sense a weakness: it means we can only prove the existence of minimal upper bounds for a restricted class of sets of degrees. However we have no idea how to contract trees otherwise.

2 Infinite Time Turing degrees

Hamkins and Lewis in [3] give the construction of these machines and develop the basic theory of this notion of computability. To summarise: an infinite time Turing machine has the hardware of an ordinary Turing machine, except that transfinite stages of computation are allowed (the action at successor stages being just that specified as usual by a finite program with a finite number of possible states). The action at limit stages ν of "time" is very definite: the machine head always returns to a specific cell on the tape, and enters a special limit state. The value in the cell C_n (thought of as a 0 or a 1) is specified as the limsup of the values at all previous "times". This specifies precisely a behaviour of a machine, and like ordinary Turing machines, such a device may "halt" with a specific output or may loop forever. The infinite amount of time available means however that such machines can accept an infinite string of integers as input (and can output the same). This implies that a machine can decide whether an input is a code of an wellfounded relation etc., etc. We list some of the basic facts that we shall need from [3], [12], and [13].

Fact 2.1 *([3] 2.5)(i) The relation $\phi_p^f(x)\downarrow$ (in α-steps) (the p'th program halts on input $x \in {}^\omega\omega$ using oracle $f \in {}^\omega\omega$ (additionally, in α-steps)) is a Δ^1_2 relation of f, p, x (and α).*
(ii) Similarly $\phi_p^f(x)\uparrow$ is also Δ^1_2. (It is only necessary to have a/all wellordered sequence(s) coding the course of a looping computation to see this.)

Fact 2.2 *([3] 3.7, 8.3) If $\mathcal{W} =_{df} \{x \in {}^\omega\omega \mid x$ is the output of $\phi_p(0)\downarrow\}$, then the "writable" ordinals, (those ordinals α so that there is $x \in \mathcal{W} \cap WO$ with $|x| = \alpha$) form an initial segment of the countable ordinals, λ. The class $H(\lambda)$ whose elements are coded by reals in $\mathcal{W}$ is a transitive admissible set, closed under hyperjump (and more).*

We use the notation $\mathcal{W}^f, \lambda^f \ldots$ for the notions relativised to a real. The following theorem states that the halting times of computations are themselves all realisable as outputs of such computations.

Fact 2.3 *([12]) Let γ be $\sup\{\alpha \mid \exists p\ \phi_p(0)\downarrow$ in precisely α steps$\}$. Then $\gamma = \lambda$ (and by relativisation $\forall f \in {}^\omega\omega \quad \lambda^f = \gamma^f$).*

The following is a corollary to the proof of the last fact:

Fact 2.4 *([12]) $H(\lambda) = L(\lambda)$, and for any $f \in {}^\omega\omega \quad H^f(\lambda^f) = L_{\lambda^f}[f]$.*

We shall need the following characterisation of the ordinal λ^f.

Fact 2.5 *([13] Thm 2.1) Let $f \in {}^\omega\omega$. Let (ζ^f, Σ^f) be the lexicographically least pair of ordinals with $L_\zeta[f] \prec_{\Sigma_2} L_\Sigma[f]$. Then $\lambda^f < \zeta^f$ and λ^f is the least $\overline{\lambda}$ with $L_{\overline{\lambda}}[f] \prec_{\Sigma_1} L_{\zeta^f}[f]$.*

Hamkins and Lewis define a notion of degree between reals:

Definition 2.1 *For $f, g \in {}^{\omega}\omega$ we write $f \leq_\infty g \longleftrightarrow \exists p \in \omega\ \phi^g_p(0){\downarrow}= f$.*

We consider the following questions.

Question 1 Are there incomparable minimal ∞-degrees?

Hamkins has shown ([4]) that there are no degrees between that of 0 and the jump of 0. The following is a natural definition for a jump operator on reals alone (they call this the "weak jump" see [3]).

Definition 2.2 *For $f \in {}^{\omega}\omega$ $f^\nabla =_{df} \{n \in \omega \mid \phi^f_n(0){\downarrow}\}$.*

One may readily show that $f \to f^\nabla$ is a Δ^1_2 function. Clearly $\mathbb{R} \cap L$ is closed under this operation, as indeed is any Σ^1_2-correct model. Let M be a transitive rudimentary closed model, with $\omega \in M$. We define Σ^1_2-correctness for M in the usual way: if Φ is Σ^1_2 and $\vec{a} \in \mathbb{R} \cap M$, then $\Phi(\vec{a}) \Longleftrightarrow \mathbb{R} \cap M \models \Phi(\vec{a})$.

Question 2 For what countable sets of ∞-degrees can one show the existence of minimal upper bounds?

Question 3 How long is the natural hierarchy of ∞-degrees? We define $\langle \boldsymbol{h_i} \mid i < \rho \rangle$ using as jump the operator $f \longrightarrow f^\nabla$, by $\boldsymbol{h_{i+1}} = \boldsymbol{h_i}^\nabla$ and taking $\boldsymbol{h_\lambda}$ as the least upper bound (if possible) of $\langle \boldsymbol{h_i} \mid i < \lambda \rangle$ at limit stages. If ρ is the least so that $\boldsymbol{h_\rho}$ is undefined, how large is ρ? Is $\rho > \omega$?

2.1 δ-degrees and minimality

The characterisation of $H^f(\lambda^f)$ in Fact 2.4 above can be reformulated as:

Fact 2.6 $x \leq_\infty y \longleftrightarrow x \in L_{\lambda^y}[y]$

Note then: $x \leq_\infty y \longrightarrow \lambda^x \leq \lambda^y$, and in fact one may easily see $x \leq_\infty y \longrightarrow (x^\nabla \leq_\infty y \longleftrightarrow \lambda^x < \lambda^y$. The assignment $x \longrightarrow \lambda^x$ then satisfies a Spector criterion.

Definition 2.3 *Let $F = \{x \in {}^{\omega}\omega \mid x \in L_{\lambda^x}\}$*

Then F, the set of "fast" reals x , consists of those x that can be computed as being in L, by an ordinal that is the length of time of an x-computation. This is by way of analogy with the largest thin Π^1_1 set of reals, $Q = \{x \mid x \in L_{\omega^x_{1ck}}\}$, the quickly constructible reals. It is easily seen that F is a strictly Δ^1_2 and thin set of reals. Those $x \notin F$ are correspondingly "slow".

Definition 2.4 *Let M be a Σ^1_2-correct model. Then f is* remote from M *if and only if, $\forall g \in {}^\omega\omega \cap M\ f \notin L_{\lambda^f}[g]$.*

It is the remote branches of our eventual tree that will provide minimal upper bounds. Clearly only slow reals can be remote. We define an auxiliary notion of degree to help establish minimality.

Definition 2.5 *Let δ be any p.r. closed ordinal. Then set $x \leq_\delta y \longleftrightarrow x \in L_\delta[y]$.*

Then again $\leq_\delta$ is a transitive partial ordering and we define the corresponding notion of "δ-degree" $\boldsymbol{x_\delta}$ as:

$$[x]_\delta = \{y \mid y \leq_\delta x \wedge x \leq_\delta y\}$$

Definition 2.6 *(i) If D is a set of δ-degrees, then f is* δ-minimal over *D, if $\forall g \in D\, g <_\delta f$, and for all h, if $(h \leq_\delta f \wedge \forall g \in D\, g \leq_\delta h)$ then either $h \leq_\delta g$ for some $g \in D$ or $f \leq_\delta h$.*
(ii) The notion of being ∞-minimal over a set of ∞-computability degrees is defined entirely similarly.

Lemma 2.1 *If f is λ-minimal (over $\mathbf{0}$), and $\lambda^f = \lambda$, then f is ∞-minimal over $\mathbf{0}$.*

Proof: Let f be λ-minimal. As $f \notin L_\lambda$, $f \not\leq_\infty 0$. Suppose $0 <_\infty h \leq_\infty f$. Then $h \in L_{\lambda^f}[f]$. As $\lambda^f = \lambda$, and $\lambda \leq \lambda^h \leq \lambda^f = \lambda$, by λ-minimality of f, $f \in L_{\lambda^h}[h]$ as required. **Q.E.D.**

This generalises straightforwardly to:

Lemma 2.2 *Let D be a countable set of (reals of) ∞-degrees. Let $\lambda^D =_{df} \sup_{f\in D} \lambda^f$. Let f be λ^D-minimal over D, with $\lambda^f = \lambda^D$. Then f is ∞-minimal over D.*

Lemma 2.3 *Let D be as above, and $\langle \delta_n \mid n < \omega\rangle$ an increasing sequence of p.r. closed ordinals with sup$(\delta_n) = \lambda^D$. Suppose $\forall n \exists h_n \exists d \in D\, h_n \leq_\infty d \wedge (f \oplus h_n)$ is δ_n-minimal over h_n. If f satisfies $\lambda^f = \lambda^D$, and $\forall d \in D\, d <_\infty f$, then f is ∞-minimal over D.*

Proof: Show f is λ^D-minimal over D, and then apply the previous lemma. **Q.E.D.**

To fix notation we let $T, T^*, \ldots$ etc. range over sets of sequence numbers in $2^{<\omega}$. Actually we shall always assume that $T, T', \ldots$ range over perfect trees. We denote the set of branches of T, by $[T]$; $f, g, \ldots$ etc. will range over such branches. A tree T is then ∞-*pointed* if for all $f \in [T]$, $T \leq_\infty f$.

By straightforward methods of Sacks ([8]) one has as usual:

Lemma 2.4 *Let T be ∞-pointed. Then:*
(i) $\{g \in {}^\omega\omega \mid T \leq_\infty g\} = \{g \in {}^\omega\omega \mid \exists h \in [T]$ with $h =_\infty g\}$.
(ii) Let $g \in {}^\omega\omega$ with $T \leq_\infty g$. Then there is an ∞-pointed tree $T^* \subseteq T$, $T^* =_\infty g$. In fact there is a uniform procedure for finding such a T^* recursively in T and g.

Lemma 2.5 *(i) Let $\delta > \omega$ be p.r.closed, and $g : \delta \longrightarrow \omega$ is (1-1). Then there is $T^* \subseteq T(T^* \in L_{\delta+1}[g, T] \wedge \forall f \in [T^*]((f \oplus T)$ is δ-minimal over $T)$.*
(ii) If T is ∞-pointed, and $\delta < \lambda^T$, then T^* can be taken ∞-pointed.

Proof: (i): This is just as in [7] 1.4 and 3.1. For (ii): just note then that we may take the $g : \delta \longrightarrow \omega$ as lying in $L_{\lambda^T}[T]$, as in the latter model, everything is countable. But then we may assume that T^* is there also. But then if $f \in [T^*]$, then $f \in [T]$, hence $T \leq_\infty f$ and $T^* \leq_\infty T$ as Fact 2.4 implies that $H^T(\lambda^T) = L_{\lambda^T}[T]$,

Q.E.D.

3 Constructing minimal upper bounds

For x a set, let x^+ denotes the ordinal height of the smallest transitive admissible set containing $\{x\}$.

We first answer Question 3 of the Introduction.

Theorem 3.1 $\rho = \omega$

Proof: Set $\lambda_0 = 0$, and for $i \geq 1$, let $\lambda_i =_{df} \lambda^{0^{i\nabla}}$ where $\mathbf{0}^{n\nabla}$ is the n'th iterate of the jump operator on $\mathbf{0}$, and λ^f is defined at Fact 2.2 above. Let $\lambda_\omega = \sup_i \lambda_i$. Then each λ_i is a recursively inaccessible ordinal. We note that

$$L_{\lambda_1} \prec_{\Sigma_1} L_{\lambda_2} \cdots \prec_{\Sigma_1} L_{\lambda_\omega} \prec_{\Sigma_1} L_\zeta \prec_{\Sigma_2} L_\Sigma$$

by Theorem 2.1 and Lemma 2.2 of [13] (where ζ is least so that L_ζ has a Σ_2 transitive end extension L_Σ - this is just a natural generalisation of Fact 2.5). Consequently $L_{\lambda_n} \models$ "T is ∞-pointed" if and only if $L_{\lambda_\omega} \models$ "T is ∞-pointed" if and only if $L_\Sigma \models$ "T is ∞-pointed". It is also easy to see that "$\forall f \in [T]\, F(e, f) = \alpha$" is also absolute between these three models, for any $T, \alpha \in L_{\lambda_n}$.

Lemma 3.2 *Let $T \in L_{\lambda_\omega}$ be perfect, $f \in [T]$, $T \leq_\infty f$ and $F(p,f) \approx \beta < \zeta$, and $f \notin \Delta^1_1(h_\beta, T)$ where h_β is the L-least code for a wellorder of length β. Let n be least with $T \in L_{\lambda_n}$. Then there is $T^* \subseteq T$, $T^* \in L_{\lambda_n}$, and there is an $\alpha < \lambda_n \, \forall f \in [T^*] \;\; F(p,f) = \alpha$. If T is ∞-pointed, then T^* can be assumed ∞-pointed.*

Proof: Let $\beta = F(p,f)$. Note that for any $\mu < \zeta$, h_μ can be assumed definable over L_μ (by virtue of the smallness of ζ). Note:

(1) "$F(p,f) = \beta$" is Σ^1_1 in f and a code for β.
Define a set T' by:

(2) $T' =_{df} \{f \,:\, F(p,f) = \beta \wedge f \in [T]\}; \quad T' \in \Sigma^1_1(h_\beta, T)$.

Moreover T' contains elements not hyperarithmetic in (T, h_β) by assumption. By the "hyperarithmetic perfect set theorem", (see, *e.g.* [9] 6.2.III) T' contains a perfect set T^* recursive in $\mathcal{O}^{h_\beta, T}$, the hyperjump of (h_β, T), and so $T^* \in L_{(\mathcal{O}^{h_\beta,T})^+}[h_\beta, T] \subseteq L_{\lambda_\omega}$. Hence

(3) $L_\zeta \models \exists h_\beta \exists$ perfect $T^*, T^* \leq_T \mathcal{O}^{h_\beta,T}[\forall f \in [T^*]\, F(p,f) = |h_\beta|]$.

But $L_{\lambda_n} \prec_{\Sigma_1} L_\zeta$ so this reflects to L_{λ_n}. Consequently there are $\alpha, h_\alpha, T^* \in L_{\lambda_n}$ satisfying (3). (Note that λ_n is recursively inaccessible, hence L_{λ_n} is closed under hyperjump.)

To conclude the lemma now suppose T is ∞-pointed. Let $f \in [T^*]$; then $f \in [T]$, hence $T \leq_\infty f$, hence $\lambda^T = \lambda_n \leq \lambda^f$. But $T^* \in L_{\lambda_n} \subseteq L_{\lambda^f}[f]$. Hence $T^* \leq_\infty f$. Hence T^* is ∞-pointed. This finishes the lemma. **Q.E.D.**

Now let $\boldsymbol{d_i} = [0^{\nabla i}]$ for $i < \omega$ (so setting $\lambda_0 = 0$, $\boldsymbol{d_0} = \boldsymbol{0}$). Let $d_i = 0^{\nabla i}$, and $E = \{d_i\}_{i<\omega}$.

Lemma 3.3 *There exists a perfect tree T_0 definable over L_{λ_ω} such that $\forall f \in [T_0](f$ slow $\wedge\, \lambda^f \leq \zeta \Rightarrow \lambda^f = \lambda_\omega = \lambda^E \wedge f$ is ∞-minimal over $E)$.*

Proof: We define a binary system of ∞-pointed trees T_s for $s \in 2^{<\omega}$. Suppose $\langle \delta_n \,|\, n < \omega\rangle$ an increasing sequence of p.r. closed ordinals with $\delta_0 = \omega$, $\lambda_n < \delta_n < \lambda_{n+1} (= \lambda^{0^{\nabla n}})$. Set $T_\emptyset = 2^{<\omega}$; assume T_s is defined for any $s \in 2^{<\omega}$ with $lh(s) \leq i$ with the properties
(i) If $lh(s) = j \leq i$ then $d_j =_\infty T_s$ (hence $T_s \in L_{\lambda_{j+1}}$);
(ii) T_s is ∞-pointed.

For $lh(s) = i$ we define $T_{s\frown 0} =_\infty T_{s\frown 1} \in L_{\lambda_{i+2}}$, two subtrees of T_s preserving (i) and (ii), according to the following four step recipe.

1) First split T_s into two ∞-pointed subtrees $T'^0_s =_\infty T'^1_s =_\infty T_s$ with $[T'^0_s] \cap [T'^1_s] = \emptyset$;
2) Find (by appealing to Lemma 2.4), ∞-pointed $T^{*j}_s \subseteq T'^j_s$ in M, with $T^{*j}_s =_\infty d_{i+1}\,(j = 0, 1)$.
3) Find $T^j_s \subseteq T^{*j}_s$, $T^j_s =_\infty T^{*j}_s$ so that $\forall f \in T^j_s((f, T^{*j}_s)$ is δ_{i+1}-minimal over $T^{*j}_s)$. Such T^j_s can be chosen ∞-pointed, and in $L_{\lambda_{i+2}}$, by Lemma 2.5;
4) If there is any $\alpha_0 < \lambda_{i+2}$ and ∞-pointed $T_{s\frown 0} \subseteq T^0_s$ so that $T_{s\frown 0} \in L_{\lambda_{i+2}}$, and also so that $\forall f \in T_{s\frown 0}\ F(i, f) = \alpha_0$, then let $T_{s\frown 0}$ be the L-least such; otherwise set $T_{s\frown 0} = T^0_s$. Similarly define $\alpha_1, T_{s\frown 1} \subseteq T^1_s$.

Note that whatever happens at 4), $T_{s\frown 0} =_\infty T_{s\frown 1} =_\infty d_{i+1}$. Now let T_0 be the fusion of the $\langle T_s \mid s \in 2^{<\omega}\rangle$. Then $[T_0] = \bigcup_{g\in 2^\omega} \bigcap_n [T_{g\restriction n}]$. We now show that T_0 is as in the lemma. Let $f \in \bigcap_n [T_{g\restriction n}]$. Clearly $\forall i\, d_i \leq_\infty f$. As $\forall i \exists j\, d_i <_\infty d_j$, actually $\forall i\, d_i <_\infty f$. Hence f is a strict upper bound for E and $\lambda^E \leq \lambda^f$. Then if $s = g \restriction i$, (f, T^j_s) is δ_{i+1}-minimal over $T^{*j}_s =_\infty T^j_s$. If we set $h_n = T^{*j}_s$ we shall have by Lemma 2.3 that f is ∞-minimal over E if $\lambda^f = \lambda^E$. So suppose $F(i, f)$ is defined, where f is slow, and by assumption $\lambda^f \leq \zeta$. Thus $F(i, f) < \zeta$ and Lemma 3.2 is applicable. Thus we indeed could have found $T_{s\frown j} \subseteq T^j_s$ for $j \in \{0, 1\}$, again with both $T_{s\frown j}$ in $L_{\lambda_{i+2}}$, and ordinals $\alpha_j < \lambda_{i+2}$, with $\forall g \in [T_{s\frown j}]\quad F(i, g) = \alpha_j$. As this is true for all $i < \omega$, we conclude that $\gamma^f \leq \lambda^E$. But then by Fact 2.3 $\lambda^f \leq \lambda^E$. This finishes the Lemma. **Q.E.D.**

To conclude the theorem we must remark that there *are* slow reals f in $[T_0]$ with $\lambda^f \leq \zeta$; in fact, there are continuum many, as follows.

Let g be generic over L_Σ for the local Cohen forcing $\mathbb{P}_{T_0}$ whose conditions are finite initial segments of T_0. ("g is generic over L_α" means that g meets all predense sets of $\mathbb{P}_{T_0}$ in L_α.) If we suitably formulate forcing, using for definiteness the ranked language of [8], forcing for Σ_1-sentences is Σ_1 over any admissible L_α. By genericity, for any τ with $\lambda_\omega < \tau < \Sigma$, if τ is admissible, it is also g-admissible. Although Σ is not necessarily an admissible ordinal here, it is a limit of admissibles (in fact of Σ_2-admissibles) and the forcing relation for Σ_2 is Π_3 over L_Σ, and we get the reflection property that $L_\zeta[g] \prec_{\Sigma_2} L_\Sigma[g]$, (and are the least pair ζ, Σ with this property). By the Fact 2.5 $\lambda^g < \zeta$; further $L_{\lambda^g}[g] \prec_{\Sigma_1} L_\zeta[g]$. Clearly then g is slow. By construction $g \in [T_0]$. **Q.E.D.Theorem 3.1**

We now refine the above argument to construct a tree T_0 definable over L_λ with continuum many branches that are minimal over $\mathbf{0}$. Let now $\langle \delta_n \mid n < \omega\rangle$ be an increasing sequence of p.r. closed ordinals cofinal in λ. We construct trees T_s as before, but all T_s are in L_λ (and so are trivially ∞-pointed as $T_s =_\infty \mathbf{0}$). We perform steps 1),3), & 4) of the above recipe (2) now being redundant), choosing $T'^j_s, T^j_s \subseteq T'^j_s$ in L_λ, and $T_{s\frown j} \subseteq T^j_s$ with $T_{s\frown j}$, $\alpha_j \in L_\lambda$.

Again let T_0 be the fusion of the trees constructed, and we obtain:

Lemma 3.4 $\forall f \in [T_0](f$ *slow* $\wedge \lambda^f < \zeta \Rightarrow \lambda^f = \lambda \wedge f$ *is* ∞*-minimal over* $\mathbf{0}$).

Now let g be $\mathbb{P}_{T_0}$-generic over L_Σ. By the same argument as before $\lambda^g < \zeta$ so g satisfies the antecedent in the statement of the lemma. Hence:

Theorem 3.5 *There are continuum many reals g ∞-minimal over* $\mathbf{0}$.

This answers Question 1 of the Introduction. We turn now to consideration of Question 2.

Let M be a countable transitive KP-model. Let us call a real f *countably remote from M*, if it is remote from M and satisfies $L_{\lambda^f} \models$ "$V = HC$" (that is, every set is countable).

Remark: 1) The assumption of countable remoteness is used to ensure that we can extend ∞-pointed trees T to ∞-pointed subtrees in the argument of Lemma 3.7 below.

Theorem 3.6 *Let $M = L_\sigma$ be a Σ^1_2-correct KP-model with $\sigma < \omega_1^L$; assume also that M is projectible (that is there is some $\Sigma_n(M)$ map of ω onto M), and that $M \models$ "$V = HC$". Then:*
(i) There is a perfect tree $T_0 \subseteq {}^\omega\omega$, so that if $f \in [T_0]$ is any branch countably remote from M, then f is a minimal upper bound for the ∞-degrees of reals $g \in M$.
(ii) There are continuum many $f \in [T_0]$ which are countably remote from M branches.

Remark: 2) Countably remote from M is equivalent here to simply being "slow", *i.e.* $f \notin L_{\lambda^f}$, and satisfying $L_{\lambda^f} \models$ "$V = HC$".

Lemma 3.7 *Let $T \in M$ be perfect and suppose for some countably remote from M $f \in [T]$, $T \leq_\infty f$ and $\varphi^f_p(0)\downarrow$. Then there is $T^* \subseteq T$, $T^* \in M$, and there is an $\alpha < On \cap M\, \forall f \in [T^*]\;\; F(p, f) = \alpha$. If T is ∞-pointed, then T^* can be assumed ∞-pointed.*

Proof: Let $\beta = F(p, f)$. Our assumption on f means we can find $h \in WO \cap Q \cap L_{\lambda^f}$ coding an ordinal $\mu > \beta$. An h satisfying this can be found definably over L_μ where $\mu \geq \beta$ is least with $L_{\mu+1} \models \mathrm{card}(\beta) = \omega$. Let $\Psi(h, \beta, n)$ stand for "h is the L-least $k \in WO$ with $|n|_k = \beta \wedge k \in L_{|k|+1}$"

Hence we may additionally assume:

(1) $\Psi(h,\beta,n)$ where $F(p,f)=\beta=|n|_h$.

Let $\Phi(f)$ stand for: "$\neg\exists\zeta'\exists\Sigma'[\zeta'<\Sigma'\wedge L_{\zeta'}[f]\prec_{\Sigma_2} L_{\Sigma'}[f]$". By the above and our choice of f, we thus have:

(2) $\exists h[h\in WO\wedge\exists f[F(p,f)=|n|_h\wedge L_{|h|+1}[f]\models$ "$\Psi(h,|n|_h,n)\wedge\Phi(f)$" $\wedge$
$\wedge f\in[T]\wedge f\notin\Delta^1_1(T,h)]]$ is a true $\Sigma^1_2(T,p,n)$ statement.

By the Σ^1_2-correctness of M, it is true in M. So pick $\tilde{h}\in M$, L-least, witnessing this with $\tilde{h}\in WO\cap L_{\delta+1}$ where $|\tilde{h}|=\delta$. Then:

(3) Let $T'=_{df}\{f\,:\,F(p,f)=|n|_{\tilde{h}}\wedge f\in[T]\wedge L_\delta[f]\models\Phi(f)\}$.

Then $T'\in\Sigma^1_1(\tilde{h},T)$.

Moreover T' contains elements not hyperarithmetic in $(T,\tilde{h})$ by the last conjunct of (2). By the hyperarithmetic perfect set theorem already referred to above, T' contains a perfect set T^* recursive in $\mathcal{O}^{\tilde{h},T}$, the hyperjump of $(\tilde{h},T)$, and so $T^*\in L_{(\mathcal{O}^{\tilde{h},T})^+}[\tilde{h},T]\subseteq M$. Further $f\in T^*\longrightarrow F(p,f)=\alpha$ where $\alpha=|n|_{\tilde{h}}$ as required.

Suppose T is ∞-pointed. Let $f\in[T^*]$; then $f\in[T]$, hence $T\leq_\infty f$, and $F(p,f)=\alpha$. By the last conjunct of the defining clause of (3), $|\tilde{h}|<\Sigma^f$. But then $\tilde{h}$ has a unique Σ_1 definition in $L_{\Sigma^f}[f]$ (as the solution for k in the predicate $\Psi(k,\alpha,n)$); hence by Σ_1-elementarity, (Fact 2.5), $\tilde{h}\in L_{\lambda^f}$ and thus $\tilde{h}\leq_\infty f$. As $L_{\lambda^f}[f]$ is closed under hyperjump, (Facts 2.2, 2.4) $T^*\leq_\infty f$. Hence T^* is ∞-pointed. This finishes the lemma. **Q.E.D.**

Now let $\boldsymbol{d_0}\leq_\infty\boldsymbol{d_1}\leq_\infty\cdots\boldsymbol{d_n}\leq_\infty\cdots(n<\omega)$ be a cofinal (in $\leq_\infty\cap M\times M$) sequence of ∞-degrees of M. We assume that $\boldsymbol{d_0}=\boldsymbol{0}$. Let $d_i\in\boldsymbol{d_i}$. Let $E=\{d_i\}_{i\in\omega}$.

Lemma 3.8 *There exists a perfect tree T_0 such that* $\forall f\in[T_0]$
(f countably remote from M $\Rightarrow f$ is ∞-minimal over $E\wedge\lambda^f=\lambda^E$).

Proof: We define a binary system of ∞-pointed trees T_s for $s\in 2^{<\omega}$. Suppose $\langle\delta_n\,|\,n<\omega\rangle$ an increasing sequence of p.r. closed ordinals with $\delta_0=\omega$, $\sup\delta_n=\lambda^E$ (which here is just σ). We may assume that $\delta_n<\lambda^{d_n}$. We adopt the same tactic as above. Set $T_\emptyset=2^{<\omega}$; assume T_s is defined for any $s\in 2^{<\omega}$ with $lh(s)\leq i$ with the properties
(i) If $lh(s)=j\leq i$ then $d_j\leq_\infty T_s$;
(ii) T_s is ∞-pointed.

For $lh(s)=i$ we define $T_{s\frown 0}=_\infty T_{s\frown 1}\in M$, two disjoint subtrees of T_s preserving (i) and (ii), according to a suitable variation on the four step recipe.

1) As in 3.3 *verbatim*.
2) Let $k(i) \geq i+1$ be the least k so that $T_s <_\infty d_k$. Find (by appealing to Lemma 2.4), ∞-pointed $T_s^{*j} \subseteq T_s^{\prime j}$ in M, with $T_s^{*j} =_\infty d_{k(i)}$ $(j = 0, 1)$.
3) Find $T_s^j \subseteq T_s^{*j}$, $T_s^j =_\infty T_s^{*j}$ so that $\forall f \in T_s^j((f, T_s^{*j})$ is $\delta_{k(i)}$-minimal over $T_s^{*j})$. Such T_s^j can be chosen ∞-pointed, and in M, by Lemma 2.5;
4) If there is any $\alpha_0 \in On \cap M$ and ∞-pointed $T_{s\frown 0} \subseteq T_s^0$ so that $T_{s\frown 0} \in M$, and also so that $\forall f \in T_{s\frown 0}$ $F(i,f) = \alpha_0$, then let $T_{s\frown 0}$ be the M-least such; otherwise set $T_{s\frown 0} = T_s^0$. Similarly define $\alpha_1, T_{s\frown 1} \subseteq T_s^1$.

If any of the $T_{s\frown j} \neq T_s^j$ (for s with $lh(s) = i$), then we may assume they are all of the same ∞-degree (by appealing to Lemma 2.4(ii) and shrinking our original choices of $T_{s\frown j}$).

Now let T_0 be the fusion of the $\langle T_s \mid s \in 2^{<\omega}\rangle$. Then $[T_0] = \bigcup_{g \in 2^\omega} \bigcap_n [T_{g\restriction n}]$. We now show that T_0 is as in the lemma. Let $f \in \bigcap_n [T_{g\restriction n}]$. Clearly $\forall i\, d_i <_\infty f$. Hence f is a strict upper bound for E and $\lambda^E \leq \lambda^f$. Then if $s = g \restriction i$, (f, T_s^j) is $\delta_{k(i)}$-minimal over $T_s^{*j} =_\infty T_s^j$. If we set $h_n = T_s^{*j}$ we shall have by Lemma 2.3 that f is ∞-minimal over E if we have that $\lambda^f = \lambda^E$. We now use our assumption that $f \in [T_0]$ is countably remote from M. So suppose $F(i,f)$ is defined. Then by Lemma 3.7 we indeed could have found $T_{s\frown j} \subseteq T_s^j$ for $j \in \{0,1\}$, again with both $T_{s\frown j}$ in M, and ordinals $\alpha_j \in M$, with $\forall g \in [T_{s\frown j}] \quad F(i,g) = \alpha_j$. As this is true for all $i < \omega$, we conclude that $\gamma^f \leq \lambda^E$. But then by Fact 2.3 $\lambda^f \leq \lambda^E$. This finishes the Lemma. **Q.E.D.**

Again we must show that there are continuum many countably remote reals in $[T_0]$. Again this is easily seen by a genericity argument. Note that such a T_0 can be defined over L_σ; let $\sigma < \zeta' < \Sigma' < \omega_1^L$ be the lexicographically least pair such that $L_{\zeta'} \prec_{\Sigma_2} L_{\Sigma'}$ Note that $L_{\Sigma'} \models$ "$V = HC$". As before, let g be generic over $L_{\Sigma'}$ for the local Cohen forcing $\mathbb{P}_{T_0}$. We shall have that $L_{\zeta'}[g] \prec_{\Sigma_2} L_{\Sigma'}[g]$, and that $\lambda^g < \zeta'$; clearly g is remote from $L_{\Sigma'}$ and so from the reals of $M = L_\sigma$; by construction $g \in [T_0]$. **Q.E.D.Theorem 3.8**

We note that this argument indeed requires some assumption about M to work: there are countable Σ_2-admissible models whose set of ∞-degrees have unique strict least upper bounds: L_ζ is such: the real $\tilde{0}$ coding the Σ_2-mastercode of L_ζ is the unique strict l.u.b. of this set (*cf* [13] Lemma 2.3). What is really being used is the extendibility of certain models to others that themselves have Σ_2-elementary end extensions. One can use the arguments above to show that more generally:

Theorem 3.9 *Let $\tau < \omega_1^L$ be such that there are $\overline{\zeta} < \overline{\Sigma}$ with $L_\tau \prec_{\Sigma_1} L_{\overline{z}} \prec_{\Sigma_2} L_{\overline{\Sigma}}$ and $L_{\overline{\Sigma}} \models$ "$V = HC$". Then there is a perfect tree $\overline{T}_0 \in L_{\overline{\zeta}}$ so that the $\mathbb{P}_{\overline{T}_0}$ generic branches $g \in [\overline{T}_0]$ are ∞-minimal over the degrees of L_τ.*

However a full characterisation of which countable sets of degrees have minimal upper bounds is still wanting.

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