On Elementary Embeddings from an Inner Model to the Universe.

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Abstract

We consider the following question of Kunen:
Does $Con(ZFC + \exists M$ a transitive inner model and a non-trivial elementary embedding $j : M \longrightarrow V)$ imply $Con(ZFC + \exists$ a measurable cardinal)?
We use core model theory to investigate consequences of the existence of such a $j : M \to V$. We prove, amongst other things, the existence of such an embedding implies that the core model K is a model of "there exists a proper class of almost Ramsey cardinals". Conversely, if On is Ramsey, then such a j, M are definable.

We construe this as a negative answer to the question above. We consider further the consequences of strengthening the closure assumption on j to having various classes of fixed points.

1 Introduction

It is quite natural to study the properties of elementary embeddings $j : V \longrightarrow M$ for M some inner model, since many such embeddings, if they exist, have first order formulations within ZFC. The question of reversing the arrow and looking at a non-trivial $j : M \longrightarrow V$ in general does not readily admit of such formulations. So we study in this paper what might be considered the ZFC consequences of the second order statement that there are proper classes j, M such that ... (This was the formulation of Kunen's question as it appears in [7], but of course, if the reader does not like this conceit, then he or she may simply translate everything into properties of elementary submodels of some $V_\lambda \models ZFC$, by replacing V by V_λ for some inaccessible λ. Hence we could equally well be motivated by investigating the possible elementary substructures of some V_λ of size λ.)

It is not hard to see that some sort of hypothesis beyond ZFC is needed to generate such situations. If there were a non-trivial $j : M \longrightarrow V$, then $j \restriction L : L \longrightarrow_e L$ is non-trivial and so $O^\#$ must exist. Indeed, by considering critical points, one can see that such an embedding could not be definable by class terms, even from parameters, or there would be an infinite regress of ordinals!

Obviously more can be extracted from this, and it is natural to ask how much. As the hypothesis is similar to a statement about On possessing Jónsson or Ramsey like properties, it is not surprising that it implies some kind of large cardinal property. Whatever the "core" of V is, denoting it by K, then $j \upharpoonright K^M : K^M \longrightarrow K^V$ and we may ask what kind of K must there be for this to happen. Kunen's question (Question 15.8 of [7]) in this vein was: can we extract an inner model with a measurable cardinal? Or more precisely, does $Con(ZFC_{\{\dot{j}\}} + \exists j : M \longrightarrow V,\ j \neq id,\ j$ elementary$) \Rightarrow Con(ZFC + \exists$ a measurable cardinal $)$? (Here $ZFC_{\{\dot{j}\}}$ means ZFC in $\mathcal{L}_{\{\dot{\in},\dot{j}\}}$ a language with a predicate symbol for j, whilst $j : M \longrightarrow V$ is considered to be Σ_1-elementary in $\mathcal{L}_{\{\dot{\in}\}}$ alone. (The latter implies full elementarity.) This question is easily answered however. Perhaps more interestingly, in §1.1 we consider the consequences of the existence of such elementary embeddings from an M into V, varying the basic question by adding closure requirements on j to see what K must be like. From just the existence of such a j we get (Theorem 1.1) that K must contain a proper class of "almost Ramsey" cardinal (defined below). We get full Ramseyness (Theorem 1.3) by asking that $dom(j)$ have a class of regular fixed points. By insisting that $dom(j)$ contain a stationary class of singular fixed points of cofinality ω, then we obtain that K must have a Mahlo class of measurables; if the fixed points have uncountable cofinality, or if M computes correctly countable cofinalities, then O^{sword} exists (Theorem 1.7) (that is, Jensen's core model for measures of order zero - see [3], is non-rigid). We should like to thank M. Foreman for saving us from a demonstrably inconsistent hypothesis for Theorem 1.7 (we had assumed that M was closed under ω-sequences); and for allowing us to include his counterexample as Theorem 2.3.

In §2 we make a first attempt at some converses by showing that if On is Ramsey, then we may define a j, M with $j : M \longrightarrow_e V$, $j \neq id$. We could formulate this as a statement about V_κ, where κ is a Ramsey cardinal, and show that definably over $\langle V_\kappa, \in, I\rangle$ (where I is a class of good indiscernibles for $\langle V_\kappa, \in\rangle$ coming from the Ramsey property) there are classes j, M satisfying our requirements. We thus have $Con(ZFC + \exists$ a Ramsey cardinal $) \Rightarrow Con(ZFC + j : M \longrightarrow_e V, j \neq id)$.

Although this is not an exact equiconsistency, it could be considered close. However, overall in this direction we have left several upper bounds open.

We make some comments on results obtainable by Woodin, using the stationary tower forcing. (We should like to thank the referee for prompting us to make such comparisons.) Woodin has shown [11], that if κ is a measurable cardinal, and $\mathbb{P}_\kappa$ the full stationary tower forcing, then if G is $\mathbb{P}_\kappa$ generic over V, then we do have an embedding $j : V_\kappa \longrightarrow V_\kappa[G]$. (Below the measurable κ there is an unbounded sequence of completely Jónsson cardinals, fixed by j.) However in this situation, we cannot have that the structure $\langle V_\kappa[G], j, \in\rangle \models ZFC_{\{\dot{j}\}}$. That is, we cannot put back the predicate V_κ. (For example, note that if $V = L[\mu]$ with μ a normal measure on κ, then $V_\kappa \models$ "$V = K$" (with K the Dodd-Jensen core model), then $V[G]$ is a model of the same, and hence some new "mouse" has been added, that must iterate past all those of V_κ!) This, then is slightly different from our requirements that we have $\langle V, j, \in\rangle \models ZFC_{\{\dot{j}\}}$. If we regard the relationship of V to a forcing extension as being "close", then perhaps we need the extra strength in order to obtain such a close relationship between the "M" and "V". Our results do not require such strength since our "M" and "V" are not at all close in this sense.

In §3.1 we reprove the Jensen Indiscernibles Lemma [2], originally done using the old-style fine-structure of the Dodd-Jensen K_{DJ}, but now for the modern hierarchy and the K under consideration. The reasons for this are that, corresponding to Jensen's results for the Dodd-Jensen K, this shows that the α-Erdős (if $cf(\alpha) > \omega$), almost Ramsey, Ramsey etc. properties are downwards absolute between V and any universal weasel. Secondly, this actually also gives a second proof of Theorems 1.1 & 1.3 which are proven in §1 by using a coarse "lift-up" technique.

Our results are all in the region of inner models with measures of order zero, K^{MOZ}, so it is appropriate to use Jensen's model of [3] in order to discuss them. This model will be K throughout the paper. (Although the Kunen question can be resolved with just the Dodd-Jensen K^{DJ}, or even the Σ_1-K of [10].) We assume familiarity with the measures of order zero K, and the techniques of that paper, some of which appear in [4]. We use the strong Covering Lemma for K, [9], [5] on singular cardinals β of V being either singular, or possibly measurable in K if $cf(\beta) = \omega$.

Mostly however it is the theory of iteration and comparison that is useful, and fine-structural results have been hived off to §3.

In §3.2 we sketch the outlines of Jensen's Σ^* fine structure theory in a general setting. It is a relatively bare-bones outline, but all the essential definitions are here. What we have omitted is a full discussion of $\Sigma_1^{(n)}$ relations and functions, and a fuller analysis of good parameters, and of witnesses. The intention here is to give an introduction to the theory, and, in §3.3, to that of the Σ^* ultrapower, to provide a modicum of self-containedness for the proof of the fine-structural Upwards Extensions of Embedding Lemma of §3.4 which is needed for Theorem 1.7. This has essentially been extracted from various parts of §3.3 of [3], but we have thought it worthwhile to give a complete version here, as firstly, the reader may not wish to do it for his or herself, (and anyway such arguments are not often in print), and secondly this is a form of "lift-up" which we wish to use elsewhere.

Some prerequisites The set-theoretic notations are standard. [6] gives an account of the basics of general elementary embeddings. Our notation for iterations of mice is that of Jensen's: an iteration of a mouse M is of the form $I = \langle\langle M_i\rangle\langle\pi^M_{i,j}\rangle\langle\nu_i,\alpha_i\rangle \mid i < \theta\rangle$ with models M_i and indices $\langle\nu_i,\alpha_i\rangle$ and commuting maps $\pi^M_{i,j} : M_i \longrightarrow M_j$, with $\omega\nu_i \leq \omega\alpha_i \leq On \cap M_i$ and if $E_i \neq \emptyset$ (the ν_i'th filter on the E^{M_i} filter sequence), then $\pi^M_{i,i+1} : M|\alpha_i \longrightarrow_{E_{\nu_i}} M_{i+1}$ where $M|\beta$ denotes $\langle J^M_\beta, E^M, E^M_{\omega\beta}\rangle$. ($M||\beta$ will denote $\langle J^{E^M}_\beta, E^M, \emptyset\rangle$), or when there is no danger of confusion, simply $|J^E_\beta|$.) If $E_{\nu_i} = \emptyset$ then $\pi^M_{i,i+1} = id \restriction M|\alpha_i$. Direct limits are taken as needed. A *truncation* occurs when $\omega\alpha_i < On \cap M_i$. We truncate if need be in order to allow a filter to become a full measure over the subsets of its critical point in $M|\alpha_i$, and thus are able to take an ultrapower.

It is a fundamental fact that in any iteration only finitely many truncations can occur. The *coiteration* of two mice or "weasels" (= proper class mice) is performed by "iterating away points of least difference". This results in a terminating double iteration of length some $\theta \leq On$, with either $M_\theta = N_\theta$, $M_\theta \in N_\theta$, or $N_\theta = M_\theta$. We shall use the following well-known lemma:

Basic Coiteration Lemma If (M, N) are premice, and they are coiterated to models (M_θ, N_θ) with iterations $\langle\langle M_i\rangle_{i\leq\theta}, \langle\alpha_i^M\rangle_{i<\theta}, \langle\pi_{i,j}^M\rangle_{i\leq j\leq\theta}\rangle$, $\langle\langle N_i\rangle_{i\leq\theta}, \langle\alpha_i^N\rangle_{i<\theta}, \langle\pi_{i,j}^N\rangle_{i\leq j\leq\theta}\rangle$ and common indices $\langle\nu_i|i < \theta\rangle$ and critical points $\langle\kappa_i\rangle_{i<\theta}$ where $\kappa_i = crit(E_{\nu_i}^{M_i/N_i})$ then $\theta < \mu^+$ where $\mu = \max\{card(M), card(N)\}$ (or if either is a weasel we may need to set $\theta = On$). If $\theta = On$ then either:
a) (i) M is a weasel (*i.e.* $On \cap M = On$), $\alpha_i^M = On$ for $i < On, \pi_{0,\theta}^M$"$On \subseteq On$ and M_θ is an "initial segment" of N_θ whilst
(ii) there is a cub class C so that $i < j \in C \Rightarrow \pi_{i,j}^N(\kappa_i) = \kappa_j = j \wedge \pi_{i,j}^M$"$\kappa_j \subseteq \kappa_j$;
or
b) the same as a) but with the roles of M and N reversed.

For a premouse M, N, we say that N is a *simple iterate* of M, if there has been no truncation, or cut down, at any stage of the iteration from M to N. For two premice (set, or class sized) we set $M <_* N$ if in the coiteration of (M, N) to final models (M_θ, N_θ), then either $M_\theta \in N_\theta$, or there is some truncation occurring at some stage in the N iteration. A weasel W is *universal* if for any set mouse M, $M <_* W$. (This definition suffices below a strong cardinal.) By O^{sword} (or O^s) we mean an iterable mouse of the form $N = \langle J_\nu^E, E_{\omega\nu}, F\rangle$ with both $E_{\omega\nu}$ and F normal measures in N with critical point κ, and if $\pi : N \longrightarrow M =_{df} Ult(N, F)$, then $M|\nu = \langle J_\nu^E, E_{\omega\nu}\rangle$. It is thus a mouse with a final predicate for a measure "of order one", and generates embeddings, or a "sharp", of Jensen's model K.

Definition 1.1 *Let $I \subseteq \mathcal{A} = L_\kappa[A, \in \vec{B}, \ldots]$ be a first order structure. Then I is a* good set of indiscernibles *for $\mathcal{A}$ if for any $\gamma \in I$:*
(i) $\mathcal{A} \restriction \gamma =_{df} L_\gamma[A \restriction \gamma, \in \vec{B} \restriction \gamma, \ldots] \prec \mathcal{A}$;
(ii) $I\backslash\gamma$ is a set of indiscernibles for $\langle\mathcal{A}, \langle\xi\rangle_{\xi<\gamma}\rangle$.

We take the following as our definition of Ramsey. It is well known to be equivalent to the usual one.

Definition 1.2 *κ is* Ramsey *if any first order structure with $\kappa \subseteq |\mathcal{A}|$ has a good set of indiscernibles of length κ.*

Definition 1.3 *κ is* almost Ramsey *if for any first order structure $\mathcal{A}$, with $\kappa \subseteq |\mathcal{A}|$, has, for any $\lambda < \kappa$, a good set of indiscernibles of length λ.*

1.1 Properties of Core Models arising from Embeddings

Theorem 1.1 *If $\exists j : M \longrightarrow {}_e V$, M a transitive proper class, $j \neq id$, then $K \models$ "There exists unboundedly many almost Ramsey cardinals."*

Proof: Assume the theorem false. Then in K there are only boundedly many measurable cardinals. Coiterate (K, K^M) to get models $\langle\langle N^i, K_i^M\rangle|i \leq \theta\rangle$ for some $\theta \leq On$, with indices $\langle\nu_i|i < \theta\rangle$. Let the critical points of the ultrapowers used be $\langle\kappa_i|i < \theta\rangle$.

Note that as the full measures of K^M are bounded in On, and K is universal, $\pi_{\gamma\gamma+1}^{K^M}$ is only non-trivial for set-many indices γ. Let δ be larger than any such γ.

Lemma 1.2 *$\theta = On$ and there is a truncation on the K-side of the coiteration, and there is a cub $C \subseteq \{\kappa_i | i < \theta\}$ so that $i < j \in C \to i = \kappa_i$, $\pi^K_{ij}(\kappa_i) = \kappa_j$.*

Proof: Suppose that K, K^M have a common simple iterate P. As K is universal, so is P, and hence K^M. As we are below a strong cardinal, K^M is a simple iterate of K, *via* some iteration map $\sigma : K \to K^M$ (*cf* for example, [8] 8.13). But then $j \circ \sigma : K \to K$ is non-trivial. Hence O^s exists. So there is no such P. Thus $\theta = On$. The Basic Coiteration Lemma above gives the cub class C. If there is no truncation, then for for all $i \in C$, N_i is a proper class model of $ZFC +$ "κ_i is a measurable cardinal ". In ZFC measurable implies Ramsey, hence as $\mathcal{P}(\kappa_i)^{N_i} = \mathcal{P}(\kappa_i)^{K^M_\theta}$ for all $\delta < i \in C$, we shall have $K^M_\theta \models$ "There is a proper class of Ramsey cardinals"

By elementarity of $\pi^{K^M}_{0\theta}$, so does K^M, and by elementarity of j, so does K and this contradicts our global assumption. **Q.E.D.Lemma 1.2**

Now let $\overline{\tau} > \delta$ be sufficiently large so that, if $\pi^K_{\gamma\gamma+1}$ involves a truncation, then $\overline{\tau} > \gamma$. Then $\rho^\omega_{N_\tau} \leq \kappa_{\overline{\tau}}$ for $\tau \geq \overline{\tau}$. Indeed we shall assume that $\rho^1_{N_\tau} \leq \kappa_{\overline{\tau}}$, and that $N_\tau \models$ "κ_τ is the largest cardinal" since otherwise, if $N_\tau \models \kappa_\tau^+$ exists then $N_\tau \models \kappa_\tau$ is Ramsey , or if $\rho^1_{N_\tau} = On \cap N_\tau$ then $N_\tau \models$ "$KP + \Sigma_1-$separation". And again this is enough to show that $N_\tau \models$ "κ_τ is Ramsey". (See the remarks in Claim 5 below). If $\overline{K} = K^M_{\overline{\tau}} = K^M_{On}$ is the final model on the K^M-side we should again have for all $\tau \geq \overline{\tau}$ $\overline{K} \models$ "κ_τ is Ramsey".

Now let $\tau > \overline{\tau}$ be such that

(i) $\tau \in C$

(ii) τ a limit cardinal, cf $(\tau) \geq \omega_2$.

(iii) cf (τ) is greater than any κ_j where κ_j is the critical point of any measure used on the K^M side.

(iv) $j``\tau \subseteq \tau$.

Claim 1 $K^M \models \tau$ is inaccessible.
Proof As $\tau \in C, \tau = \kappa_\tau$ and so $\overline{K} \models \tau$ is inaccessible. Note by (iii) $\pi^{K^M}_{0\overline{\tau}}(\tau) = \tau$, and the claim follows. QED(Claim 1)

As $j \restriction K^M : K^M \overrightarrow{_e} K$, we may perform a standard copying of the iteration $K^M \to \overline{K}$ *via* j to get an iteration $\langle \widetilde{K}^i | i \leq \overline{\tau} \rangle$, where $\widetilde{K}^0 = K$ with commuting maps $j^i : K^{M_i} \longrightarrow_e \widetilde{K}^i$ as follows:

Set $j^0 = j, \widetilde{K}^0 = K$. Suppose $\widetilde{K}^i, j^i$ defined. Set $\pi = \pi^{K^M}_{ii+1}$. Suppose $\pi : K^{M_i} \to K^{M_{i+1}} = Ult(K^{M_i}, E_i))$ with crit$(E_i) = \kappa_i$. Set $\widetilde{K}^{i+1} = Ult(\widetilde{K}^i, j^i(E_i))$, and $\sigma_{ii+1} : \widetilde{K}^i \to \widetilde{K}^{i+1}$ the ultrapower map. If $\pi = id$, then we define $K^{M_{i+1}} = K^{M_i}$, $j^{i+1} = j^i$, $\sigma_{i,i+1} = id$.

Then define $j^{i+1}(\pi(f)(\kappa_i)) = \sigma_{ii+1}(j^i(f))(j^i(\kappa_i))$. At limit stages one takes the appropriate direct limits. Let $\overline{\jmath} = j^{\overline{\tau}}$, $\widetilde{K} = \widetilde{K^{\overline{\tau}}}$.

Claim 2 $\bar{\jmath}``\tau \subseteq \tau$ for $\tau \in C$.

Proof This is immediate from $\pi_{0\theta}^{K^M}(\tau) = (\tau)$, as in the last claim and using the definition of $\overline{\tau}$. QED Claim 2

Claim 3 There exists a Σ_0, cofinal $\tilde{\jmath} : \langle \overline{K}|\tau^+, F\rangle \longrightarrow \langle \widetilde{M}, \widetilde{F}\rangle$, with $\tilde{\jmath} \supseteq \bar{\jmath} \restriction \overline{K}|\tau : \overline{K}|\tau \longrightarrow \widetilde{K}|\tau$, where $\tau^+ = \tau^{+K_\tau^M} = On \cap N_\tau$, and $F = E_{\tau^+}^{N_\tau}$, the final measure of N_τ.

Proof We use a standard lift up technique - see for example, [4] Claim 9, forming a pseudo-ultrapower

$$D = \{\langle \xi, f\rangle \mid f \text{a function} \wedge \text{dom} f \subset \gamma < \tau, \quad f \in \overline{K}|\tau^+ \wedge \ \xi \in \bar{\jmath}(\text{dom}(f))\},$$

with an $\in^D$-relation

$$\langle \xi', f'\rangle \in^D \langle \xi, f\rangle \Leftrightarrow \langle \xi', \xi\rangle \in \bar{\jmath}(\{\langle x, y\rangle \mid f'(x) \in f(y)\}).$$

Standard arguments using the fact that $\text{cf}(\tau) > \omega$ show that $\in^D$ is wellfounded, and D is isomorphic to some premouse $\widetilde{M} \supseteq \widetilde{K}|\tau$.

(For wellfoundedness, suppose not and let $\langle \xi_{n+1}, f_{n+1}\rangle \in^D \langle \xi_n, f_n\rangle$ yield an infinite descending chain in the lift-up. Let $A_n = \{\langle x, y\rangle \in \gamma \times \gamma \mid f_{n+1}(x) \in f_n(y)\}$ where γ is sufficiently large below τ so that $\text{dom}(f_n) < \gamma$ for all n. Let $\gamma < \delta < \tau$ be sufficiently large, so that for all n $f_n \in \text{ran}\, \pi_{\delta,\tau}^K$. Let $\pi(\overline{f}_n) = f_n$. Then A_n also equals $B_n = \{\langle x, y\rangle \in \gamma \times \gamma \mid \overline{f}_{n+1}(x) \in \overline{f}_n(y)\}$ using the $\overline{f}_n$'s, all of which are in the model $N_\delta \cap \overline{K}|(\delta^+)^{\overline{K}}$. (So we have pulled down into $\overline{K}$ witnesses to "illfoundedness".) But now applying $\bar{\jmath}$ to B_n we get for all n, $\bar{\jmath}(\overline{f}_{n+1})(\xi_{n+1}) \in \bar{\jmath}(\overline{f}_n)(\xi_n)$ - the usual contradiction.)

We define as usual $\widetilde{F} = \bigcup_{X \in \overline{K}|\tau^+} \bar{\jmath}(F' \cap X)$.
Then $\langle \widetilde{M}, \widetilde{F}\rangle$ is a premouse. QED Claim 3

We note that F is determined by the iteration points $\langle \kappa_i | i \in C \cap \tau\rangle$, and so F is $\omega-$complete. Another standard argument (for example, *cf* [4] Claim 10) shows that $\widetilde{F}$ is determined by $\langle \bar{\jmath}(\kappa_i) | i \in C \cap \tau\rangle$. Hence

Claim 4 $\widetilde{F}$ is $\omega-$ complete.

Lastly:

Claim 5 $M^+ = \langle \widetilde{M}, \widetilde{F}\rangle$ is a mouse and $\rho_{M^+}^\omega = \rho_{M^+}^1 = \tau$.

Proof As $\tilde{\jmath}$ is Σ_0-cofinal, it is easily seen that all proper initial segments $\widetilde{M}|\nu$ for $\nu < On \cap \widetilde{M}$ are mice.
(1) $\quad \Gamma = \{\nu < On \cap \widetilde{M} \mid \langle \widetilde{M}|\nu, \widetilde{F} \cap \widetilde{M}|\nu\rangle$ is amenable$\}$ is bounded in $On \cap \widetilde{M}$.

Proof The usual proof that τ measurable $\Rightarrow \tau$ Ramsey, for any $f \in \widetilde{M}|\nu$, with $\langle \widetilde{M}|\nu, \widetilde{F} \cap \widetilde{M}|\nu\rangle$ amenable, proceeds by taking the given function $f : [\tau]^{<\omega} \longrightarrow \tau$, and building by induction on n a sequence $X_n \supset X_{n+1}$ of homogeneous sets so that $f``[X_n]^n$ has cardinality 1. One may do this over an amenable $\widetilde{M}|\nu$ (which is a model of Σ_0-comprehension), and get such a sequence $\langle X_n \,|\, n<\omega\rangle$ $\Sigma_1(\widetilde{M}|\nu, \widetilde{F}\cap\widetilde{M}|\nu)$. If $\nu < On\cap\widetilde{M}$ then $X=\cap_n X_n \in \widetilde{F}\cap\widetilde{M}$. If Γ were unbounded in $\tau^{+\overline{K}}$, then τ would be Ramsey in $\widetilde{M}$ and so in $\overline{K}$, and so in N_τ. Again then all κ_i are Ramsey in $\overline{K}$, contradicting our global assumption. QED (1).

But the boundedness of Γ shows there is a failure of Δ_1-comprehension over $\langle\widetilde{M}|\nu, \widetilde{F}\cap\widetilde{M}|\nu\rangle$. But this implies $\rho^1_{M^+}\leq\tau$, as $M^+\models\tau$ is the largest cardinal. QED (Claim 5)

As $\widetilde{M}|\tau=\widetilde{K}|\tau$ these conditions show *via* an easy comparison argument that a code, $A^1_{M^+}$, for M^+ is in $\widetilde{K}$, and so $M^+\in\widetilde{K}$.
Now suppose $\mathcal{A}=\langle A, \underset{\sim}{(f_n)}_{n<\omega}\rangle$ is a first order structure on τ in $\overline{K}$. Then $\overline{\jmath}(\mathcal{A})$ is a structure with domain $\overline{\jmath}(\tau)$ in $\widetilde{K}$ whose natural restriction to $\tau, \overline{\jmath}(\mathcal{A}) \restriction \tau$ is $\widetilde{\jmath}(\mathcal{A})\restriction\tau$. But $\widetilde{\jmath}(\mathcal{A})\restriction\tau\in\widetilde{M}$ and $\widetilde{F}$ is an ω-complete measure on $P(\tau)\cap\widetilde{M}$. Hence definably over M^+ there is a homogeneous set X for $\widetilde{j}(\mathcal{A})\restriction\tau$ of order type τ. (This is the set $X=\cap_n X_n$ constructed as indicated in the proof of (1) above. Each $X_n\in\widetilde{F}$ and so contains a final segment of the $\langle\widetilde{\jmath}(\kappa_i)|i\in C\cap\tau\rangle$, and thus has order type τ, guaranteeing the same for X.) But $\mathrm{M}^+\in\widetilde{K}$, and hence $\forall\lambda<\kappa\, X\cap\lambda\in(H_\tau)^{\widetilde{K}}$. As $\overline{\jmath}(\mathcal{A})\restriction\tau$ is an elementary subalgebra of $\overline{\jmath}(\mathcal{A})$, $X\cap\lambda$ are good indiscernibles for $\overline{\jmath}(\mathcal{A})$. Hence for any $\lambda<\tau$

$$(2)\quad \widetilde{K}\models \text{“}\overline{\jmath}(\mathcal{A}) \text{ has a good set of indiscernibles of order type} \geq\lambda\text{”}$$

Hence by elementarity $\overline{K}$ thinks the same. As $\mathcal{A}\in\overline{K}$ was arbitrary, $\overline{K}\models$ “τ is almost Ramsey” and again by elementarity of $\pi^{K^M}_{0\tau}$ and j, K thinks the same. Hence all $\tau\in C$, $\tau\geq\overline{\tau}$, are almost Ramsey in K. **Q.E.D.Theorem 1.1**

Theorem 1.3 *If $\exists j: M \underset{e}{\rightarrow} V$, $j\neq id$ with unboundedly many regular cardinals fixed by j, then*

$$K\models \text{“}\exists \textit{ unboundedly many Ramsey cardinals”}.$$

Proof: We use the machinery and notation of the last theorem. Again we assume the theorem false, and hence that there are only boundedly many measurable cardinals in K.

Claim 1 Unboundedly many regular cardinals fixed by j are inaccessible.

Proof Note first that as, again, $N_{\overline{\tau}}$ iterates past $\overline{K}$, for each regular cardinal $\eta>\overline{\tau}$, there is $C_\eta\subseteq\eta$ with $\pi^N_{i,k}(\kappa_i)=k=\kappa_k$ for $i<k\in C_\eta$, and $N_\eta\models$ “η is measurable”. So there is $F=F_\eta$, on the N_η sequence, with F a normal measure on η in N_η, and with F the final segment filter determined by C_η. Hence η is inaccessible in $\overline{K}$. Now assume that only V-successor cardinals are fixed unboundedly in On by j. Then for any V-successor cardinal satisfying $j(\eta)=\eta$ we see that $j(\beta)=\beta$ where $\beta^{(n)+}=\eta$, β the maximal limit cardinal below η. By assumption β is singular in M (if it were regular in M, it would be so in V, contradicting our assumption that no limit regular cardinals of V are fixed by j). By the Covering Lemma in M and elementarity $\beta^{+M}=j(\beta^{+M})=\beta^{+V}$. Continuing with the notation of the last theorem, in the coiteration of K^M with K, by induction on $\alpha<\overline{\tau}$ one may check that $\beta^{+K^M}=\beta^{+K^M_\alpha}=\beta^{+\overline{K}}$. So β^+ is not inaccessible in $\overline{K}$ - a contradiction.

We then have the following strengthening of Claim 2 of Theorem 1.1:

Claim 2 $\bar{\jmath}(\eta) = \eta$ for η any sufficiently large inaccessible fixed point of j.

Proof As all ultrapowers for $\gamma < \overline{\tau}$ taken on the K^M side are by meansures in $K^M_{\overline{\tau}}$, any such ultrapower fixes η. (Recall that $\overline{\tau} > \eta$ is above any γ with $\pi^{K^M}_{\gamma,\gamma+1} \neq id$.) Hence the same is true of the direct limits. Hence $\forall \alpha \leq \overline{\tau} \pi^{K^M}_{0,\alpha}(\eta) = \eta$. Similarly by induction on $i < \overline{\tau}$ it is easy to see that the maps completing the commutative diagram of the i'th stage of the copy iteration, $j^i : K^{M_i} \longrightarrow \widetilde{K}^i$, also satisfy $j^i(\eta) = \eta$, and hence $j^{\overline{\tau}}(\eta) = \eta$. $\bar{\jmath}$ was defined as $j^{\overline{\tau}}$. QED Claim 2

(Although we do not make use of these facts, note that the linear iteration on the K^M side can be embedded into a universal iteration definable inside K^M, all of the action of which takes place below $K^M_{\overline{\tau}}$. This implies that for η as in Claim 2 $(\eta^+)^{K^M} = (\eta^+)^{K_i^M} = (\eta^+)^{\overline{K}}$ for all $i < \overline{\tau}$. Further $(\eta^+)^{\overline{K}}$ is fixed by the "completion maps" j^i for $i \leq \overline{\tau}$. As $cf((\eta^+)^{\overline{K}}) = cf(On \cap N_{\overline{\tau}})$ ($= \omega$ by our assumption on $N_{\overline{\tau}}$) we also have that $(\eta^+)^M = (\eta^+)^{K^M} = (\eta^+)^{\overline{K}} < (\eta^+)^K$ by the fact that $N_{\overline{\tau}}$ goes past $\overline{K}$ and by the Covering Lemma applied inside M.)

Hence we may repeat the proof of Theorem 1.1, using $F = F_\eta$ as F (and η for τ) there, but because $j^\eta(\eta) = \eta$ we argued at (2) of the last theorem above, that $\widetilde{K}^\eta \models$ "$j(\mathcal{A}) = \bar{\jmath}(\mathcal{A})$ has a good set of indiscernibles $I \subseteq \eta$ of size η", and ultimately that $K \models$ "$\exists$ unboundedly many Ramsey cardinals". **Q.E.D.Theorem 1.3**

Theorem 1.4 *If $\exists j : M \underset{e}{\longrightarrow} V$, $j \neq id$ with stationary many fixed points of cofinality ω then $\{\alpha \,|\, K \models$ "α is measurable"$\}$ is a Mahlo class in K.*

Proof: Suppose not. It will suffice to show that K^M is universal by Lemma 1.5.

Lemma 1.5 *If there exists a stationary class S such that for all $\tau \in S$ τ is singular in a weasel W then W is universal.*

Proof: Suppose W is not universal. Then there exists a mouse M which iterates past W. By the Basic Coiteration Lemma of the introduction there is a cub class C of cardinals left behind as inaccessibles in the model W_{On}. But for any sufficiently large $\tau \in C \cap S$, we have $\pi^W_{0,\tau}$"$\tau \subseteq \tau$. By induction on $\gamma < \tau$ one sees that $(\tau \text{ singular})^W \Rightarrow (\tau \text{ singular})^{W_\gamma}$. This therefore holds in the direct limit at W_τ and as $crit(\pi^W_{\tau,On}) > \tau$ this means that $(\tau \text{ singular})^{W_{On}}$ - a contradiction. **Q.E.D.Lemma 1.5**

But for $\tau \in S$, by the Covering Lemma, τ is singular or measurable in K. Since $j(\tau) = \tau$, τ is singular or measurable in K^M. If the class of $\tau \in S$ that is measurable in K is non-stationary, by Lemma 1.5, K^M is universal. As we are below $O^{\P}$, K iterates simply up to K^M without K^M moving, with iteration map π_{KK^M}. But then $j \circ \pi_{KK^M}$ is a non-trivial elementary embedding from K to K which implies O^s.

Q.E.D.Theorem 1.4

A similar argument shows that if we weaken the assumption of the last theorem to that of a class of singular cardinals fixed by j, then K contains a proper class of measurable cardinals.

Theorem 1.6 *If $\exists j : M \xrightarrow[e]{} V$ $j \neq id$, with stationary many fixed points of cofinality $> \omega$ then O^s exists.*

Proof: As in the proof of the last theorem, it would suffice for a contradiction to show that K^M is universal. But $j(\tau) = \tau$ so τ is singular in K by the Covering lemmas of [9] and [5]. So $K^M \models \tau$ is singular. The result follows by Lemma 1.5. **Q.E.D.**

Theorem 1.7 *If $\exists j : M \xrightarrow[e]{} V$, $j \neq id$ and $\forall \alpha (cf(\alpha) = \omega \Rightarrow cf^M(\alpha) = \omega)$ then O^s exists.*

Proof: We first note that the assumption implies that there is a stationary class S of cardinals τ with $cf(\tau) = \omega$ which are fixed points of j. Consider the comparison of K with K^M with resulting coiteration $\langle \langle N_i, K_i^M \rangle | i \leq \theta \rangle$ for some $\theta \leq On$, with indices $\langle \nu_i | i < \theta \rangle$. Let the critical points of the ultrapowers used be $\langle \kappa_i | i < \theta \rangle$. Let $\alpha = crit(j)$. Clearly $cf(\alpha) > \omega$.

(1) On the K side of the coiteration, the very first index used, ν_0, is less than $\alpha^+ =_{df} \alpha^{+K}$, and hence involves a truncation to a set mouse $N_0 = K|\nu_0$.

Proof: Suppose the first index used on the K side were $\nu_0 \geq \alpha^+$. Then $\alpha^+ = \alpha^{+K^M}$ and by our assumption, and the Covering Lemma in M, $cf(\alpha^+) > \omega$. Then $\mathcal{P}(\alpha)^K = \mathcal{P}(\alpha)^{K^M}$ and if $\mathcal{U}$ is the derived measure from the embedding j (meaning that $\mathcal{U} = \{X \subseteq \alpha \,|\, X \in K \wedge \alpha \in j(X)\}$), then consider $\pi : K \longrightarrow \widetilde{K} =_{df} Ult(K, \mathcal{U})$. $\widetilde{K}$ is wellfounded as $\alpha^+ = \alpha^{+K}$, and $\mathcal{U}$ ω-complete. As we are below $O^{\P}$, π is an iteration map, $\pi_{0,\mu}$ for an iteration by full measures $\langle E_\nu | \nu \leq \mu \rangle$. The first measure, E_0, used in this iteration must have critical point $\alpha = crit(\mathcal{U})$, and so $E_0 = E^K_{\alpha^+}$; and it agrees with $\mathcal{U}$ on $\mathcal{P}(\alpha) \cap K$; so $\mathcal{U}$ must be $E^K_{\alpha^+}$.
Let $\sigma : K^M \longrightarrow K' =_{df} Ult(K^M, \mathcal{U})$, then $E^{K'}_{\alpha^+} = \emptyset$, by coherence. Defining $k : K' \longrightarrow_{\Sigma_0} K$ by $k([f]_{\mathcal{U}}) = j(f)(\alpha)$ we have as usual that $k \restriction \alpha^+ + 1 = id \restriction \alpha^+ + 1$, whilst $E^K_{\alpha^+} \neq \emptyset$, $E^{K'}_{\alpha^+} = \emptyset$ - a contradiction. QED (1)

We have as usual a cub class C and N_1 iterating past K^M, with $\pi_{N_i N_k}(\kappa_i) = \kappa_k$ for $i < k \in C$, whilst $\pi_{K_0^M, K_i^M}$"$\kappa_i \subseteq \kappa_i$.

However $C \cap S$ cannot form a stationary class of cardinals singular in K^M, as then we should have K^M universal - a contradiction as in the last theorem. Again the only other possibility is that it forms a class of measurables of K^M.

Let $\omega_2 < \kappa \in C \cap S$ with κ greater than the last truncation point on the iteration on the K side. Let $\kappa' = \kappa^{+K^M_\kappa}$. Then $K^M_\kappa | \kappa'$ is an initial segment of N_κ, and the latter is sound above κ. Then $\kappa' = \kappa^{+N_\kappa} = \sup \pi_{K^M, K^M_\kappa}$"$\kappa^{+K^M}$. Note

(2) $\rho^\omega_{N_\kappa} < \kappa$, and $\kappa > cf(\kappa') > \omega$.
Proof: By (1) there is at least one truncation on the K-side, and from this the first clause follows: since from the last truncation point i_0 say, we have $\rho^n_{N_{i_o}} \leq \kappa_{i_0}$ for some n. For this n we have $\rho^n_{N_\kappa} < \kappa$. As M calculates countable cofinalities correctly, by the Covering Lemma in M, $cf(\kappa^{+K^M}) = cf(\kappa^{+M}) > \omega$. QED (2)

Let the iteration $\langle K_i^M \mid i \leq \kappa\rangle$ with indices $\langle \nu_i | i < \kappa\rangle$ be copied (as in *Claim 1* of Theorem 1.1) *via* $j = \sigma_0$ to an iteration of $K = \widetilde{K}^0$ as $\langle \widetilde{K}^i \mid i \leq \kappa\rangle$ with commuting maps $\sigma_i : K_i^M \longrightarrow_e \widetilde{K}^i$ and indices $\langle \sigma_i(\nu_i) | i < \kappa\rangle$.

Let $\delta = \sup \sigma_\kappa$ "κ'. Then $\sigma_\kappa : K_\kappa^M|\kappa' \longrightarrow_{\Sigma_0} \widetilde{K}^\kappa|\delta$. Note that $\sigma_\kappa(\kappa') = \kappa^{+\widetilde{K}^\kappa} = \kappa^{+K} = \kappa^{+V}$ by the Covering Lemma in V, and the fact that the copy iteration below κ fixes κ and κ^{+K}. So we see $\delta < \kappa^{+\widetilde{K}^\kappa}$.

We now have to do a fine-structural lift-up argument, as this time we know that $\rho^{n+1}_{N_\kappa} < \kappa < \rho^n_{N_\kappa}$ for some $n \geq 0$. Fix this n.

(3) There is a lift-up map $\widetilde{\sigma} : N_\kappa \longrightarrow_{\Sigma^*} \widetilde{M}$ for some mouse $\widetilde{M}$ with $\widetilde{\sigma} \supseteq \sigma_\kappa \restriction K_\kappa^M|\kappa'$ where $\widetilde{M}$ has the following properties: (i) $\widetilde{M}|\delta = \widetilde{K}^\kappa|\delta$; (ii) $\rho^{n+1}_{\widetilde{M}} < \delta \leq \rho^n_{\widetilde{M}}$; (iii) $\widetilde{M}$ is the $\Sigma_1^{(n)}$-closure of $\kappa \cup \{p\}$ for some $p \in R^{n+1}_{\widetilde{M}}$.

Proof: This is a somewhat standard argument (*cf* [3] §3.3.2) exploiting the fact that $cf(\kappa') > \omega$ to get a wellfounded fine-structural "pseudo-ultrapower". Instead of the coarse pseudo-ultrapower of Claim 3 of Theorem 1.1 we define

$$D = \{\langle a, f\rangle \mid dom(f) \subset \tau < \kappa' \text{ and either } f \in \boldsymbol{\Sigma_1^{(m)}}(N_\kappa)(m < n) \vee f \in H^{N_\kappa}_{\rho^n_{N_\kappa}}\}$$

and then set

$$\langle a, f\rangle \, {}^{e}_{I} \langle b, g\rangle \Longleftrightarrow \langle a, b\rangle \in \sigma_\kappa(\{\langle u, v\rangle \mid f(u) \mathrel{\substack{\in \\ =}} g(v)\}$$

and

$$\dot{G}(\langle a, b\rangle) \Longleftrightarrow a \in \sigma_\kappa(\{u \mid f(u) \in G\}) \qquad (\text{for } G \in \{F, E\}).$$

We then define $\mathcal{D} = \langle D, I, e, \dot{E}, \dot{F}\rangle$ and proceed to prove a Łos Theorem for Σ_0 formulae. The fact that $cf(\kappa') > \omega$ results in e being wellfounded, and then $\mathcal{D}$ is isomorphic to $\widetilde{M}$ for some premouse $\widetilde{M}$. The canonical embedding $\widetilde{\sigma} : N_\kappa \longrightarrow \widetilde{M}$ is defined by $\widetilde{\sigma}(x) = [\langle 0, \{\langle x, 0\rangle\}\rangle]_I$. That $\widetilde{M}$ is iterable, and so a mouse, is an amplification of the argument that it is wellfounded (see [3] §3.3.3). Using that κ' is the N_κ-successor of κ enables one to prove that $\widetilde{\sigma}$ is a $\Sigma_1^{(n)}$ preserving embedding, and ultimately properties (ii) and (iii) ((i) holds by construction). The full proof is in §3.4.

QED (3)

(4) $\widetilde{M} \in \widetilde{K}^\kappa$; thus $\rho^\omega_{\widetilde{M}} = \rho^{n+1}_{\widetilde{M}} = \kappa$.

Proof: There is a $\boldsymbol{\Sigma_1^{(n)}}(\widetilde{M})$ subset of κ that codes $\widetilde{M}$. As $\widetilde{K}^\kappa$ is universal (being just a simple iterate of K), in their coiteration the first point of difference between $\widetilde{M}$ and $\widetilde{K}^\kappa$ is not less than δ, so we see that this code is in $\widetilde{K}^\kappa$. As κ is obviously a cardinal of $\widetilde{K}^\kappa$ we must have $\rho^\omega_{\widetilde{M}} = \rho^{n+1}_{\widetilde{M}} = \kappa$. QED (4)

(5) $core(\widetilde{M})$ is an initial segment of $\widetilde{K}^\kappa$. $core(\widetilde{M})|\delta + 1 = \widetilde{M}|\delta + 1$; hence $E^{\widetilde{M}}_\delta = E^{\widetilde{K}^\kappa}_\delta$.

Proof: Let $P =_{df} core(\widetilde{M})$. Then $P \cong \Sigma_1^{(n)\widetilde{M}}\mathrm{Hull}(\omega\rho^{\mathrm{n+1}}_{\widetilde{\mathrm{M}}} \cup \{\mathrm{p}_{\widetilde{\mathrm{M}}}\})$. (Where $p_{\widetilde{M}}$ is the standard parameter of $\widetilde{M}$.) But as $E^{\widetilde{M}}_\delta \neq \emptyset$, $\widetilde{M}$ is an iterate of P above δ. Let $\gamma \geq \delta$ be least so that $\rho^\omega_{\widetilde{K}^\kappa|\gamma} \leq \kappa$. Let $N = \widetilde{K}^\kappa|\gamma$. Then $N||\delta = \widetilde{M}||\delta = P||\delta$. But over both P and N there are Σ^*

definable subsets of κ coding wellorders of type δ. As $\delta = \kappa^+$ (or is the height of the ordinals) in both models, neither model contains such a code. So a simple comparison argument shows that P and N coiterate without truncation to the same model. But then by soundness, in fact $P = N$. QED (5)

Remark: With a little more work on parameters, one can show that in fact $\widetilde{M}$ is itself sound and an initial segment of $\widetilde{K^\kappa}$.

Let $\overline{F} = E_{\kappa'}^{K_\kappa^M}$. Let $\overline{M} = |K_\kappa^M||\kappa'| = J_{\kappa'}^{E^{K_\kappa^M}}$. Let F be defined by: $F = \bigcup_{X \in \overline{M}} \sigma_\kappa(X \cap \overline{F})$. Then:

(6) σ_κ extends to: $\sigma_\kappa \restriction \langle \overline{M}, \overline{F} \rangle \longrightarrow_{\Sigma_0} \langle \widetilde{K^\kappa}|\delta, E_\delta^{\widetilde{K^\kappa}} \rangle \ \wedge \ F = E_\delta^{\widetilde{K^\kappa}}$.
Proof: σ_κ is a cofinal map into $|\widetilde{K^\kappa}||\delta|$. And $F = |\widetilde{K^\kappa}||\delta| \cap E_{\sigma_\kappa(\kappa')}^{\widetilde{K^\kappa}}$. By the initial segment condition $F = E_\delta^{\widetilde{K^\kappa}}$. QED (6)

(6) shows, as $\tilde{\sigma} \supseteq \sigma_\kappa$, that $E_\delta^{\widetilde{K^\kappa}} = E_\delta^{\widetilde{M}} \Rightarrow E_{\kappa'}^{K_\kappa^M} = E_{\kappa'}^{N_\kappa}$. So the conclusion of this consequence holds by (5). But $E_{\kappa'}^{K_\kappa^M} \neq E_{\kappa'}^{N_\kappa}$ by virtue of the comparison process a contradiction!

Q.E.D.Theorem 1.7

Let K' be a core model for sequences of measures with $o(\kappa) < (\kappa)^{++}$ (for example Mitchell's Core Model).

Question If $\exists j : M \xrightarrow[e]{} V$ and $j \neq id$, α is a regular cardinal, and there exists stationary many cardinals of cofinality α fixed by j, and M computes correctly cofinalities $\leq \alpha$, then: (a) is this consistent?; or, (b), does $K' \models$ "$\exists$ unboundedly many measurable cardinals κ with $o(\kappa) = \alpha$" (where we set $o(\kappa) = 0$ if $\alpha = \omega$)?

2 Some converses

We briefly consider here some partial converses to the above. It might be thought at first that with some extra work, one could obtain from a non-trivial elementary embedding $j : M \to V$, an inner model with a measurable cardinal. The following indicates that this is not possible. Suppose that κ is a strongly inaccessible Jónsson cardinal, and that there is $A \subseteq \kappa$ with $H_\lambda = L_\lambda[A]$ for cardinals $\lambda < \kappa$, and (by the Jónsson property) that there is a non-trivial $j : L_\kappa[\overline{A}] \longrightarrow L_\kappa[A]$, then we clearly have candidates for a "$j : M \longrightarrow V$" by taking $M = L_\kappa[\overline{A}]$ and V as $L_\kappa[A]$. And such a Jónsson cardinal is known to be equiconsistent with a Ramsey cardinal. See [6]. Notice also that this argument shows that if κ, $j : L_\kappa[\overline{A}] \longrightarrow L_\kappa[A]$ are as above, then we can perform a "lift-up" argument (as in Claim 3 of Theorem 1.1) to extend j to $\tilde{j} : L[\overline{A}] \longrightarrow L[A]$ (using the regularity of κ). We can also think of this argument as taking an extender ultrapower of $L[\overline{A}]$, $Ult(L[\overline{A}], E_j)$ where E_j is the long extender derived from the embedding j; by the regularity of κ this is wellfounded. But then V cannot be $L[A]$ since otherwise $E_j \in V$ and the latter would be an ultrapower of the inner model $L[\overline{A}]$! Hence:

Lemma 2.1 *Suppose $V = L[A]$ for some set $A \subseteq \kappa$ with κ strongly inaccessible.*

(i) If $j : M \longrightarrow V$, (or even if $j \upharpoonright L_\kappa[\overline{A}] \longrightarrow L_\kappa[A]$) then $j = id$.

(ii) κ is not a Jónsson cardinal.

Instead of arguing in terms of the Jónsson property, one can frame a hypothesis of the form "Suppose On is Ramsey ...." (or again reformulation in terms of some V_κ where κ is a Ramsey cardinal etc.).

Definition 2.1 *(i) On is* Ramsey *if there is a class $I \subseteq On$, unbounded, of good indiscernibles for $\langle V, \in \rangle$,*

*(ii) On is 1-*Ramsey*, if the class I in (i) is Mahlo.*

In (ii) we mean that I has non-empty intersection with any closed and unbounded class $C \subset On$, definable (with parameters allowed) over $\langle V, \in, I \rangle$.

Notice that if κ is a Ramsey cardinal, then we have an $I \subseteq \kappa$ so that in $\langle V_\kappa, \in I \rangle$, On is Ramsey in this sense. Of course, if On is Ramsey we have, definably over $\langle V, \in, I \rangle$, a satisfaction relation for $\langle V, \in \rangle$, since $\alpha < \beta \in I \Rightarrow V_\alpha \prec V_\beta$. Further, using AC, we may define skolem functions, and so skolem hulls for $\langle V_\alpha, \in \rangle$, so that $\alpha < \beta \in I \Rightarrow \mathrm{Hull}_{V_\beta}(\gamma) = \mathrm{Hull}_{V_\alpha}(\gamma)$ for $\gamma < \alpha$. A prototypical situation is again where κ is Ramsey in K and then all the above hold over $\langle K_\kappa, \in I \rangle$.

Theorem 2.2 *(i) Suppose $I \subseteq On$ witnesses On is Ramsey. Then, definably over $\langle V, \in, I \rangle$, there is a transitive class M, and an elementary embedding $j : \langle M, \in \rangle \longrightarrow \langle V, \in \rangle$, with $j \neq id$.*

(ii) Additionally, if I witnesses On is 1-Ramsey, then j can be taken with unboundedly many regular fixed points.

Proof: The proof of the proposition is straightforward. Let $\alpha_0 = \min I$. By the remarks above we may define in $\langle V, \in, I \rangle$ the hulls $H_\beta =_{df} \mathrm{Hull}_{V_\beta}((I \cap \beta \backslash \{\alpha_0\}) \cup \alpha_0)$ for $\beta \in I$, and their transitive collapses with maps $\pi_\beta : M_\beta \longrightarrow H_\beta$. By the usual properties of good indiscernibles, it is easy to verify that crit $\pi_\beta = \alpha_0$, and that $\alpha < \beta \in I \Rightarrow \pi_\beta \upharpoonright M_\alpha = \pi_\alpha$. Hence we may set $j = \bigcup_{\beta \in I} \pi_\beta$, $M = \bigcup_{\beta \in I} M_\beta$. This concludes (i). For (ii), simply note that there must be, stationarily often, be inaccessible $\delta \in I$ with $j(\delta) = \delta$. **Q.E.D.**

Part (i) of the last theorem can be rephrased as a relative consistency result from the existence of a Ramsey cardinal. This shows the answer to the question of Kunen is negative.

A number of questions about the possibilities of finding models which satisfy further closure properties are immediate. For example:

Question 1 Can one arrange for there to be an inner model M and a non-trivial $j : M \longrightarrow V$, with stationary many fixed points of cofinality $\geq \omega$?

Question 2 The same question as for 1, but with M correctly computing the cofinality of ordinals β with $cf(\beta) = \omega$?

Our original Theorem 1.7 had the hypothesis that ${}^\omega M \subseteq M$. Matt Foreman pointed out that this was impossible. We thank him for letting us include his proof here.

Theorem 2.3 *(Foreman) If* $\exists j : M \longrightarrow_e V,\ j \neq id$ *then* ${}^{\omega}M \not\subseteq M$.

Proof: Suppose for a contradiction that there exists such an M and j satisfying also the conclusion. Let $\alpha = crit(j)$. By the supposed closure on M it is easy to see that there are $\alpha < \lambda_0 \ldots < \lambda_n < \ldots$ for $n < \omega$ with each λ_n a singular cardinal of cofinality ω and $j(\lambda_n) = \lambda_n$. Then $\langle \lambda_n \rangle_{n<\omega} \in M$ (again by ω-closure). We show that there is a regular fixed point. Consider any ultrapower of $\langle \lambda_n \rangle_{n<\omega}$ by a non-principal ultrafilter $\mathcal{U} \in M$ on ω. Notice that by ω-closure of M it makes no difference whether this ultrapower is considered as taken in M or V. Let κ be its cofinality.

(1) κ is a regular fixed point of j.

Proof: Note that κ is a supremum of fixed points of j. Also κ is definable in M from $\langle \lambda_n \rangle_{n<\omega}$ and $\mathcal{U}$. Hence $j(\kappa) = \kappa$. But $cf(\kappa) = \kappa$ in M, and hence also in V. **Q.E.D.(1)**

We now construct in M an ω-Jónsson algebra on κ. Let $S = \{\beta < \kappa \mid cf(\beta) = \omega\}$. Split S into κ many disjoint stationary pieces of size κ: $S = \cup_{\delta<\kappa} X_\delta$ with $\gamma \neq \delta \longrightarrow X_\gamma \cap X_\delta = \emptyset$; and X_γ stationary in κ. For any $\langle \beta_n \rangle_{n<\omega}$ with $\beta_n < \beta_{n+1}$ for all n, define $F(\langle \beta_n \rangle) =_{df}$ that γ_0 so that $\sup \beta_n \in X_{\gamma_0}$. This defines $F : {}^{\omega}[\kappa] \to \kappa$.

(2) F is an ω-Jónsson algebra in M.

Proof: Let $Y \subseteq \kappa$ be arbitrary. Let $Y^* =_{df} \{\sup f``\omega \mid f \in {}^{\omega}[Y]\}$. The $Y^*$ is an $\omega$-club in $\kappa$, (*i.e.* is unbounded and closed under $\omega$ sequences) and so $Y^* \cap X_\gamma \neq \emptyset$ for any $\gamma < \kappa$. Hence $F``{}^{\omega}[Y] = \kappa$. **Q.E.D.(2)**

Clearly $\tilde{F} =_{df} j(F)$ is an ω-Jónsson algebra in V. Now look at $Y = j``\kappa$. By the above there is $\beta_n = j(\gamma_n)$ for $n < \omega$ with $\langle \gamma_n \rangle \in M$, and $\tilde{F}(\langle \beta_n \rangle) = \alpha$. This is absurd as it implies $\alpha = j(F(\langle \gamma_n \rangle))$ whereas $\alpha = \mathrm{crit}(j) \notin \mathrm{ran}(j)$. **Q.E.D.**

3 Some fine structure

3.1 The Jensen Indiscernibles Lemma

We outline here a proof of the following lemma, due to Jensen, using the fine-structural hierarchy of [3]. The original lemma was proven using the older-style mice. It will establish (Cor. 3.3) not unsurprisingly, the downwards absoluteness of various Ramsey and Erdős properties between V and any universal weasel. We shall also use it to give variant proofs of Theorems 1.1 & 1.3. We assume throughout §3.1 that O^s does not exist (otherwise every cardinal is measurable in K and our results become trivially true).

Lemma 3.1 *Let κ be a cardinal and suppose $A \in K \cap \mathcal{P}(\kappa)$, and that there is I, an infinite good set of indiscernibles for $\mathcal{A} = \langle L_\kappa[E], E, A\rangle$, (where $E = E^K$) and that cf $(otp(I)) > \omega$. Then there is $I' \in K$, $I' \supseteq I$ a set of good indiscernibles for $\mathcal{A}$.*

Proof: Standard arguments show that each $\gamma \in I$ is strongly inaccessible in K, and hence $\mathcal{A} \upharpoonright \gamma = \langle L_\gamma[E], E \upharpoonright \gamma,\ A \cap \gamma\rangle \models ZFC$. Closely following the original proof of [2], we set $\widetilde{\mathcal{A}}^n_\gamma = \mathrm{Hull}_{\mathcal{A}}(\gamma + 1 \cup \vec{\gamma})$ where $\vec{\gamma}$ is any member of $[I \backslash \gamma + 1]^n$; $\mathcal{A}^n_\gamma = \widetilde{\mathcal{A}}^n_\gamma \cap K_{\gamma^+}$ $\quad (\gamma^+ = \gamma^{+K}$ throughout this proof); $\mathcal{A}_\gamma = \cup_n \mathcal{A}^n_\gamma$; $\delta^n = On \cap \mathcal{A}^n_\gamma$.

Note δ^n makes sense as $\mathcal{A}^n_\gamma$ is transitive, and δ^n is definable in $\mathcal{A}^{n+1}_\gamma$, and $(\mathcal{A}^n_\gamma \prec \mathcal{A}^{n+1}_\gamma \prec \mathcal{A}_\gamma)$. Hence:

(1) $\delta^n < \delta^{n+1}$.

For $\gamma < \gamma' \in I$ we define $\pi^n_{\gamma\gamma'} : \mathcal{A}^n_\gamma \longrightarrow_e \mathcal{A}^n_{\gamma'}$ by applying shift maps on indiscernibles, to terms in the language appropriate to $\mathcal{A}$: $\quad \pi^n_{\gamma\gamma'}(t_{\mathcal{A}}(\vec{\nu}, \gamma, \vec{\gamma})) = t_{\mathcal{A}}(\vec{\nu}, \gamma', \vec{\gamma})$ for any choice of $\vec{\gamma} \in [I \backslash \gamma' + 1]^{<\omega}$.

Define $\mathcal{U}^n_\gamma = \Big\{X \in \mathcal{P}(\gamma) \cap \mathcal{A}^n_\gamma | \gamma \in \pi^n_{\gamma\gamma'}(X), \text{ for some } \gamma' > \gamma,\ \gamma' \in I\Big\}$. Then $\mathcal{U}^n_\gamma \in \mathcal{A}^{n+1}_\gamma$ and, if we set $\mathcal{U}_\gamma = \cup_n \mathcal{U}^n_\gamma$, then we conclude:

(2) $\langle \mathcal{A}_\gamma, \mathcal{U}_\gamma\rangle$ is amenable.

Set $\gamma^* = \sup I$, and

$$\langle\langle A_{\gamma^*}, \mathcal{U}_{\gamma^*}\rangle, \langle \pi_{\gamma\gamma^*}\rangle_{\gamma \in I}\rangle = \varinjlim \langle\langle A_\gamma, \mathcal{U}_\gamma\rangle, \langle \pi_{\gamma\gamma'}\rangle_{\gamma < \gamma' \in I}\rangle$$

where $\pi_{\gamma\gamma'} = \cup_n \pi^n_{\gamma\gamma'}$. As $\mathcal{U}_{\gamma^*}$ is generated by I and $\mathrm{cf}(otp(I)) \geq \omega_1$:

(3) $\mathcal{U}_{\gamma^*}$ is ω-complete.

Let $\mathcal{A}_{\gamma^*} = \langle L_{\beta^*}[E^*],\ E^*,\ A^*\rangle$. Clearly $A^* \cap \gamma^* = A \cap \gamma^*$, $E^* \upharpoonright \gamma^* = E \upharpoonright \gamma^*$, and, using (2), $\langle L_{\beta^*}[E^*],\ \mathcal{U}_{\gamma^*}\rangle$ is amenable, and, using (3), iterable. Further $E \upharpoonright \gamma,\ A \cap \gamma \in \ \mathcal{A}^*_\gamma \models ZF^-$; hence:

(4) $H^K_{\gamma^*} = |L_{\gamma^*}[E^*]|$.

For $\eta \leq \beta^*$, let $\mathcal{U}_\eta = \mathcal{U}_{\gamma^*} \cap J^{E^*}_\eta$ and let $N_\eta = \langle J^{E^*}_\eta,\ E^* \upharpoonright \eta,\ \mathcal{U}_\eta\rangle$. Now let $\beta \leq \beta^*$ be least so that a) $A \cap \gamma^* \in J^{E^*}_\beta$ b) $N = \langle J^{E^*}_\beta,\ E^* \upharpoonright \beta,\ \mathcal{U}_{\gamma^*}\rangle$ is amenable. Then recall that N_η is a *quasi-premouse* if it satisfied all conditions for premousehood except possibly the Initial Segment Condition. A quasimouse is an iterable quasi-premouse.

(5) $N = N_\beta$ is a quasimouse.
Proof: $N \models \gamma^*$ is the largest cardinal, and iterability of N comes from the ω-completeness of $\mathcal{U}_{\gamma^*}$ If $\pi : N \longrightarrow P =_{df} Ult(N, \mathcal{U}_{\gamma^*})$ then by amenability $\gamma^{*+P} = On \cap N = \omega\beta$. For coherency notice that $E^P_{\omega\beta}$ must be empty else $\langle J^{E^P}_\beta,\ E^P,\ E^P_{\omega\beta}, \mathcal{U}_{\gamma^*}\rangle$ is a "sword" mouse. QED (5)

But then N actually is a mouse as its terminating coiteration with K (or more precisely, the ω-full weasel W built over $K|\kappa = N|\kappa$ (*cf* [3] §3.2) shows). By definition of β, $\rho^1_N \leq crit(\mathcal{U}_{\gamma^*}) = \kappa$. The coiteration just mentioned shows A^1_N, and so N, are in K. As $A \cap \gamma^* \in |N|$, and $\mathcal{U}_{\gamma^*}$ is ω-complete, we now follow the sketch at Claim 5 of Theorem 1.1 above, and we can define over the amenable N a sequence of sets $X_n \in \mathcal{U}_{\gamma^*}$ whose intersection $X \supseteq I$ and X is a good set of indiscernibles for $\langle L_{\gamma^*}[E \restriction \gamma^*],\ A \cap \gamma^* \rangle$. (For the full details here see the argument of [1] p.131 for example). Then again $I' \in K$, and we are done.

Q.E.D.Lemma 1.10

Corollary 3.2 *Let $\overline{K} = L[\overline{E}]$ be any universal weasel. Then the last lemma remains true with $\overline{K}$, $\overline{E}$ replacing K, E.*

Proof: The previous construction works perfectly well using the predicate $\overline{E}$. Universality of K was used to argue that the N there constructed was in K. As $\rho^1_N \leq \gamma^*$, and $N|\gamma^* = \overline{K}|\gamma^*$ here, we shall have here $N \in \overline{K}$, and so this suffices.

Q.E.D.Cor.3.2

Corollary 3.3 *The following are downward absolute between V and any universal weasel W: "κ is Ramsey", "κ is almost Ramsey", "κ is τ-Erdős" where $cf^V(\tau) \geq \omega_1$.*

We now show how Theorem1.1 can be derived using the indiscernibles lemma: in the notation of the proof of Theorem 1.1 we have a mouse $N_{\overline{\tau}}$ iterating past the final model $\overline{K}$ on the K^M-side. Let $\tau > \overline{\tau}$ fulfill conditions (i) - (iv). Let $I = \{\kappa_\gamma | \overline{\tau} < \gamma < \tau\}$ be the cub in τ set of iteration points used in the iteration of $\pi^N_{\overline{\tau},\tau} : N_{\overline{\tau}} \longrightarrow N_\tau$. These are then good indiscernibles for $\langle \overline{K}_\tau, \overline{E}, \overline{A} \rangle$ where $\overline{A} \in \mathcal{P}(\tau)^{\overline{K}}, \left(= \mathcal{P}(\tau)^{N_\tau}\right)$, $\overline{K} = L[\overline{E}]$. Continuing with the notation of Theorem1.1, if $\overline{\jmath} : \overline{K} \longrightarrow \widetilde{K}$, $\overline{\jmath}(\overline{A}) = A$, $\overline{\jmath}``I = I' \subseteq \tau$, then I' is a set of good indiscernibles for $\langle \widetilde{K}_{\overline{\jmath}(\tau)}, \widetilde{E}, A \rangle$ (setting $\overline{\jmath} = j^{\overline{\tau}}$, $\widetilde{E} = E^{\widetilde{K}}$) with $\mathrm{cf}(otp(I')) > \omega$.

Applying Corollary 3.2 we have $\widetilde{K} \models \forall \tau' < \tau \exists\ I' \subseteq \tau\ (otp(I') \geq \tau' \wedge I'$ is good for $\widetilde{\mathcal{A}} = \langle \widetilde{K}_{\overline{\jmath}(\tau)}, \widetilde{E}, A \rangle)$.

As $\overline{\jmath}``\tau \subseteq \tau$, by elementarity we have $\overline{K} \models \forall \tau' < \tau \exists I' \subseteq t(otp(I') \geq \tau' \wedge I'$ is good for $\mathcal{A} = \langle \overline{K}, \overline{E}, \overline{A} \rangle)$.

As $\overline{A}$ was arbitrary we thus have $\overline{K} \models$ "τ is almost Ramsey". There are thus arbitrarily large τ for which this is true. By elementarity, the same is true K^M and, via j, hence in K.

To see that Theorem 1.3 is true, pick a sufficiently large regular cardinal $\tau > \overline{\tau}$, satisfying (i)-(iv), and so that $j(\tau) = \tau$. Now in the above analysis $\overline{\jmath}(\tau) = \tau$ and so in $\widetilde{K}$ we shall have a τ-sequence of good indiscernibles for $\langle \widetilde{K}_\tau, \widetilde{E}, A \rangle$. Hence $\widetilde{K} \models$ "τ is Ramsey". By elementarity of $\pi^{K^M}_{0\tau}$ and j, τ is then Ramsey in K.

3.2 Σ^* Fine Structure

In what follows we assume that $M = \langle J^A_\alpha, \in, A, B\rangle$ is an amenable structure with J^A_α a level of a relativised J-hierarchy, (a "J-model"), that satisfies the axiom of acceptability.

Definition 3.1 *A J-model $M = \langle J^A_\alpha, \in, A, B\rangle$ as above is* acceptable *if* $\forall \gamma < \alpha\, \mathcal{P}(\omega\tau) \cap J^A_{\gamma+1} \not\subseteq J^A_\gamma \longrightarrow J^A_{\gamma+1} \models \mathrm{card}(\gamma) \leq \tau$.

Definition 3.2 *ρ_M (the Σ_1-projectum of M) $=_{df}$ the least ρ such that for some $p \in M$ there is a $\Sigma_1^M(\{p\})$ definable class A with $A \cap \omega\rho \notin M$.*

Remark: ρ_M is a Σ_1-cardinal of M.

Definition 3.3 *$P_M =_{df} \{p \in M \mid$ there is such a class A as above with $\Sigma_1^M(\{p\})$*
$R_M =_{df} p \in M \mid h_M$"$(\omega \times ([\omega\rho]^{<\omega} \times \{p\}) = |M|\}$.

Remark: We use here the notation that h_M is the canonical Σ_1-skolem function for such a J-model M, and we abbreviate by $h(X)$ the set h"$(\omega \times [X]^{<\omega})$.

Let $\langle \varphi_i \mid i \in \omega\rangle$ enumerate the Σ_1-formulae of the language $\mathcal{L}_{\{\dot\in, \dot=, \dot A, \dot B\}}$ with two free variables.

Definition 3.4 $A^p_M =_{df} \{\langle i, x\rangle \in \omega \times J^A_{\rho_M} \mid M \models \varphi_i(x, p)\}$

By using Gödel pairing and the remark above one can easily see that $p \in P_m \iff A^p_M \cap (\omega \times \omega\rho) \notin M$. Note also that $R_M \subseteq P_M \neq \emptyset$.

Definition 3.5 $M^p =_{df} \langle J^A_{\rho_M}, A^p_M\rangle$ *for any $p \in M$.*

We iterate this idea and define simultaneously by recursion on $n < \omega$:

Definition 3.6 *The n-*th projectum *$\rho^n = \rho^n_M$, the set $\Delta^n = \Delta^n_M$ of possible* parameter sequences*, and the n-*th reduct *$M^{n,p}$ of M for $p \in \Delta^n$, and $H^n = H^n_M$.*

$$\rho^0_M = \omega\alpha = On \cap M; \quad \Delta^0 = \{\emptyset\}; \quad M^{0,\emptyset} = M; \quad H^0_M = |M|; \quad A^{0,p} = \emptyset;$$
$$\rho^{n+1} = \rho^{n+1}_M = min\{\rho_{M^{n,p}} \mid p \in \Delta^n\}; \quad \Delta^{n+1} = \Pi_{i<n+1} H^i;$$

For $p \in \Delta^{n+1}$:

$$M^{n+1,p} = (M^{n,p\restriction n})^{p(n)} \mid \rho^{n+1} =_{df} \langle J^A_{\rho^{n+1}}, A^{p(n)}_{M^{n,p\restriction n}} \cap J^A_{\rho^{n+1}}\rangle$$
$$H^{n+1} = |J^A_{\rho^{n+1}}|; \quad A^{n+1,p} =_{df} A^{p(n)}_{M^{n,p\restriction n}} \cap J^A_{\rho^{n+1}};$$
$$\rho^\omega_M = \min_n \rho^n_M.$$

We then organise the possible parameter sequences as follows:

Definition 3.7 $R^0 = P^0 = \{\emptyset\}$;
$$P^{n+1} = \{p \in \Delta^{n+1} \mid p \restriction n \in P^n \wedge \rho^{n+1} = \rho_{M^{n,p\restriction n}} \wedge p(n) \in P_{M^{n,p\restriction n}}\};$$
$$R^{n+1} = \{p \in \Delta^{n+1} \mid p \restriction n \in R^n \wedge \rho^{n+1} = \rho_{M^{n,p\restriction n}} \wedge p(n) \in R_{M^{n,p\restriction n}}\}.$$

The sequences $p \in P^n_M$ are those parameter sequences where additional components $p(i)$ are "good" for defining new subsets of $\omega\rho^{i+1}$ over the ith reduct with respect to the previous components $p \restriction i$, $M^{i,p\restriction i}$. These always exist. Those $p \in R^n_M$ perform a similar function, but in addition, the Σ_1 Skolem function of this i'th reduct, acting on $\omega\rho^{i+1} \cup \{p(i)\}$ recovers all of that reduct. These are clearly "very good" parameters, and will not exist in general. We do have the following facts:

Fact 3.1 *(i)* $R^n \subseteq P^n \neq \emptyset$;
(ii) if $p \in R^n$ *and* $q \in \Delta^n$ *then* $A^{n,q}_M$ *is rudimentary in* $A^{n,p}_M$ *in parameters from* H^n_M.
(iii) If $p \in R^n$ *then* $\rho^{n+1} = \rho_{M^{n,p}}$; *if* $p \in R^{n+1}$ *then* $M^{n+1,p} = (M^{n,p})^{p(n)}$;
(iv) $p \in R^n \longleftrightarrow p \in \Delta^n \wedge \forall i < n\, p(i) \in R_{M^{i,p\restriction i}}$. *(v)* $\rho^{n+1} = \min\{\rho_{M^{n,p}} \mid p \in P^n\}$.

Definition 3.8 *If* $R^n = P^n$ *then we say* M *is* n-sound. M *is* sound *if it is* n*-sound for all* n. *It is* sound above μ *if*
(a) $\rho^n_M \geq \mu \longrightarrow M$ *is* n*-sound, and*
(b) if $\rho^n_M \geq \mu > \rho^{n+1}_M \wedge p \in P^{n+1}_M$ *then* $M^{n,p\restriction n} = h_{M^{n,p\restriction n}}(\mu \cup \{p(n)\})$.

Notions of "fine-structural" preserving maps (and ultimately ultrapowers) can be smoothly presented in terms of Jensen's Σ^* hierarchy of formulae. This is a hierarchy of definability over a J-model, with a different order of stratification than the usual Levy hierarchy of formulae. Although the relations Σ^*-definable over M are also the usual Σ_ω-definable relations, the intermediate levels of Σ_n-definability do not in general correspond to those within Σ^*. More complex initially, the Σ^*-hierarchy possesses nice properties the Levy hierarchy lacks: for example, over L, Jensen's Σ_n-uniformization Theorem states that such relations may be Σ_n-uniformized albeit *in a parameter*. When given the Σ^*-analysis, $\Sigma^{(n)}_1(J_\alpha)$ relations (to be defined) enjoy uniform $\Sigma^{(n)}_1(J_\alpha)$ Uniformizing functions. (See the two lemmas stated at the end of this subsection.)

We now define this language $\mathcal{L}^*_{\{\dot\in, \dot=, \dot A, \dot B\}}$. This has variables v^i_j $(i, j < \omega)$ of *type* i. The atomic formulae are those of the form $v^i_j \in \dot A(\dot B)$, $v^i_j \in v^k_l$, $v^i_j = v^k_l$. The formulae are those obtained from the atomic formulae by closing under $\neg, \wedge$ and typed quantification $\exists v^i_j, \forall v^i_j$. The stratification we referred to earlier is as follows:

Definition 3.9 *(a) The* $\Sigma^{(n)}_0$ *formulae, is the smallest class* $\Sigma \subset \mathcal{L}^*$ *such that:*
(i) Σ *contains the atomic formulae, and the formulae* $\Sigma^{(m)}_1$ *for any* $m < n$;
(ii) Σ *is closed under* $\neg, \wedge$, *and quantification which binds variables of type* n *by a higher type, that is if* $\varphi \in \Sigma$, *so is* $\forall x^n \in y^m\varphi$, $\exists x^n \in y^m\varphi$, *where* $m \geq n$;
(b) $\Sigma^{(n)}_k$ *formulae are obtained by alternating* k *blocks of quantifiers of type* n: $\exists\vec{x}^n_1\forall\vec{x}^n_2 \ldots Qx^n_k\varphi$ *where* $\varphi \in \Sigma^{(n)}_0$.
We set $\Pi^{(n)}_0 = \Sigma^{(n)}_0$ *and* $\Sigma^* = \bigcup_n \Sigma^{(n)}_0 \quad (= \bigcup_n \Sigma^{(n)}_1)$.

The variables v^n_m are intended to range over $H^n_M =_{df} H^M_{\omega\rho^n_M}$ of an acceptable J-model M. Note that $n \leq m \longrightarrow \rho^n_M \geq \rho^m_M$ and hence that v^m_i will vary over a smaller domain than v^n_j. The notion of $\Sigma^{(n)}_k$ relation and function are defined fairly straightforwardly, using these formulae. A full analysis of such relations and functions would take us too far afield at this point, however we shall refer to the *good* $\Sigma^{(n)}_1$*-functions* as those functions that may be substituted into a $\Sigma^{(n)}_1$ relation and still yield a $\Sigma^{(n)}_1$ relation. We shall not use the following definition, but they may be defined as follows:

Definition 3.10 *The* good $\Sigma_1^{(n)}(M)$*-functions form the smallest class S, such that, setting $i, j_1, \ldots, j_k \leq n$*
(i) Each partial $\Sigma_1^{(i)}(M)$ function $F(x_k^{j_k}, \ldots, x_1^{j_1}) = x^i \in S$;
(ii) If $F(x_k^{j_k}, \ldots, x_1^{j_1}) = x^i \in S$ and $G_i(\vec{y}) = z^{j_i} \in S$ (and the $\vec{y}$ all have type $\leq n$), then $F(G_k(\vec{y}), \ldots, G_1(\vec{y})) \in S$.

We may define $\Sigma_k^{(n)}$-preserving embeddings in a natural way:

Definition 3.11 *Let $\overline{M}, M$ be J-models, $n, l < \omega$.*
(i) $\pi : \overline{M} \longrightarrow_{\Sigma_l^{(n)}} M$ iff $\pi : \overline{M} \longrightarrow M$ and whenever $\varphi(v_1^{j_1}, \ldots v_m^{j_m}) \in \Sigma_l^{(n)}$, $x_i \in H_{\overline{M}}^{j_i}(1 \leq i \leq m)$ then $\pi(x_i) \in H_M^{j_i}$, and $\overline{M} \models \varphi(\vec{x}) \longrightarrow M \models \varphi(\vec{\pi(x)})$.
(ii) $\pi : \overline{M} \longrightarrow_{\Sigma^*} M$ iff $\pi : \overline{M} \longrightarrow_{\Sigma_l^{(n)}} M$ for all $n < \omega$.
(iii) $\pi : \overline{M} \longrightarrow_{\Sigma_0^{(n)}} M$ cofinally *iff $\pi : \overline{M} \longrightarrow_{\Sigma_0^{(n)}} M$ and $H_M^n = \bigcup \pi``H_{\overline{M}}^n$.*

Fact 3.2 *If F is a good $\Sigma_1^{(n)}(\overline{M})$ definable function, then one may show that F has such a definition that is "functionally absolute", that is a definition that is robust under $\Sigma_1^{(n)}$ preserving maps. In other words, the definition applied over M where $\pi : \overline{M} \longrightarrow_{\Sigma_l^{(n)}} M$ also yields a $\Sigma_1^{(n)}$ function.*

Some of the fruits of this analysis are seen in the following two lemmas.

Lemma 3.4 *Let $R(y^n, x_1^{j_1}, \ldots x_m^{j_m}) \in \Sigma_1^{(n)}(M)$ with $j_i \leq n$. Then there is a good $\Sigma_1^{(n)}(M)$ function F (uniformly definable with respect to any such M) into H_M^n uniformising R. Namely*
a) $domF = \{\langle x_1^{j_1}, \ldots x_m^{j_m}\rangle \in M \mid \exists y^n \in M R(y^n, \vec{x})\}$.
b)$\exists y^n R(y^n, \vec{x}) \longleftrightarrow R(F(\vec{x}), \vec{x})$.

Lemma 3.5 *Let $n < \omega$. There is a $\Sigma_1^{(n)}$ formula defining a good $\Sigma_1^{(n)}(M)$ function $F_n(u, v)$ into M, (definable uniformly with respect to any J-model M) so that*

$$\forall p(p \in R_M^{n+1} \longrightarrow |M| = F_n``H_M^{n+1} \times \{p\}).$$

3.3 Σ^*-ultrapowers

We give an account of the formation of a fine-structure preserving ultrapower. We shall develop this theory for ultrapowers by extenders that are weakly amenable with respect to the models concerned. This is more than one needs for the models in the previous sections, as these were all dealing with premise with measures of order 0. However, this greater generality now will enable us to quickly dispose of "pseudo-ultrapowers" in the next subsection. In any case it is still basically the account of such ultrapowers from [3]. Suppose that $\overline{M} = \langle J_\alpha^A, A, B\rangle$ is a J-model and E is an extender with critical point $crit(E) = \kappa$ over $\overline{M}$ and $E = \langle E_a \mid \alpha \in [\nu]^{<\omega}$. We assume that E is *weakly amenable* with respect to $\overline{M}$. This is defined as for any

$a \in [\nu]^{<\omega}$, for any $\langle X_\alpha \mid \alpha < \kappa \rangle \in \overline{M}$, $\{\alpha \mid X_\alpha \in E_a\} \in \overline{M}$. This is equivalent to saying that $\mathcal{P}(\kappa) \cap \overline{M} = \mathcal{P}(\kappa) \cap Ult(\overline{M}, E)$ (where the latter is the usual "Σ_0-ultrapower" of $\overline{M}$ using functions belonging to $\overline{M}$).

As is well known there is a Σ_0 and cofinal embedding $\pi : \overline{M} \longrightarrow_E M = Ult(\overline{M}, E)$. In general this is as much definable preservation that one could hope for. The notion of Σ^*-ultrapower involves using functions that lie outside of $\overline{M}$ but that are definable classes using formulae of the language $\mathcal{L}^*$. Let $\overline{M}$ and E be as above.

Definition 3.12 *The relation $\pi : \overline{M} \longrightarrow_E^* M$ holds iff the following conditions are met:*
(i) $\pi : \overline{M} \longrightarrow_{\Sigma_0^{(m)}} M$; where M is transitive, for any m so that $\omega\rho_M^m > \kappa$.
(ii) Let $\rho_{\overline{M}}^n = \min\{\rho_M^m \mid \rho_M^m > \kappa\}$. Let $H = \bigcup_{x \in H_{\overline{M}}^n} \pi(x)$. Then $\pi \restriction H_{\overline{M}}^n : H_{\overline{M}}^n \longrightarrow_E H$.
(iii) Additionally if there is k with $\kappa < \omega\rho_{\overline{M}}^{k+1}$, then M is the closure of $H \cup ran(\pi)$ under good $\Sigma_1^{(k)}(M)$ functions.

In most situations we shall want M to be wellfounded (hence the definition), but sometimes it is convenient to ask only that the wellfounded core of M (which will be assumed transitive) contains ν. Just by the notation one has straightaway that

Fact 3.3 *(i) If $\omega\rho_{\overline{M}}^1 < \kappa$ then $\pi : \overline{M} \longrightarrow_E M$ iff $\pi : \overline{M} \longrightarrow_E^* M$;*
(ii) If $\omega\rho_{\overline{M}}^\omega > \kappa$ then $\pi : \overline{M} \longrightarrow_E^* M$ implies $\pi : \overline{M} \longrightarrow_{\Sigma^*} M$.

We shall see that the existence of such a π, and an M with $\pi : \overline{M} \longrightarrow_E^* M$ satisfying the above, implies that such π and M are unique.

Definition 3.13 $\Gamma = \Gamma(\kappa, \overline{M})$ *is the set of functions f with dom $(f) = [\kappa]^n$ (some $n < \omega$) and either f is a good $\Sigma_1^{(n)}(\overline{M})$ function for some n with $\omega\rho_{\overline{M}}^{n+1} \geq \kappa$, or $f \in \overline{M}$.*

The extension of π to elements of Γ is effected as follows: if $f \in \Gamma$ and $dom(f) = \kappa$, and f is good $\boldsymbol{\Sigma_1^{(m)}}(\overline{M})$, where $\omega\rho_{\overline{M}}^{m+1} > \kappa$ then the statement that any two functionally absolute definitions define the same function is $\boldsymbol{\Pi_0^{(m+1)}}$. Under the definition above, then, this will carry over to M, and define a unique function f' over M. We then may set $\pi(f) = f'$. This will then imply that

(1) $M = \{\pi(f)(a) \mid f \in \Gamma,\, a \in [\nu]^{<\omega}\}$.

From this (using a Łos theorem for $\Sigma_0^{(m+1)}$-formulae) one then derives immediately the uniqueness mentioned above.

We define a suitable domain $\mathcal{D}$ with relations $e, I, \dot{A}$, whose wellfoundedness of e as usual will give us a transitive ultrapower, and the existence of the $\pi : \overline{M} \longrightarrow_E^* M$.

Definition 3.14 $D = \{\langle a, f\rangle | f \in \Gamma\}$. *Set:* $\langle a, f\rangle \genfrac{}{}{0pt}{}{e}{I} \langle b, g\rangle$ *iff* $\{u | f(u) \genfrac{}{}{0pt}{}{\in}{=} g(v)\} \in E_{a\cup b})$;
$\dot{G}(\langle a, f\rangle)$ *iff* $\{u | f(u) \in G\}) \in E_a$ *(for $G = A$, B);* $\mathcal{D} = \langle D, I, e, \dot{A}, \dot{B}\rangle$.

One can then prove a Łos theorem for Σ_0 formulae:

(2) Let $\langle a_1, f_1\rangle, \cdots, \langle a_n, f_n\rangle \in D$, and let $b \supseteq \cup_i a_i$, and let φ be a Σ_0 formula. Then $\mathcal{D} \models \varphi[\langle\overrightarrow{\langle a_i, f_i\rangle}\rangle] \iff \{u \mid \overline{M} \models \varphi[\overrightarrow{f_i^{a_i b}(u)}]\} \in E_b$

(We use the notation that if $a \subseteq b \in [\nu]^{<\omega}$, with card(a) $= n$, then if $f : [\kappa]^n \longrightarrow V$ then $f^{ab}(u) = f(u^{ab})$ where $u^{ab} = \langle u_{i_1}, \ldots u_{i_n}\rangle$ for u with card(u) = card(b), and $a = \langle b_{i_1}, \ldots b_{i_n}\rangle$.) Then I satisfies the axioms for equality, and $\mathcal{D}$ satisfies extensionality. If e is wellfounded then there is an isomorphism $[\ \ ] : \mathcal{D} \longleftrightarrow M$ where M is transitive and $[x] \overset{=}{\underset{\in}{}} [y]$ iff $x \overset{I}{\underset{e}{}} y$. We can define $\pi : \overline{M} \longrightarrow_{\Sigma_0} M$ by $\pi(x) = [\langle \emptyset, c_x\rangle]$ (the latter c_x the constant function x). M will be of the form $\langle J_\beta^{A'}, A'\rangle$. and we shall show that π satisfies the properties of the definition. We assume the wellfoundedness of e.

We first note:

(3) Let n be chosen least to satisfy the hypothesis of 3.12 (ii). Then condition (ii) holds. Hence $\kappa = crit(\pi)$ and $[\langle a, f\rangle] = \pi(f)(a)$ for $f \in \overline{H} = H^n_{\overline{M}}$, $a \in [\nu]^{<\omega}$.
Proof: By the definition of Γ, whenever $f \in \Gamma$ and $ran(f) \subseteq v \in \overline{H}$, then in fact $f \in \overline{M}$. **Q.E.D.**

We now prove

Lemma 3.6 Ultrapower Lemma *(i) M is an acceptable J-model.*
(ii) $\pi : \overline{M} \longrightarrow_{\Sigma_0^{(n)}} M$ if $\omega\rho^n_{\overline{M}} > \kappa$. *(iii) $\pi : \overline{M} \longrightarrow_{\Sigma_2^{(n)}} M$ if $\omega\rho^{n+1}_{\overline{M}} > \kappa$.*
(iv) Let φ be a $\Sigma_0^{(n)}$ (or in $\Sigma_1^{(n)}$, if $\kappa < \omega\rho^{n+1}_{\overline{M}}$). Let $b \supseteq \cup_i a_i$.

$$M \models \varphi(\pi(f_1)(a_1), \ldots, \pi(f_n)(a_n)) \iff \{u \mid \overline{M} \models \varphi(\overrightarrow{f_i^{a_i b}(u)}\} \in E_b$$

The proof of the lemma is in two stages. The main difficulty is in showing that the maps are between the relevant reducts. What this amounts to is showing that the elements in the natural "strata" that one defines from the functions in Γ are actually those obtained by using the iterated definition of projectum over M. The "strata" referred to arise as the H_n in the following definition: Set

$$\begin{aligned}
\Gamma_n &= \{f \in \Gamma \mid \mathrm{ran}(f) \subseteq H^n_{\overline{M}}\} \text{ if } \omega\rho^{n+1}_{\overline{M}} \geq \overline{\kappa} \\
&= \{f \in \Gamma \mid \mathrm{ran}(f) \in H^n_{\overline{M}}\} \text{ if } \omega\rho^{n+1}_{\overline{M}} < \overline{\kappa} \leq \omega\rho^n_{\overline{M}}. \\
H_n &= \{[\langle a, f\rangle] \mid f \in \Gamma_n \wedge \langle a, f\rangle \in D\} \text{ if } \Gamma_n \text{ is defined. } \omega\rho_n = On \cap H_n.
\end{aligned}$$

Note that H_n is transitive. For any i with $\omega\rho^i_{\overline{M}} > \kappa$, we shall interpret on the M-side the variables v^i_j as varying over H_i, For such an i we can take $\varphi \in \Sigma_k^{(i)}$ and say that $M \models \varphi(\vec{r})$ in the *pseudo-interpretation* if the variables v^i_j are so interpreted as $r^i \in H_i$. We show first the remark:

Lemma 3.7 *(iv) in the Ultrapower Lemma holds.*

Proof: This is an induction on n and then for each n by complexity of the formula φ. **Q.E.D.**

Corollary 3.8 *(ii) and (iii) of the Ultrapower Lemma hold.*

Proof: (ii) is immediate. (iii): Suppose $M \models \exists x^n \varphi(x^n, \pi(\vec{z}))$ where φ is $\Pi_1^{(n)}$. Let $\langle a, g\rangle \in \mathcal{D}, g \in \Gamma_n$ such that $M \models \varphi([\langle a, g\rangle], \pi(\vec{z}))$. By the last lemma $\{u \in [\kappa]^{\mathrm{card(a)}} \mid \overline{M} \models \varphi(g(u), \vec{z}) \in E_a$. Hence $\exists u_0 \overline{M} \models \varphi(g(u_0), \vec{z})$. As $ran\, g \subseteq H^n_{\overline{M}}$, $\overline{M} \models \exists^n \varphi(x^n, \vec{z})$. **Q.E.D.**

Corollary 3.9 *Q-properties are preserved: $\pi : \overline{M} \longrightarrow_Q M$. In particular M is an acceptable J-model.*

Proof: (Recall that a Q-formula was one expressible as $\forall x \exists y (x \in y \wedge \mathrm{Trans(y)} \wedge \psi(\mathrm{y}))$ where x does not occur free in $\psi \in \Sigma_1$. The axiom of acceptability is expressible by a Q-condition.) If $\omega\rho^1_{\overline{M}} \leq \kappa$ then $\pi : \overline{M} \longrightarrow_{\Sigma_0} M$ cofinally, so this suffices. If $\omega\rho^1_{\overline{M}} > \kappa$ then $\pi : \overline{M} \longrightarrow_{\Sigma_2} M$ since π is $\Sigma_2^{(0)}$ preserving, whilst $H^0_M = M$. **Q.E.D.**

Corollary 3.10 *If $M = \langle J^{A'}_{\alpha'}, B'\rangle$, then $H_n = |J^{A'}_{\rho_n}|$.*

Proof: $\pi \restriction H^n_{\overline{M}} \longrightarrow_Q H_n$ for all $n < \omega$. If $\omega\rho^{n+1}_{\overline{M}} \leq \kappa < \omega\rho^n_{\overline{M}}$, then $\pi : H^n_{\overline{M}} \longrightarrow \langle H_n, A'\rangle$ is Σ_0-cofinal, and again if $\omega\rho^{n+1}_{\overline{M}} > \kappa$, then π is Σ_2-preserving. For $\rho^n < \kappa$ $\pi = id \restriction H^N_{\overline{M}}$. **Q.E.D.**

In the second stage we show that this interpretation is the correct one: *i.e.* H_i does indeed equal $H^M_{\omega\rho^n_M}$. The point is now to show that the ρ_n, which were defined from the H_n are the "right" ordinals. This is stated in the next lemma. Actually for hierarchies involving measures, as here, stronger statements are provable (which are below). We have divided the preservations properties into two at this, as for hierarchies involving extenders further amenability assumptions are required to gain this extra strength. For measures we have in any case full amenability which we can exploit.

An embedding $\pi : \overline{M} \longrightarrow_{\Sigma_0^{(n)}} M$ is *cofinal* if π is $\Sigma_0^{(n)}$ preserving and $H^n_M = \cup \pi``H^n_{\overline{M}}$. This implies that $\pi \restriction H^n_{\overline{M}} : H^n_{\overline{M}} \longrightarrow_{\Sigma_0} H^n_M$ cofinally in the usual sense, and that $\pi : \overline{M} \longrightarrow_{\Sigma_0^{(n)}} M$.

It only remains to show that the projecta above κ correspond to the ordinal ρ_n.

Lemma 3.11 $\quad \rho_m = \rho^m_M$ *if* $\kappa \leq \omega\rho^{m+1}_{\overline{M}}$*;* $\quad \rho_m \leq \rho^m_M$ *if* $\kappa \leq \omega\rho^m_{\overline{M}}$.

Proof: By induction on m. This is trivial for $m = 0$. We show $\rho_m \leq \rho^m_M$ by showing that

(4) $\quad \langle H_m, A\rangle$ is amenable for $A \in P(\omega\rho_m) \cap \mathbf{\Sigma}_1^{(m-1)}(M)$.
Suppose $A \subseteq \omega\rho_m$ and is $\Sigma_1^{(m-1),M}([\langle a, f\rangle])$, say $A(x) \iff M \models \varphi(x, [\langle a, f\rangle])$ with $\varphi \in \Sigma_1^{(m-1)}$. Let $w = [\langle b, g\rangle] \in H_m$. We need $A \cap w \in H_m$. Let card(a) $= n$. Define $h : [\kappa]^n \longrightarrow H_{\overline{M}^m}$ by $h(u) = \{t \in g(u) \mid \overline{M} \models \varphi(t, f(u))\}$. Then $h \in \Gamma_m$. So $[\langle a, h\rangle] \in H_m$, and using 3.6(iv) $[\langle a, h\rangle] = [\langle b, g\rangle] \cap A$.

We now show $\rho_m \geq \rho^m_M$ if $\kappa \leq \omega\rho^{m+1}_{\overline{M}}$. We show:
(5) $\quad$ There is $A \subseteq \omega\rho_m$, $A \in \mathbf{\Sigma}_1^{(m-1)}(M)$ with $A \notin M$.

Suppose $\overline{A}$ is $\mathbf{\Sigma}_1^{(m-1)}(\overline{M})$ definable in a parameter $\overline{p}$.

Let A be $\mathbf{\Sigma}_1^{(m-1)}(M)$ in $\pi(\overline{p}) = p$ by the same definition. We show that $A \cap \omega\rho_m \notin M$. Suppose otherwise. Let $A \cap \omega\rho_m = [\langle a, f\rangle] \in M$. Then

$$A \cap \omega\rho_m = x \Longleftrightarrow \forall z^m(z_m \in A \Longleftrightarrow z^m \in x).$$

The right hand side here is a $\Pi_1^{(m)}$, in p, formula. $M \models \varphi[[\langle a, f\rangle], \pi(\overline{p})]$ so $\{u|\overline{M} \models \varphi[f(u), \overline{p}]\} \subseteq \{u|f(u) = \overline{A}\} \in E_a$, by 3.6 (iv). So $\exists u f(u) = \overline{A} \in \overline{M}$. A contradiction. **Q.E.D.**

Corollary 3.12 *(i) Let $\omega\rho_{\overline{M}}^{n+1} > \kappa$. Then $\pi``P_{\overline{M}}^n \subseteq P_M^n$.*
(ii) $\pi : \overline{M} \longrightarrow_E^* M$; in particular for $f \in \mathcal{D}$, $[f] = \pi(f)(\kappa)$.

Proof: (i) This was shown in the last part of the last proof. For (ii) all that we have left to show is (iv) in Def 3.12, that M is the closure of $ran\pi$ under good $\Sigma_1^{(n)}$-functions, for $\omega\rho_{\overline{M}}^{n+1} < \kappa$. We've extended π to such functions so it is enough to show $M = \{\pi(f)(a) \mid f \in \mathcal{D}, a \in [\nu]^n, \mathrm{dom}(f) = [\kappa]^n\}$. By (3) above $[c_\alpha] = \alpha$ for $\alpha < \kappa$. As $\{u \mid \overline{M} \models f(u) = f(id(u))\} \in E_a$ by the Łos theorem for $f \in \Gamma$, we have $\mathcal{D} \models f = c_f(id)$ and so $M \models [\langle a, f\rangle] = \pi(f)(a)$. **Q.E.D.**

If we are using measures to take ultrapowers, we can build mouse hierarchies with the measures fully amenable with respect to the structures concerned. This enables us to improve the last inequality above, and indeed, continue the analysis down below the critical point. However we shall continue to assume that we have an extender ultrapower and impose some additional requirements on E in order to do this.

We now use weak amenability to enable us to improve the last lemma and corollary at the point where the projectum crosses the critical point.

Lemma 3.13 *Suppose $\omega\rho_{\overline{M}}^n > \kappa \geq \omega\rho_{\overline{M}}^{n+1}$, and that $\mathcal{P}(\omega\rho_{\overline{M}}^{n+1}) \cap \overline{M} = \omega\rho_{\overline{M}}^{n+1} \cap M$. Then:*
(i) $\pi : \overline{M} \longrightarrow_{\Sigma_0^{(n)}} M$ cofinally; (ii) $\omega\rho_{\overline{M}}^n = \omega\rho_M^n$;
(iii) $\omega\rho_M^{n+1} \leq \omega\rho_{\overline{M}}^{n+1}$.

Proof: Assume $n > 0$.

Claim Let $p \in P_{\overline{M}}^{n+1}$. Then there is a set $A \in \Sigma_1^{(n-1)M}(\{\pi(p)\})$ such that $A \subseteq \omega\rho_n$ with $A \notin M$.
Proof: Let $\overline{A} = A_{\overline{M}^{n-1,p\restriction n-1}}^{p(n-1)}$. Then $\overline{A} \in \Sigma_1^{(n-1)}(\overline{M})$ and $\overline{M}^{n,p\restriction n} = \langle H_{\omega\rho^n}^{\overline{M}}, \overline{A}\rangle$ is amenable. Let $B \subseteq H_n$ be defined by the same $\Sigma_1^{(n-1)}$ formula φ in parameter $\pi(p)$. Let $\overline{H} = H_{\omega\rho_{\overline{M}}^n}^{\overline{M}}$. $\pi \restriction \overline{H} : \overline{H} \longrightarrow_{\Sigma_0} H_n$ cofinally.

(1) $B = \bigcup_{x \in \overline{H}} \pi(x \cap \overline{A})$.

Proof: Pick $x^n \in \overline{H}$. Let $y^n = x^n \cap \overline{A} \in \overline{H}$. Then

$$\overline{M} \models \forall z^n \in x^n(z^n \in y^n \longleftrightarrow \varphi(z^n, p)) \Longleftrightarrow M \models \forall z^n \in \pi(x^n)(z^n \in \pi(y^n) \longleftrightarrow \varphi(z^n, \pi(p))$$

(As π is Σ_0 preserving in the pseudo-interpretation.) This then simply says $\pi(x^n \cap \overline{A}) = \pi(x^n) \cap B$. As $\pi \restriction \overline{H}$ is cofinal (1) follows. **Q.E.D.(1)**.

(2) $B \notin M$

Proof: Assume $B \in M$. Then $\langle H_n, B\rangle \in M$ (by Corollary 3.10.) As $p \in P^{n+1}_{\overline{M}}$, there is $D \in \mathbf{\Sigma_1}(\overline{M}^{n,p\restriction n}) \cap \mathcal{P}(\omega\rho^{n+1}_{\overline{M}})$ with $D \notin \overline{M}$. As $\pi \restriction \langle \overline{H}, \overline{A}\rangle \longrightarrow_{\Sigma_0,cof} \langle H_n, B\rangle$, and $\pi \restriction \omega\rho^{n+1}_{\overline{M}} = id$, this D is $\Sigma_1(\langle H_n, B\rangle) \subseteq M$. But by assumption $\mathcal{P}(\omega\rho^{n+1}_{\overline{M}})$ is the same in both $\overline{M}$ and M. Contradiction! **Q.E.D.(2)**

Now just code B onto $\omega\rho_n$ to get a suitable A. As ρ_n is p.r.closed, and $H_n = J^{A'}_{\rho_n}$, let f be $\Sigma_1(M)$ with $f : \omega\rho_n \longrightarrow H_n$ be onto; let $A = \{\xi \mid f(\xi) \in B\}$. Then A is as required. **Q.E.D.Claim**

From the Claim we see that $\omega\rho^n_M \leq \omega\rho_n$. The reverse inequality we already established in the last lemma. This gives us (ii) here; and we can now assert (i).

The D of (2) is $\Sigma_1^{(n)\overline{M}}(p)$, and the same definition over M in $\pi(p)$ yields a D' with $D' \cap \omega\rho^{n+1}_{\overline{M}} = D \notin M$. Hence $\omega\rho^{n+1}_M \leq \omega\rho^{n+1}_{\overline{M}}$ which is (iii). **Q.E.D.**

Remark: It is possible to replace the assumption on weak amenability with just that $R^n_{\overline{M}} \neq \emptyset$ and get the same conclusions.

Definition 3.15 *E is* Σ_1-amenable *with respect to $\overline{M}$, if E_a is $\mathbf{\Sigma_1}(\overline{M})$ for any $a \in [\nu]^{<\omega}$.*

Note that the parameters involved in the definition are not assumed to be given uniformly. The reader may like to check that this notion yields that $\mathbf{\Sigma_1}(M) \cap \mathcal{P}(\kappa) \subseteq \mathbf{\Sigma_1}(\overline{M}) \cap \mathcal{P}(\kappa)$.

In addition to weak amenability we now make the following assumption on the extender E, that it is Σ_1-amenable. Now we obtain:

Lemma 3.14 *Suppose $\omega\rho^n_{\overline{M}} > \kappa \geq \omega\rho^{n+1}_{\overline{M}}$, and that E is $\mathbf{\Sigma_1}$-amenable. Then $\mathcal{P}(\kappa) \cap \mathbf{\Sigma_1^{(n)}}(M) \subseteq \mathbf{\Sigma_1^{(n)}}\overline{M}$.*

Proof: Let $B \in \mathcal{P}(\kappa) \cap \mathbf{\Sigma_1^{(n)}}(M)$. Suppose $B(z) \longleftrightarrow \exists x^n B'(x^n, z, \pi(f)(a))$ with $B' \in \Sigma_0^{(n)}(M)$. Using $\Sigma_0^{(n)}$ cofinality, we can write

$$B(z) \longleftrightarrow \exists u \in H^n_{\overline{M}} \exists x \in \pi(u) B'(x, z, \pi(f)(a)) \longleftrightarrow \exists u \in H^n_{\overline{M}} D(\pi(u), z, \pi(f)(a))$$

(where $D \in \Sigma_0^{(n)}(M)$) Let $\overline{D} \in \Sigma_0^{(n)}(\overline{M})$ have the same definition.

$$B(z) \longleftrightarrow \exists u \in H^n_{\overline{M}} \{v \mid \overline{D}(u, z, f(v))\} \in E_a \longleftrightarrow \exists u^n \exists x^n (x^n = \{v \mid \overline{D}(u^n, z, f(v))\} \wedge x^n \in E_a)$$

for any $z < \kappa$. The matrix here is $\mathbf{\Sigma_0^{(n)}} \cap \mathbf{\Sigma_1^{(0)}}$. **Q.E.D.**

Remark: There is a non-uniformity in the above due to the final "$x^n \in E_a$", but that is the only one. We now go down below κ.

Lemma 3.15 *Let E be Σ_1-amenable, and assume $\omega\rho^m_{\overline{M}} \leq \kappa$. Then*
(i) $H^m_M = H^m_{\overline{M}}$; *(ii)* $\pi : \overline{M} \longrightarrow_{\Sigma_1^{(m)}} M$;
(iii) $\mathbf{\Sigma_1^{(m)}}(M) \cap \mathcal{P}(H^m_M) = \mathbf{\Sigma_1^{(m)}}(\overline{M}) \cap \mathcal{P}(H^m_{\overline{M}})$;
(iv) $\pi``P^m_{\overline{M}} \subseteq P^m_M$.

Proof: By induction on m. We first show (i). Note that H^m_M by definition is $J^{A'}_{\omega\rho^m_M}$, and $\omega\rho^m_M$ is a cardinal of $M = \langle J^{A'}_{\alpha'}, B'\rangle$. Similarly for $\overline{M} = \langle J^A_\alpha, B\rangle$. We also know that $J^A_\kappa = J^{A'}_\kappa$. So it suffices for (i) to show that $\omega\rho^m_{\overline{M}} = \omega\rho^m_M$. This will also show (ii), since $\pi \restriction H^m_{\overline{M}} = id$. Suppose that $m = k+1$.
$\rho^m_{\overline{M}} \geq \rho^m_M$: since there is $B \subseteq \omega\rho^m_{\overline{M}}$, $B \in \mathbf{\Sigma_1^{(k)}}(\overline{M})\backslash\overline{M}$. By the inductive hypothesis, as $\pi : \overline{M} \longrightarrow_{\Sigma_1^{(k)}} M$ and $\pi \restriction H^m_{\overline{M}} = id$, B is thus also $\mathbf{\Sigma_1^{(k)}}(M)$. But also $B \notin M$, since $\mathcal{P}(\omega\rho^m_M)$ is the same in both models.

$\rho^m_M \geq \rho^m_{\overline{M}}$: suppose not. Then there is $B \subseteq \omega\rho^m_M < \omega\rho^m_{\overline{M}}$ with $B \in \mathbf{\Sigma_1^{(k)}}(M)\backslash M$. But then $B \in \mathbf{\Sigma_1^{(k)}}(\overline{M})$ by the previous lemma, if $\omega\rho^k > \kappa$, and by the inductive hypothesis (iii) for k otherwise. But B cannot be in $\overline{M}$, since $\mathcal{P}(\kappa) \cap \overline{M} \subseteq M$. So $\omega\rho^m_M \not< \omega\rho^m_{\overline{M}}$. This proves (i) and as remarked, (ii).

For (iii): let $H = H^m_{\overline{M}} = H^m_M$ and suppose $B \subseteq H$.

$$\begin{aligned} B \in \Sigma_1^{(m)}(M) &\longleftrightarrow \exists D \in \Sigma_1^{(k)}(M)\, B \in \Sigma_1(\langle H, D\rangle) \\ &\longleftrightarrow \exists D \in \Sigma_1^{(k)}(\overline{M})\, B \in \Sigma_1(\langle H, D\rangle) \longleftrightarrow B \in \Sigma_1^{(m)}(\overline{M}). \end{aligned}$$

where the second equivalence uses the I.H. (iii) or the last lemma. This is (iii) for m. We consider (iv). Let $\overline{p} \in P^m_{\overline{M}}$, $p = \pi(\overline{p})$. We need to show:

Claim $A^{k,p\restriction k}_M \notin M$ for $1 \leq k \leq m$.

We prove this by induction on $m - k$. If $m = k$, then as $\pi \restriction H = id$, and $\pi : \overline{M} \longrightarrow_{\Sigma_1^{(m)}} M$, so $A^{k,\overline{p}\restriction k}_{\overline{M}} = A^{k,p\restriction k}_M$, and the latter is not in $\overline{M}$, and hence not in M either, because $\mathcal{P}(H) \cap M = \mathcal{P}(H) \cap \overline{M}$. Suppose it holds for $k+1 \leq m$. Then $A^{k+1,p\restriction k+1}_M = B \cap H^{k+1}_M$ where $B \in \Sigma_1(\langle H^k_M, A^{k,p\restriction k}_M\rangle)$ in parameter $p(k)$. But $A^{k,p\restriction k}_M \in M \Rightarrow \langle H^k_M, A^{k,p\restriction k}_M\rangle \in M$ which contradicts the inductive hypothesis for k. **Q.E.D.**

Collecting together the pieces of the above one obtains (with a small amount swept under the carpet in (ii) for $\omega\rho^m_M < \kappa$, as the argument requires some more analysis of the $\Sigma_1^{(n)}$ relations than we have catered for here):

Corollary 3.16 *Let E be Σ_1-amenable. Then*
(i) $\pi : \overline{M} \longrightarrow_{\Sigma^*} M$
(ii) $\mathcal{P}(\kappa) \cap \mathbf{\Sigma_1^{(n)}}(\overline{M}) = \mathcal{P}(\kappa) \cap \mathbf{\Sigma_1^{(n)}}(M)$ *for all $n < \omega$.*

3.4 An Upwards Extension of Embeddings Lemma

Lemma 3.17 *Suppose $\overline{M} = \langle J_{\overline{\beta}}^{E^{\overline{M}}}, E^{\overline{M}}, F^{\overline{M}} \rangle$ is a mouse, and $\overline{Q} = \langle J_{\overline{\nu}}^{E^{\overline{M}}}, E^{\overline{M}} \restriction \overline{\nu} \rangle$ and there is a cofinal embedding $\sigma : \overline{Q} \longrightarrow_{\Sigma_0} Q$. We suppose there is $\overline{\kappa} < \overline{\nu}$ with $\overline{Q} \models$ "$\overline{\kappa}$ is the largest cardinal", and either $\overline{\beta} = \overline{\nu}$ or $\overline{M} \models \overline{\nu} = \overline{\kappa}^+$. Suppose that $cf(\overline{\nu}) > \omega$. Then there is $\tilde{\sigma} \supseteq \sigma$ and $M = \langle J_{\beta}^{E^M}, E^M, F^M \rangle$ with M a mouse and $\tilde{\sigma} : \overline{M} \longrightarrow_{\Sigma_1^{(n)}} M$, for any n with $\omega\rho_{\overline{M}}^{n+1} \geq \kappa$, where $\kappa = \sigma(\overline{\kappa})$. Further, if $\overline{M}$ is sound above $\overline{\kappa}$, and for some $m < \omega$, $\omega\rho_{\overline{M}}^{m+1} \leq \overline{\kappa} < \omega\rho_{\overline{M}}^{m}$, then $\omega\rho_M^{m+1} \leq \kappa < \omega\rho_M^m$, M is the closure of $\kappa \cup \{p\}$ under good $\Sigma_1^{(m)}$-functions for a $p \in M$, and $\tilde{\sigma} : \overline{M} \longrightarrow_{\Sigma_1^{(m)}} M$.*

We first remark that in many situations one may use the lemma with κ a cardinal (or at least the cardinal of a model) which we are not assuming here. One can often improve the conclusion to require that M is sound above κ (and that $\tilde{\sigma}(p_{\overline{M}}) = p_M \backslash \kappa$ where $p_{\overline{M}}$ is the standard parameter of $\overline{M}$) - but this requires more work on parameters, which we have eschewn. The proof of this lemma involves forming a "pseudo-ultrapower" M of $\overline{M}$ that has also a fine-structure preserving extension of σ mapping $\overline{M}$ into M. The construction is very similar to that for forming an ultrapower by an extender (more precisely it is that for the "long" extender or "derived" extender from the map σ). So naturally it bares many similarities to what we have just done in the previous subsection. Much of what we shall do is redefinition of the previous notions, and many of the changes in the proofs require only a change of notation. We omit some detail therefore. We shall first redefine the class of functions Γ, the stratified functions Γ_n, and form the term model $\mathcal{D}$, and the "strata" H_n. There is little that can be said about the H_n for $n > m+1$ and indeed we shall not define these. The "lift-up" structure M is wellfounded (and iterable) by the crucial use of the uncountable cofinality of ν.

Definition 3.16 *Γ is the set of functions f with dom $(f) \subseteq u$ for some $u \in \overline{Q}$ and either f is a good $\Sigma_1^{(n)}(\overline{M})$ function for some n with $\omega\rho_{\overline{M}}^{n+1} \geq \overline{\nu}$, or $f \in H_{\overline{M}}^n$ where $\omega\rho_{\overline{M}}^n \geq \overline{\nu}$.*

We define $e, I, \dot{E}$, and $\mathcal{D}$ as indicated §1.

Definition 3.17 *$D = \{\langle a, f\rangle | f \in \Gamma \wedge a \in \sigma(f)\}$; set:*
$\langle a, b\rangle {}_I^e \langle b, g\rangle$ iff $\langle a, f\rangle \in \sigma(\{\langle u, v\rangle | f(u) \overset{\in}{=} g(v)\})$
$\dot{G}(\langle a, f\rangle)$ iff $a \in \sigma(\{u | f(u) \in G\})$ for $G = E,\, F$; *set: $\mathcal{D} = \langle D, I, e, \dot{E}(\dot{F})\rangle$.*

One can then again prove a Los theorem for Σ_0 formulae:

(1) Let $\langle a_1, f_1\rangle, \cdots, \langle a_n, f_n\rangle \in D$ and let φ be a Σ_0 formula. Then
$\mathcal{D} \models \varphi[\overrightarrow{\langle a_i, f_i\rangle}] \Longleftrightarrow \vec{a_i} \in \sigma(\{\vec{u} | \overline{M} \models \varphi[\vec{f_i}(\vec{u})]\})$

Then I satisfies the axioms for equality, and $\mathcal{D}$ satisfies extensionality. If e is wellfounded then there is an isomorphism $[\ \ ] : \mathcal{D} \longleftrightarrow M$ where M is transitive and $[x] \overset{=}{\underset{\in}{}} [y]$ iff $x {}_e^I y$. We can define $\tilde{\sigma} : \overline{M} \longrightarrow_{\Sigma_0} M$ by $\tilde{\sigma}(x) = [\langle 0, \{\langle 0, x\rangle\}\rangle]$. M will be of the form $\langle J_\beta^E, E, F\rangle$, and we shall show that it is a mouse and $\tilde{\sigma}$ satisfies the properties of the Lemma. Since wellfoundedness of e can be considered a special case of the proposed iterability of M, we omit the proof of the wellfoundedness of e, and simply assume it for the moment.

We first note:

(2) $\tilde{\sigma} \upharpoonright \overline{Q} = \sigma$

Proof: Note that for $f, g \in \overline{Q}$ $[\langle a, f\rangle] \in [\langle b, g\rangle] \Leftrightarrow \sigma(f)(a) \in \sigma(g)(b)$. Hence for such f we may define a map $[\langle a, f\rangle] \mapsto \sigma(f)(a)$. This map is onto and so by the above is the identity. Hence $[\langle a, f\rangle] = \sigma(f)(a) = \tilde{\sigma}(f)(a)$ and so $\tilde{\sigma}(x) = [\langle 0, \{\langle 0, x\rangle\}\rangle] = \sigma(x)$.

QED

Define

$$\begin{aligned}
\Gamma_n &= \{f \in \Gamma \,|\, ran(f) \subseteq H^n_{\overline{M}}\} \text{ if } \omega\rho^{n+1}_{\overline{M}} \geq \overline{\nu} \\
&= \{f \in \Gamma \,|\, ran(f) \in H^n_{\overline{M}}\} \text{ if } \omega\rho^{n+1}_{\overline{M}} < \overline{\nu} \leq \omega\rho^n_{\overline{M}}. \\
H_n &= \{[\langle a, f\rangle] \,|\, f \in \Gamma_n \wedge \langle a, f\rangle \in D\} \text{ if } \Gamma_n \text{ is defined. } \omega\rho_n = On \cap H_n.
\end{aligned}$$

Note that H_n is transitive as before.

(3) $\tilde{\sigma} : \overline{M} \longrightarrow_{\Sigma_0^{(n)}} M$ cofinally for any n with $\omega\rho^n_{\overline{M}} \geq \overline{\nu}$.
Proof: We shall prove (3) in three stages. The first two are just the manœuvres of Lemma 3.11. The third stage will use the extra assumption on the cofinality of ν to check that (3) holds for the m of crossing projectum of Lemma 3.17. In the first step we interpret the variables of type k with $\omega\rho^k_{\overline{M}} \geq \overline{\nu}$ on the right hand side as varying over H_k. We prove (3) using this pseudo-interpretation. Again we then shall show in the second step that this interpretation is "correct" in that H_k in fact is H^k_M for any $k \leq m$

The proofs of (4),(5) & (6) are, with minor alterations, those of the last subsection.
(4) Let $\langle \overrightarrow{\mathrm{a_i}, \mathrm{f_i}}\rangle \in D$ and let $\varphi \in \Sigma_0^{(n)}$ where $\omega\rho^n_{\overline{M}} \geq \overline{\nu}$. Then
$M \models \varphi[[\langle \overrightarrow{\mathrm{a_i}, \mathrm{f_i}}\rangle]] \Longleftrightarrow \overrightarrow{\mathrm{a_i}} \in \sigma(\{\overrightarrow{\mathrm{u}} \,|\, M \models \varphi[\overrightarrow{\mathrm{f}}(\overrightarrow{\mathrm{u}})]\})$.

(5) $\tilde{\sigma} : \overline{M} \longrightarrow_{\Sigma_0^{(n)}} M$ cofinally, for $\omega\rho^n_{\overline{M}} \geq \overline{\nu}$.
Proof: We mean here in the pseudo-interpretation. By cofinally, we mean $\tilde{\sigma} \upharpoonright H^m_{\overline{M}}$ is cofinal into H^m_M, in case m is defined. But this is straightforward. **Q.E.D.(5)**

As $\tilde{\sigma}$ is Σ_0 and cofinal into M, acceptability of $\overline{M}$ translates to acceptability of M, and so $H_n = J^M_{\rho_n}$ for $\omega\rho^n_{\overline{M}} \geq \overline{\nu}$. It remains to show that $\rho_n = \rho^n_M$ for $\omega\rho^n_{\overline{M}} \geq \overline{\nu}$.

(6) $\quad \rho_n = \rho^n_M$ if $\overline{\nu} \leq \omega\rho^{n+1}_{\overline{M}}$; $\quad \rho_n \leq \rho^n_M$ if $\nu \leq \omega\rho^n_{\overline{M}}$.

Proof: The reader is referred to (4) and (5) of Lemma 3.11. The notational changes are simply that in (4) the function h now has domain dom(g). The map between the two structures is now $\tilde{\sigma}$ (not π); and in (5) we need to conclude with: $a \in \sigma(\{u|\overline{M} \models \varphi[f(u), \overline{p}]\}) = \sigma(\{u|f(u) = \overline{A}\})$ rather than claim that the set is in E_a. **Q.E.D.(6)**

We now use the extra condition on $\overline{Q}, \overline{\nu}$.

(7) If $\omega\rho_{\overline{M}}^{m+1} < \overline{\nu} \leq \omega\rho_{\overline{M}}^{m}$ then
(i) $\omega\rho_M^{m+1} < \nu \leq \omega\rho_M^m$; (ii) $\rho_m = \rho_M^m$.
(iii) M is the closure of $\kappa \cup \{r\}$ under good $\Sigma_1^{(m)}(M)$-functions, for some $r \in R_M^{m+1}$.

Proof: We first show $\tilde{\sigma}(\overline{\nu}) = \nu = \sup \tilde{\sigma}``\overline{\nu}$ (setting $\tilde{\sigma}(On \cap M) = On \cap M$). Let $\xi = \tilde{\sigma}(f)(a) < \tilde{\sigma}(\overline{\nu})$ with $[\langle a, f\rangle] \in D$. Without loss of generality we can assume $ran(f) \subseteq \overline{\nu}$. By the definition of Γ, and using $\overline{\nu}$ is a cardinal of $\overline{M}$, $ran(f)$ cannot be cofinal in $\overline{\nu}$. Hence there is η with $ran(f) \subseteq \eta < \overline{\nu}$ and so $\tilde{\sigma}(f)(a) < \sigma(\eta) < \nu$.

If $\sigma(\overline{\kappa}) = \kappa$, then one may simply check that $\kappa^{+M} = \nu$. By (6) $\nu \leq \omega\rho_M^m$. Suppose $\omega\rho_{\overline{M}}^{m+1} < \overline{\nu} \leq \omega\rho_{\overline{M}}^{m}$. Let $\tilde{\sigma}(\overline{p}) = p$ and $\overline{p} \in P_{\overline{M}}^{m+1}$.

For clause (ii) we only have left to show $\rho_M^m \leq \rho_m$. We do this now. Let A be defined by the cofinal map $\tilde{\sigma} \restriction H_{\overline{M}}^{m} : \langle M^{m,\overline{p}}, A_{\overline{M}}^{m,\overline{p}}\rangle \longrightarrow_{\Sigma_0} \langle H_m, A\rangle$. That is, let $A = \bigcup_{x \in H_{\overline{M}}^m} \tilde{\sigma}(x \cap A_{\overline{M}}^{m,\overline{p}})$. Then $\langle H_m, A\rangle \prec_{\Sigma_0} M^{m,p}$. By the Downwards Extensions of Embeddings Lemma, there are $\widetilde{M}$, $\widetilde{p}$ such that $\widetilde{M}^{m,\widetilde{p}} = \langle H_m, A\rangle$ and $\widetilde{p} \in R_{\widetilde{M}}^m$. We have the further properties that there is $\pi : \widetilde{M} \longrightarrow_{\Sigma_0^{(n)}} M$ with $\pi \restriction H_m = id$ and $\pi(\widetilde{p}) = p$. Since $\overline{p} \in R_{\overline{M}}^m$ we can factor through, and see that there is $\hat{\sigma} : \overline{M} \longrightarrow_{\Sigma_1^{(n)}} \widetilde{M}$ such that $\hat{\sigma} \restriction H_{\overline{M}}^m = \tilde{\sigma} \restriction H_{\overline{M}}^M$ and $\hat{\sigma}(\overline{p}) = \widetilde{p}$. Then $\pi(\hat{\sigma}(f)(a)) = \tilde{\sigma}(f)(a)$ for $\langle a, f\rangle \in \mathcal{D}$. Hence $\pi\hat{\sigma} = \tilde{\sigma}$ and π is onto; hence $\pi = id$, $\hat{\sigma} = \tilde{\sigma}$, and $\widetilde{M} = M$. Then A cannot be in M whilst it is $\Sigma_1^{(m)}(M)$. Hence $\rho_M^m \leq \rho_m$.

Let $\overline{p} \in P_{\overline{M}}^{m+1}$ and let $p = \tilde{\sigma}(\overline{p})$. Any $x \in H_M^m$ has the form $\tilde{\sigma}(f)(a)$ where $f \in \Gamma_m(\overline{M})$ and $a \in \sigma(dom(f))$ If $f \in h_{\overline{M}^{m,\overline{p}}}(i, \langle \overline{\xi}, \overline{p}(m)\rangle)$ some $\overline{\xi} < \overline{\kappa}$, then $\tilde{\sigma}(f)(a) = h_{M^{m,p}}(j, \langle\langle \xi, a\rangle, p(m)\rangle)$ for some $j < \omega$, $\xi = \sigma(\overline{\xi})$; hence $M^{m,p} \subseteq h_{M^{m,p}}(\nu \cup \{p(m)\})$. But as at the beginning of this proof, $h_{M^{m,p}}(\kappa \cup \{p(m)\})$ is cofinal in ν. As $\nu = (\kappa^+)^{M^{m,p}}$, (or equals $On \cap M^{m,p}$), this suffices to show $h_{M^{m,p}}(\kappa \cup \{p(m)\}) \supseteq M^{m,p}$. This shows that both that M is the closure of $\kappa \cup \{p(m)\}$ under good $\Sigma_1^{(m)}$–functions, and that $\omega\rho_M^{m+1} \leq \kappa$, which are clauses (i) and (iii).

Q.E.D.(7) & (3)

Claim M is wellfounded and iterable above κ.

Proof: Suppose not; let $I = \langle M_\alpha | \alpha < \theta\rangle$ be an iteration of $M_0 = M$ above κ that cannot be continued. Then there is a set $\{x_i\}_{i\in\omega} \subseteq M$ of supports of functions coding an illfounded $\in_\theta$-chain through the "ordinals" of a model M_θ. We treat wellfoundness of M as a special case of iterability. In this case set $x_{i+1} \in x_i$ as a descending e-chain through $M = M_0$. Let H be some sufficiently large ZFC^--model that is transitive and contains $I, \{x_i\}$. Let $I \cup \{x_i\} \subseteq Y \prec H$ where Y is countable. Let $k : \overline{H} \Longleftrightarrow Y$ be the inverse of the transitive collapse. Let $k(\langle N, \kappa', \overline{x}_i, \overline{I}\rangle) = \langle M, \kappa, x_i, I\rangle$. Then in $\overline{H}, \{\overline{x}_i\}$ is a set of supports for functions coding illfoundedness of the last model $\overline{M}_{\overline{\theta}}$ of $\overline{I} = \langle \overline{M}_\alpha | \alpha < \overline{\theta}\rangle$. Let $\{y_i\}_{i\in\omega}$ enumerate N. Then each $k(y_i)$ is of the form $\tilde{\sigma}(f_i)(a_i)$ for some $f_i \in H_{\overline{M}}^n$, or f_i a good $\Sigma_1^{(n)}(\overline{M})$ function with $dom f_i \in \overline{Q}$.

We define by cases a model P and a map $\pi : P \longrightarrow \overline{M}$.

Case 1 $n \geq 1$ and $\overline{\nu} < \rho^n_{\overline{M}}$.
Let $\mu < \overline{\nu}$ be chosen so that

(i) $\overline{P} =_{df} \langle \overline{M}^{n,\overline{p}} | \mu, A^{n,\overline{p}} | \mu \rangle$ is amenable, $\rho^1_{\overline{P}} \leq \overline{\kappa}$.

(ii) $\sup_i dom f_i < \mu$

(iii) $id : \overline{P} \longrightarrow_{\Sigma_1} \langle \overline{M}^{n,\overline{p}}, A^{n,\overline{p}}_{\overline{M}} \rangle$

(i) & (ii) are clearly possible since $cf(\overline{\nu}) > \omega$. As $\overline{\nu} < \rho^n_{\overline{M}}$, $\overline{\nu}$ is a cardinal of $\overline{M}^{n,\overline{p}}$, so $\langle \overline{M}^{n,\overline{p}} | \overline{\nu}, A^{n,\overline{p}} | \overline{\nu} \rangle \prec_{\Sigma_1} \langle \overline{M}^{n,\overline{p}}, A^{n,\overline{p}} \rangle$. But there are arbitrarily large $\mu < \overline{\nu}$ with such a $\overline{P} \prec_{\Sigma_1} \langle \overline{M}^{n,\overline{p}} | \overline{\nu}, A^{n,\overline{p}}_{\overline{M}} | \overline{\nu} \rangle$. Hence $\overline{P}$ exists. Now any f_i is either in $H^n_{\overline{M}} = \overline{M}^{n,\overline{p}}$ or has a good $\Sigma_1^{(k)}(\overline{M})$ definition involving parameters $\vec{\xi_i} < \omega\rho^k_{\overline{M}}$. By the soundness of $\overline{M}$ above $\overline{\kappa}$, either way such an f_i has a good $\Sigma_1^{(n)}(\overline{M})$ definition involving parameters $\vec{\xi_i}$ below $\overline{\kappa}$.
By the Extensions of Embeddings Lemma there is a premouse P, iterable above $\overline{\kappa}$, and $q \in R^n_P$, and a $\pi : P \longrightarrow_{\Sigma_0^{(n)}} \overline{M}$, with $\overline{P} = P^{n,q}$ and $\pi(q) = \overline{p}$, and $A^{n,q}_P = A^{n,\overline{p}} | \mu$. As $A^{n,q}_P \in H^n_{\overline{M}}, P \in H^n_{\overline{M}}$. Let $\overline{f}_i$ have the same good $\Sigma_1^{(n)}$ definition over P from $\vec{\xi_i}, q$ as f_i did over $\overline{M}$ in $\vec{\xi_i}, \overline{p}$. Note that π is in fact $\Sigma_1^{(n)}$ preserving. Let $\sigma(p) = \tilde{p}$. But as $dom f_i = dom \overline{f}_i$ and π is $\Sigma_1^{(n)}$ -elementary $\pi(\overline{f}_i(u)) = f_i(u)$ for any $u \in dom \overline{f}_i$.

Case 2 $n \geq 1$ and $\overline{\nu} = \rho^n_{\overline{M}}$.

$\overline{\mu}$ may be chosen as in *Case 1*, however in (iii) we may only have that $\overline{P}$ is Σ_0-embedded into $\langle \overline{M}^{n,\overline{p}}, A^{n,\overline{p}}_{\overline{M}} \rangle$ by id. However this suffices to still extend id to a $\pi : P \longrightarrow_{\Sigma_0^{(n)}} \overline{M}$, but which is nevertheless $\Sigma_1^{(n-1)}$ preserving. (This is just part of the Extensions of Embeddings Lemmas.) However we may now at no additional cost impose on μ, using $\mathrm{cf}(\overline{\nu}) > \omega$, and the Case hypothesis:

(iv) Any f_i that has a good $\mathbf{\Sigma}_1^{(n-1)}(M)$ definition in parameters $\vec{\xi} < \rho^n_{\overline{M}}$, then such a $\vec{\xi}$ is below μ.

Again we have $\pi : P \longrightarrow_{\Sigma_0^{(n)}} \overline{M}$, with P a premouse iterable above $\overline{\kappa}, P \in H^{\overline{M}}_{\overline{\nu}}$. As π is $\Sigma_1^{(n-1)}$ preserving, we can let $\overline{f}_i$ be the functions defined over P with the same definitions as that of f_i over M, with $sup_i dom f_i < On \cap \overline{P}$. In both cases we define $l = N \longrightarrow \tilde{P}$, where $\tilde{P} = \sigma(P)$, by $l(y_i)\tilde{\sigma}(\overline{f}_i)(a_i)$. l will embed, in particular, the supports for a descending sequence of ordinals of an iterate of $\tilde{P}$ above κ. This will contradict the latter's iterability above κ.

(9) l is $\Sigma_0^{(n)}$ preserving.

Proof:

$$\begin{aligned}
\tilde{P} \models \varphi(l(y_i)) &\iff \tilde{P} \models \varphi(\tilde{\sigma}(\overline{f}_i)(a_i)) \\
&\iff a_i \in \{u \mid \tilde{P} \models \varphi(\sigma(\overline{f}_i)(u))\} \\
&\iff a_i \in \sigma(\{u \mid P \models \varphi(\tilde{f}_i(u))\}) \text{ (as } \sigma \restriction P \text{ is } \Sigma_\omega) \\
&\iff a_i \in \sigma(\{u \mid \overline{M} \models \varphi(\overline{f}_i(u))\}) \text{ (as } \pi \text{ is } \Sigma_0^{(n)} \text{ or } \Sigma_1^{(n)} \text{ depending on the case)} \\
&\iff a_i \in \{u \mid \overline{M} \models \varphi(\tilde{\sigma}(f_i)(u))\} \text{ (as } \tilde{\sigma} \text{ is } \Sigma_0^{(n)}) \\
&\iff M \models \varphi(\tilde{\sigma}(f_i)(a_i)) \iff N \models \varphi(y_i) \text{ (as } k \text{ is } \Sigma_\omega).
\end{aligned}$$

QED(9)

Note that $\tilde{P}$ is truly iterable above κ, since "P is iterable above κ" holds in some admissible set $\mathcal{A}$ with $P \in \mathcal{A} \in \overline{\mathcal{M}}$, and hence by elementarity holds in $\tilde{\mathcal{A}} = \sigma(\mathcal{A})$ of $\tilde{P}$. But $\omega_1 \subseteq \tilde{\mathcal{A}}$ so this is absolute. But this is now a contradiction since $\{l(\overline{x}_i)\}_{i \in \omega}$ contains supports coding an illfounded iteration of $\tilde{P}$ above κ.

QED *Cases 1 & 2*

Case 3 $n = 0$.
A standard argument shows that $cf(\overline{\nu}) > \omega \Longrightarrow cf(On \cap \overline{M}) > \omega$. As $\tilde{\sigma}$ is Σ_0 -cofinal the same is true of M. There is thus $\tilde{\tau} = \sigma(\tau)$ with

(i) $\forall y \; k(y) \in J^M_{\tilde{\tau}}$

(ii) $\exists \; f_n \in J^{\overline{M}}_\tau \; \exists a_n \in \overline{Q} \quad \tilde{\sigma}(f_n)(a_n) = k(y_n)$.

We can assume that $\tilde{\tau}$ is chosen sufficiently large so that each f_n can be taken in $h_{\overline{M}|\tau}(\overline{\kappa} \cup p_{\overline{M}})$. Let $X \prec_{\Sigma_1} J^{\overline{M}}_\tau$ be such that $\overline{\nu} \cup \{p_{\overline{M}}\} \subseteq X$, let $\pi : P \overset{\sim}{\longleftrightarrow} X$. Let $\pi(q) = p_{\overline{M}}$, and let $\overline{f}_n$ have the same $\Sigma_1^P(\{q, \vec{\xi_n}\})$ definition as f_n had in $\Sigma^{\overline{M}|\tau}(\{p_{\overline{M}}, \vec{\xi}\})$. Note that $\sup_i \mathrm{dom} f_i \leq \overline{\nu} \cap X$. Also note that $P \in J^{\overline{M}}_{\overline{\nu}}$ (since $A^1_P \subset J^P_\kappa$, as $\rho^1_P \leq \overline{\kappa}$, and then a simple comparison argument shows $A^1_P \in \overline{M}$).

Again define $l : N \longrightarrow \tilde{P} =_{df} \sigma(P)$ by $l(y_n) = \tilde{\sigma}(\overline{f}_n)(a_n)$.

(10) l is Σ_0-preserving.
Proof The equivalences are exactly those of (9), noting in this case that $\pi : P \longrightarrow_{\Sigma_1} \overline{M}|\tau$, and so $\pi : P \longrightarrow_{\Sigma_0} \overline{M}$. QED (10)

But $\tilde{P}$ is iterable above κ as before, so l embeds a supposedly non-iterable premouse above κ' into one that is iterable above $l(\kappa') = \kappa$! This is a contradiction. QED.

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