Countable Unions of Simple Sets in the Core Model

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Abstract

We follow [8] in asking when a set of ordinals $X \subseteq \alpha$ is a countable union of sets in K, the core model. We show that, analogously to L, an X closed under the canonical Σ_1 Skolem function for K_α can be so decomposed, provided K is such that no ω-closed filters are put on its measure sequence, but not otherwise. This proviso holds if there is no inner model of a weak Erdős type property.

Introduction

In [8], Magidor proved that if there are no ω_1-Erdős cardinals in the core model K, then for every β there is a K-definable algebra of functions on β, so that every set $X \subseteq \beta$, closed under these functions, is a countable union of sets in K. His algebras are uniformly K-definable in β-in a Σ_2^{ZF} way.

Magidor's theorem came about as a generalisation of another theorem in that paper, where he proves that if $\neg O^\sharp$, then every primitively set closed subset of β is a countable union of sets in L. The statement above is his generalization to K. As he notes there, a theorem of Baumgartner [1] that if there is an ω_1-Erdős cardinal, then in a forcing extension there is a club in $\mathcal{P}_{\omega_2}(\omega_3)$ of size $\omega_3^{\omega_1} > \omega_3^{\omega}$-shows that his result is best possible. So the only possible refinement is to ask whether the algebras can be defined in some simpler way. Magidor's algebra for a β essentially consists of canonical skolem functions for $\langle H_{\beta^+}, \in, F, D\rangle$ where F and D code (for $\gamma \leq \beta$) canonically least functions $f_\gamma : [\omega\gamma]^{<\omega} \to \gamma$ without an uncountable homogeneous set. (All the above defined inside K.) So without searching through $\mathcal{P}(\beta)$ can one define simpler algebras on β? The simplest possible algebra - (similar to the situation for L) one could ask for is just the algebra of rudimentary functions plus the function $\alpha \longrightarrow K_\alpha$ (closed under composition) for K_β, whose range is contained in β.

So, for any $\beta \geq \omega$ in On define a "Magidor" algebra on β as follows: let $\langle \tilde{g}^\beta_n \mid n < \omega\rangle$ be a sequence of Σ_1 skolem functions for K_β. Define $g^\beta_n : [\beta]^{k_n} \to \beta$ by: $g^\beta_n(\alpha_1, \ldots, \alpha_{k_n}) = \tilde{g}^\beta_n(\alpha_1, \ldots, \alpha_{k_n})$ if the latter is defined and is an ordinal $< \beta$; and let it be zero otherwise. As K_β (as defined below) is a model of the form $M = \langle J^{E^K}_\beta, \in, E^K\rangle$, there is a natural way to define the g^β_n's from the canonical Σ_1-skolem function h_M for this structure.

In fact closure under g^β_n's is characterised via ω-closed mice as below. We define $M = \langle J^E_\beta, E, E_\beta\rangle$ is an *ω-closed mouse* if, in L[M], E_β is an ω-closed filter. We say simply that E_β is *ω-closed* if it is so in $L[M]$.

Theorem 1 *Suppose no initial segment of K is ω-closed then every subset of β closed under the functions $\langle g_n^\beta \rangle$ is a countable union of sets in K.*

This is a precise characterisation, shown as follows.

Theorem 2 *If $K|\alpha = \langle J_\alpha^{E^K}, E^K, E_\alpha^K \rangle$ is an ω-closed mouse , E_α measuring β, then in a set forcing extension of the core model K, there is a subset of β, closed under the $\langle g_n^\beta \rangle$, not a union of countably many subsets in K.*

This follows by the same argument of Magidor to show that his no ω_1-Erdős cardinal assumption is necessary.

In the first section we shall give a brief overview of the construction of a model, that we might call "the core model below a Σ_2-mouse", in the sense that if $M = \langle |M|, \in, \mathcal{U} \rangle$ is any mouse structure with $M \models$ "$\mathcal{U}$ is a normal measure on κ" then there is a $\Sigma_1(M)$ map of a subset of κ onto M. We shall see that only mice of this kind are needed for the theorem. In fact almost all the applications of K in forms of consistency of Weak Chang conjecture's, various Erdős properties, all the Ramsey cardinals of [5], the failure of the transversal hypothesis at regular $\kappa > \omega_1$ of [9], only need this form of Σ_1-K. Indeed any of the "classical" large cardinals weaker than a measurable can be exemplified in such a model. (One can cook up some *ad hoc* properties involving Σ_2 mice, but to the author's knowledge, the only large cardinal - type property that implies the existence of a larger core model, involves the determinacy of a particular pointclass; doubtless there are others of such a nature.) So the simplified model we construct provides an easily defined laboratory for working on many principles at this level of consistency strength.

We shall exploit this mercilessly in the definitions and lemmas that follow; since all ultrapowers will be "Σ_0 ultrapowers" only using functions from the domains under consideration, many fine-structural considerations can be bypassed. Readers familiar with Jensen's paper [7] will recognise that the following account is based heavily on his construction there. Although some of the concepts here were developed on the path to the construction of much larger core models, there is also a modest secondary aim in perhaps giving a simple, first-time, introduction to such ideas in a simpler context. The form of covering lemma proof we use in §3 follows Magidor's proof for L: we shall define generalised Σ_n Skolem functions and hulls without using any fine structural considerations; hence we can dispense with the theory of "iterated reducts" and Σ^*-preserving ultrapowers. [Our ultrapowers are of course Σ^*-preserving, but in this simple context the concepts are the same.] We only have to worry about parameters for the definitions of soundness, and for core mouse considerations - so standard parameter definitions are also kept to a minimum. Again, this is only because of the very simple form of covering we are going to prove: a full covering lemma for such a core model would use the full apparatus of fine structure if we were to do this *au* Dodd-Jensen. §2 gives the body of the argument. The structure of the proof is very similar to that of [8] and we use the same numbering of cases as there. Although the organisation of K gives a much smoother theory and makes the whole argument like that for L, we have an extra case as our models are only closed under Σ_1 skolem functions and are not ZF^- models as they are in [8]. We do one case in full detail, others we really only do if they differ in any way from the [8] case, and refer the reader to that paper for further details if needed. The kind of ω-closed mouse considered gives rise to a "strong weak" ω_1-Erdős type property and subsequent to writing this we noticed that using some ideas from [3] we can make some simple-minded remarks (in §3) to locate this with respect to other such cardinals. The set-theoretical nomenclature is standard. *ot* denotes order type. All relevant information of the rudimentary functions, the J-hierarchy, and Σ_1-skolem functions etc. can be found in [2].

1 The Σ_1-Core Model

Definition 1 An *acceptable J-model* is an amenable structure of the form $N = \langle J_\beta^{\vec{A}}, \in, \vec{A}, \vec{B}\rangle$ where $\langle J_\beta^{\vec{A}}, \in, \vec{A}\rangle$ is the β'th level of the J-hierarchy relativised to predicates $\vec{A}$; *and* where

$$\forall\xi < \beta\forall\tau(\omega \leq \tau \leq \omega\xi \wedge \mathcal{P}(\tau) \cap J_{\xi+1}^{\vec{A}} \not\subseteq J_\xi^{\vec{A}} \Rightarrow (\overline{\overline{\omega\xi}} \leq \tau)_{J_{\xi+1}^{\vec{A}}}) .$$

Definition 2 Let N as above be acceptable.

$$\rho = \rho_N(\text{the } \Sigma_1\text{-projectum of } N) = \text{least } \rho \leq \beta \text{ s.t.} \underline{\Sigma}_1(N) \cap \mathcal{P}(\omega\rho) \not\subset N .$$

The underline indicates that parameters are allowed in the Σ_1 formulae defining the relations. The following facts are standard.

Fact 1. $\omega\rho$ is a Σ_1 cardinal of N, and $\omega\rho = \rho$ for $\rho > 1$. $\tau < \omega\rho \rightarrow \mathcal{P}(\tau) \cap N \subset J_\rho^{\vec{A}}$ (by acceptability);
Let $H_{\omega\rho}^N$ be the class of sets in N hereditarily of cardinality $< \rho$ in N. Then $H_{\omega\rho}^N = |J_\rho^{\vec{A}}|$.

Definition 3 Let h_N be the canonical Σ_1-skolem function for N. So dom $h_N \subseteq \omega \times |N|$ and

$$N \models \varphi_i(y, \vec{x}) \Rightarrow N \models \varphi_i(h(i, \vec{x}), \vec{x})$$

Fact 2. $h_N``(\omega \times X^{<\omega}) \prec_{\Sigma_1} N$ for any $X \subseteq |J_\beta^{\vec{A}}|$. $h_N``(\omega \times [\omega\beta]^{<\omega}) = N$.

We let $h_N(X)$ denote $h_N``(\omega \times X^{<\omega})$.

Definition 4 Let N be acceptable. Let $\rho = \rho_N$.

$$\begin{aligned} P_N &= \{p \in N \mid \exists B \subseteq \omega\beta,\ B \in \Sigma_1(N) \text{ in parameter } p,\ B \cap \omega\rho \notin N\} \\ R_N &= \{p \in N \mid |N| = h_N(\omega\rho \cup \{p\})\} \end{aligned}$$

Fact 3. $R_N \subseteq P_N \neq \emptyset$

Definition 5 N is *sound* if $R_N = P_N$.

The theory of (Σ_0)-ultrapowers is straight forward and we do not go over this again here. It suffices to say that if N is acceptable, $\langle N, \in, \mathcal{U}\rangle \models$ "$\mathcal{U}$ is a normal measure on κ" is amenable, then we may form the (Σ_0)-ultrapower $\mathrm{Ult}(N, \mathcal{U})$ using equivalence classes modulo $\mathcal{U}$ of functions $f \in N$. If wellfounded we identify $\mathrm{Ult}(N, \mathcal{U})$ with its transitive collapse and we have (using π as the usual embedding into the ultrapower).

Fact 4. $\pi : N \to_{\mathcal{U}} \mathrm{Ult}(N, \mathcal{U})$ is a Σ_0 cofinal (and hence) Σ_1 embedding; $\mathcal{P}(\kappa)^N = \mathcal{P}(\kappa)^{Ult(N,\mathcal{U})}$.

Definition 6 Let $M = \langle J_\alpha^{E^M}, \in, E^M, E_\alpha^M\rangle$ be an acceptable J-model. M is a *premouse* iff

a) $E^M = \{\langle X, \nu\rangle \mid \nu \leq \alpha \wedge X \in E_\nu^M\}$ such that $M \mid \nu =_{df} \langle J_\nu^E, \in, E^M, E_\nu^M\rangle$ is sound for $\nu < \alpha$.

b) For $\nu \leq \alpha$, if $E_\nu \neq \emptyset$, then

(i) there is $\kappa < \nu$ such that $M \mid \nu \models$ "E_ν is a normal measure on $\kappa \wedge \kappa$ is the largest cardinal",

(ii) $\rho^1_{M|\nu} \leq \kappa$.

c) If $\pi : (M \mid \nu) \to_{E_\nu} \mathrm{Ult}(M \mid \nu, E_\nu) = N$ then $E^N \restriction \nu + 1 = E^M \restriction \nu$.

The last is thus a coherency condition on premice. We use the notation $M||\nu$ to mean the structure $\langle J_\nu^{E^M}, \in, E^M\rangle$, that is with the last predicate E_ν^M removed. If an $M = \langle J_\nu^{E^M}, \in, E^M, E_\nu^M\rangle$ is a premouse with $E_\nu^M \neq \emptyset$ then we say M is *active* and set κ_M to be the critical point of E_ν^M. If $N = \mathrm{Ult}(M \mid \nu, E_\nu)$ is wellfounded, then $\nu = \kappa^{+N}$ (by Fact 4), and $E_\nu^N = \emptyset$ (by clause c).

Definition 7 Let M be a premouse. An *iteration* of M with indices $\langle \nu_i \mid i + 1 < \theta\rangle$ is a sequence of transitive iterates $\langle M_i \mid i < \theta\rangle$ and maps $\langle \pi_{ij}^M \mid i \leq j < \theta\rangle$ such that

(i) $M_0 = M$

(ii) The π_{ij} commute on the parts of the domains on which they are defined

(iii) If $E_{\nu_i} = \emptyset$, then $M_{i+1} = M_i$ and $\pi_{ii+1} = \mathrm{id} \restriction M_i$, or if $E_{\nu_i} \neq \emptyset$ then $M_{i+1} = \mathrm{Ult}(M_i \mid \nu_i, E_{\nu_i})$ and $\pi_{ii+1} : (M_i \mid \nu_i) \to_{E_{\nu_i}} M_{i+1}$

(iv) For at most finitely many i is $\nu_i < \mathrm{On} \cap M_i$

(v) $\mathrm{Lim}(\lambda) \Rightarrow \langle M_\lambda, \langle \pi_{i\lambda} \mid i < \lambda\rangle\rangle = \underrightarrow{\mathrm{Lim}}_{i \leq j < \lambda} \langle M_i, \pi_{ij}\rangle$.

Note. In (iii) we allow $E_{\nu_i} = \emptyset$, as a "do-nothing" step for later when we define the co-iteration of two premice. In (iv) we see the possibility of abandoning the top part of M_i, and "cutting down" or truncating the structure to a smaller initial segment. If this is $(M \mid \nu_i)$ the point is that ν_i is such that $M_i \mid \nu_i \models$ "E_{ν_i} is ultrafilter on $\mathcal{P}(\kappa_i)$" (where κ_i is the largest cardinal in $M_i \mid \nu_i$). For our mice we have immediately that $J_{\nu_i+1}^{E^{M_i}} \models$ "$\overline{\overline{\nu_i}} \leq \kappa_i$" and so E_{ν_i} is no longer an ultrafilter at one step beyond ν_i. (Recall $\rho^1_{M|\nu_i} \leq \kappa_i$).

Definition 8 An iteration is *simple* iff $\forall i+1<\theta \quad \nu_i = \mathrm{On}\cap M_i$, or ν_i is a point such that $E^{M_i}_{\nu_i} = \emptyset$.

Definition 9 A *mouse* is an iterable premouse where a premouse M is iterable if every iteration can be *continued*; that is

a) if $\theta = k+1$, $E^{M_k}_{\nu} \neq \emptyset$, then $\mathrm{Ult}(M_k \mid \nu, E^{M_k}_{\nu})$ is wellfounded

b) if $\mathrm{Lim}(\theta)$, then $\langle\langle M_i\rangle\langle\pi_{ij}\rangle_{i\leq j<\theta}\rangle$ has a well founded direct limit.

Fact 5. For *mice* M, clause (iv) in Definition 7 is redundant: there can be no "iteration" of M that involves infinitely many "truncation" or cut-down stages, to a $\nu_i < On \cap M_i$.

Various other facts about iterability are again straightforward.

Lemma 1 *If M is a mouse, $\bar{M}$ a premouse and $\sigma = \bar{M} \to_{\Sigma_0} M$. Then $\bar{M}$ is a mouse.*

Proof. See for example [4] Lemma 8.3. Q.E.D.

Lemma 2 *If every countable length iteration of a premouse M can be continued then M is a mouse.*

Lemma 3 ("*The Dodd-Jensen Lemma* ") *If M is an iterate of the mouse N (so the final model in some iteration) and $\pi : N \to M$ is the iteration map, and if $\sigma : N \to_{\Sigma_1} M$, then the iteration from N to M is simple; and $\sigma(\xi) \geq \pi(\xi)$ for all $\xi \in N$.*

This fundamental lemma allows the following corollary.

Corollary 4 *M cannot be both a simple and a non-simple iterate of a mouse N. Let M be a simple iterate of $\bar{M}$; then the iteration map is unique.*

We denote by $\pi_{\bar{M}M}$ that unique iteration map given above.

Definition 10 $\langle\langle M_i\rangle, \langle\nu_i\rangle\langle\pi^M_{ij}\rangle\rangle_{i+1<\theta}$ is a *normal* iteration if, whenever $i<j$ then $\kappa_j > \nu_i$ if $E^{M_j}_{\nu_j} \neq \emptyset$ and measures κ_j, and $\nu_j > \nu_i$ if $E^{M_j}_{\nu_j} = \emptyset$.

A normal iteration is then where ascending measurables are used. Two mice M, N are said to be *comparable* if either $M = N$, or $\exists\nu \in N(N \mid \nu = M)$ or $\exists\nu \in M(M \mid \nu = N)$.

Definition 11 The *coiteration* of two mice (M, N) is the iteration pair $\langle M_i, \pi^M_{ij}\rangle_{i\leq j\leq\theta}$, $\langle N_i, \pi^N_{ij}\rangle_{i\leq j\leq\theta}$ with *common indices* $\langle\nu_i\rangle_{i+1<\theta}$ where

$$\nu_i \simeq \text{ the least } \nu \text{ so that } E^{M_i}_{\nu} \neq E^{N_i}_{\nu}$$

where $M_i = \langle J^{E^{M_i}}_{\beta}, E^{M_i}, E^{M_i}_{\beta}\rangle$ and $N_i = \langle J^{E^{N_i}}_{\gamma}, E^{N_i}, E^{N_i}_{\gamma}\rangle$ say. θ is least such that M_θ is an initial segment (not necessarily proper) of N_θ, or *vice versa*.

The following is easily verified. (Invoking Fact 5 to ensure there will not be infinitely many truncations in any coiteration of mice.)

Lemma 5 *(i) The coiteration (M,N) exists and results in each iteration $\langle M_i, \pi^M_{ij}, \nu_i\rangle$, $\langle N_i, \pi^N_{ij}, \nu_i\rangle$ being normal.*

(ii) Let λ be a cardinal $> \overline{\overline{M}}$, $\overline{\overline{N}}$. Then the coiteration of (M,N) has length $\theta < \lambda$.

Proof. (Sketch of (ii)). If not the coiteration continues for λ steps. After the last truncation stage $i_0 < \lambda$ for either side, only the same top measure is used on both sides. After iterating through each others ordinals at least ω^2 many times it is easy to see that the resulting structures have a common generating sequence and so are comparable. Q.E.D.

Lemma 6 *If M, N are two mice and they coiterate to (M_θ, N_θ) then either $M_\theta = N_\theta$*

$$\text{or}\quad M_\theta \text{ is a proper initial sequent of } N_\theta, \text{ and } M_0 = M_\theta\,.$$
$$\text{or}\quad N_\theta \text{ is a proper initial segment of } M_\theta, \text{ and } N_0 = N_\theta\,.$$

Proof. Suppose equality fails, and without loss of generality M_θ is a proper initial segment of N_θ. But all initial segments of N_θ are sound. So no ultrapower at all was taken on the M side. Q.E.D.

Corollary 7 *If (M,N) coiterate to (M_θ, N_θ) and "both sides move"* (i.e. $M \neq M_\theta, N \neq N_\theta$) then $M_\theta = N_\theta$.

We shall see below that in a coiteration of mice *only* one side moves. Thus for mice in K coiteration is particularly simple.

Definition 12 Let N be a mouse, the *core* of N is the unique sound mouse $\bar{N} = \text{core}(N)$ so that N is a simple iterate of $\bar{N}$.

Even for very simple mice *some* work must be done to show that cores exist.

Definition 13 Let N be a mouse, the *standard* (Σ_1) *parameter* p_N is the $<^*$-least $p \in P_N$ where $a <^* b \Leftrightarrow \exists\nu(a\backslash(\nu+1) = b\backslash(\nu+1) \wedge \nu \in b\backslash a))$.

Lemma 8 *Let N be a mouse; let M be a simple iterate of N with map π_{NM}. Then $\pi_{NM}(p_N) = p_M; \rho_M = \rho_N$.*

Here again the proof is quite straight forward: as in [4] chapters 9 and 10; $h_M(\kappa_M \cup \pi_{NM}(p_N)) = M$, hence $\rho_M \leq \kappa_M$, but then $\pi_{NM}(p_N) \neq p_M$ leads to a contradiction as in [4] Lemma 10.4. We note that $\Sigma_1(M) \cap \mathcal{P}(\kappa_N) = \Sigma_1(N) \cap \mathcal{P}(\kappa_N)$.

Lemma 9 *Let N be a mouse then core(N) exists.*

$\mathrm{Core}(N)$ is simply the transitive collapse of $X \prec N$ where $X = h(\rho_N \cup \{p_N\})$. The comparison of two mice then has the following form.

Lemma 10 *Let (M, N) be mice that coiterate to (M_θ, N_θ). Then at most one side moves: i.e. $M_\theta = M$ or $N_\theta = N$.*

Proof. Suppose for a contradiction both sides move: then by Corollary 7, $M_\theta = N_\theta$. Suppose both sides of the coiteration are simple: $\mathrm{core}(M_\theta) = \mathrm{core}(N_\theta) = \tilde{N}$. say. But then both M and N are simple iterates of $\tilde{N}$, and so M is a simple iterate of N or *vice versa* and we are done. So let j be the last index at which a truncation is performed, without loss of generality suppose $N_j \mid \nu_j \to N_{j+1}$ where $\nu_j < \mathrm{On} \cap N_j$. Then $\mathrm{core}(N_\theta) = N_j \mid \nu_j = \mathrm{core}(M_\theta)$. We claim that no truncation is performed on the M side. Suppose otherwise and that $j' \leq j$ with $M_{j'} \mid \nu_{j'} \to M_{j'+1}$ is the last such truncation point. Then $\mathrm{core}(M_\theta) = M_{j'} \mid \nu_{j'} = N_j | \nu_j$. But then $\nu_{j'} = \nu_j$ and $j = j'$ and this contradicts the definition of coiteration. So M_θ is a simple iterate of M_0, and so M_0 is a simple iterate of $N_j|\nu_j = \mathrm{core}(M_\theta)$. So $M_\theta = M_0$. Q.E.D.

Corollary 11 (i) *If $\langle J^E_\alpha, F\rangle$ and $\langle J^E_\alpha, G\rangle$ are mice, with $F, G \neq \emptyset$ then $F = G$.*

(ii) *For any two mice M, N, one is an initial segment of an iterate of the other.*

Definition 14 Let $M =_* N$ if M, N are mice with a common simple iterate. Let $M <_* N$ if M is a proper initial segment of an iterate of N.

Remark. From the above, if $M =_* N$ then either M is a simple iterate of N (or *vice versa*). And if $M <_* N$, then we may find a common iterate P by performing the coiteration of (M, N) (in which M does not move), and if need be, after the termination of the official coiteration, performing a final truncation on the N side.

From the Dodd-Jensen Lemma we can obtain:

Lemma 12 *$\leq_*$ is a prewellordering on mice.*

The theory of Σ_0-ultrapowers yields the following indiscernibility property

Lemma 13 *If M is a simple iterate of N with map $\pi_{NM} : N \to M$ and if $\langle \kappa_i \mid i < \theta\rangle$ are the critical points of the iteration, then they are Σ_1-indiscernibles for $\langle M, \mathrm{ran}\ \pi_{NM}\rangle$.*

Definition 15 Let M be a mouse with $\mathrm{On} \cap M = \alpha$. M is a *strong mouse* if whenever $N \supseteq M$ is a mouse then $\mathrm{core}(N) \mid \alpha = M$.

Lemma 14 *If M, N are mice, $M \leq_* N$ then M is a strong mouse iff for all sound mice $N \geq_* M$, $M = N|(\mathrm{On} \cap M)$.*

Proof. ($\Rightarrow$) Let $\alpha = \mathrm{On} \cap M$. First note that if M is strong, then $\mathrm{core}(M) = M$. Suppose $N \geq_* M$. Coiterate (M, N) to (M_θ, N_θ). By Lemma 10 $M = M_\theta$. Let $j_1 < \ldots < j_k$ be the truncation indices (if any) in the iteration of N. Since $M \subseteq N_\theta$ by the strength of M, $M = \mathrm{core}(N_\theta)|\alpha$; but $\mathrm{core}(N_\theta) = N_{j_k}|\nu_{j_k}$. Hence $M \subseteq N_{j_k}$ and $M = \mathrm{core}(N_{j_k})|\alpha$ where $\mathrm{core}(N_{j_k}) = N_{j_{k-1}}|\nu_{j_{k-1}}$. Proceeding in this way we have $M = N_{j_1}|\alpha$ and hence $M = \mathrm{core}(N_{j_1})|\alpha = N|\alpha$ as required. The ($\Leftarrow$) direction is trivial using $N \supseteq M \Rightarrow M \leq_* \mathrm{core}(N)$. Q.E.D.

Corollary 15 *If M, N both strong mice, then one is an initial segment of the other; if M is strong then all initial segments of M are strong.*

Proof. The first statement is immediate. Let $N = M|\beta$ be an initial segment. Let $P \geq_* N$ be sound, but suppose $P|\beta \neq N$. If $P \geq_* M$, the result follows from Lemma 14. If $P =_* N$ then, as N is sound, $P = N$ and we are done. Hence suppose $N <_* P <_* M$. Consider the coiterations of (N, P) and of (P, M). In the former only P moves, and in the latter only M. If ν_0 is the first index used in the coiteration (N, P), it must also be the first index used in (P, M) and it moves P there and not M. Contradiction! Q.E.D.

Definition 16 $K = \langle K, \in, E^K \rangle$ where E^K_ν is defined as follows: $E^K_0 = E^K_{\nu+1} = \emptyset$;

For any ν set $E^K \restriction \nu = \{\langle X, \mu \rangle | \mu < \nu \cap X \in E^K_\mu\}$; then for $Lim(\lambda)$, assuming E^K_ν defined for $\nu < \lambda$, and writing E for $E^K \restriction \lambda$, set:

$$E^K_\lambda \simeq \begin{cases} F & \text{if } F \text{ is any filter so that } \langle J^E_\lambda, E, F \rangle \text{ is a strong mouse} \\ \emptyset & \text{if there is no such } F. \end{cases}$$

Set $E^K = \cup_\lambda E^K \restriction \lambda$. $K|\alpha = \langle J^{E^K}_\alpha, E^K \restriction \alpha, E^K_\alpha \rangle$ for all α; $K_\alpha = \langle J^{E^K}_\alpha, \in, E^K \restriction \alpha \rangle$.

By Corollary 11, if there is such an F a candidate for $E^{K_\lambda}_\lambda$ then it is unique. Hence K is properly defined. By Corollary 15 every strong mouse is an initial segment of K: hence K is the union of all strong mice. We may define the coiteration of a mouse M with K just as for two mice. Properly formulated we may define "class-sized" mice of the form $W = \langle J^{E^W}_{On}, E^W \rangle$, and the notions of their comparison, and we may extend the $\leq_*$ notation to allow such classes in its field.

Lemma 16 *K is universal, that is for any mouse M, $M <_* K$.*

Proof. If not the coiteration of M with K to (M_∞, K_∞) - that is On many times, has M "iterating past" all of K with K being unmoved (by the argument of Lemma 10). Let θ be any cardinal, $\theta > \overline{\overline{M}}$. Then $On \cap M_\theta > \theta$ and $M_\theta \supset K|\theta$ (as M_θ is a direct limit of small structures). But $\mathrm{core}(M_\theta)|\theta \neq K|\theta$ contradicting the latter's strength.

Lemma 17 *Let M be any mouse. Then $M \in K$.*

Proof. Note that if N is any mouse, N_θ any iterate of N, then $N_\theta \in L[N]$: the iteration $N \to N_\theta$ is definable from a finite sequence of points where truncations are made, together with a finite sequence of ordinals giving the lengths of the iteration (which can only be of the top measure of N_i at any point) between the truncation stages. The lemma is trivial if M is an initial segment of K, so suppose otherwise. Now in the comparison of M with K, to $(M_\theta, (K)_\theta)$ we have $M = M_\theta$; hence $M \in L[K_{\theta}+]$. Q.E.D.

Using simplified versions of the methods of [7] one obtains:

Lemma 18 *If $j : K \to_{\Sigma_\omega} K$ then $j = id$ (unless there exist Σ_2-mice).*

Lemma 19 *K is the unique class W, such that $W >_* M$ for all mice M, and that has all initial segments sound mice.*

Proof. Let $W = L[E]$ be another such universal class. By the universality of K the comparison of $(W \mid \alpha, K)$ has $W \mid \alpha$ unmoved, and symmetrically, by assumption, for all α the comparison of $(W, K \mid \alpha)$ has $K \mid \alpha$ unmoved. But now let α be least so that $E^W_\alpha \neq E^K_\alpha$ -one of the two is empty by Corollary 11. If $E^W_\alpha \neq \emptyset$, say, then $(W \mid \alpha, K)$ has $W \mid \alpha$ moving - a contradiction. But $E^K_\alpha \neq \emptyset$ implies $K \mid \alpha$ moves in the comparison of $(W, K \mid \alpha)$! Q.E.D.

Lemma 20 (i) "x codes a countable initial segment of K" is Π^1_3; (ii) $\mathbb{R} \cap K$ is Σ^1_3.

Proof. (i) "x codes a strong mouse M_x" is "$\forall N(N \supseteq M_x \Rightarrow$ (if N is an iterable premouse $\Rightarrow M_x = \mathrm{core}(N)|\mathrm{On} \cap M_x$))" N is an iterable premouse is Π^1_2 as ever. For (ii): $r \in \mathbb{R} \cap K \Leftrightarrow \exists M$ (M a mouse $\wedge r \in M$), by Lemma 17. Q.E.D.

We make some general comments on the modifications to the above arguments needed to define the full core model below a measurable cardinal. The main point naturally is that "Σ_2" (and more) mice exist, and so we must define notions of generalised Σ_n-preserving embeddings and ultrapowers ("Σ^*-preserving") which requires the notions of iterated projecta and parameters. The notion of premouse removes clause bii) and clause c) is then modified to allow for such embeddings, as is the notion of iteration at clauses (iii) and (v). Up to Lemma 9, but excepting Lemma 6 and Cor. 7, no further modifications to the *statements* of the lemmas and definitions are needed. But in general coiterations involve movement on both sides without resulting in a necessary equality between N_θ and M_θ (Cor. 7). (But if both sides of a coiteration are non-simple then one still concludes that $N_\theta = M_\theta$.) However if we call an active mouse a "Σ_ω-mouse" if there is a collapsing map definable over it, from its measurable cardinal onto the whole structure, and if all active mice are Σ_ω-mice, then Lemma 6 and Cor. 7 remain valid as stated. Lemma 10 and Cor.11 again are only true if all active mice are Σ_ω. Otherwise they are false. The notion of core mouse is tied up with the more generalised parameters required, but Lemma 9 is valid. The definition of $\leq_*$ is correct for larger mice. The definition of strength in Definition 15 is good if all active mice are Σ_ω but otherwise the full definition from [7] is needed. The characterisation in Lemma 14 fails once there are active mice that are not Σ_ω. However Cor. 15 and the definition of K are valid in the wider context. K is naturally still universal in the sense of Lemma 16, but there may be mice not members of K (however all Σ_ω-mice are in K). The characterisations of Lemmas 19 & 20 go through. Returning to our situation, if there are no Σ_2-mice, then elementary covering lemma arguments would show that the K as defined here would have the same domain as that of [4], as indeed would be the case for the full K^{DJ} below a measurable cardinal, when compared to the full K defined according to this new style construction.

1.1 Skolem functions

Set $\mathcal{A} = \langle J_\alpha^{E^K}, E^K, E^K_\alpha \rangle$. Let $\mathcal{L} = \mathcal{L}(\dot{\in}, \dot{=} \dot{E}, \dot{F})$ be the appropriate language for such structures. Define Σ_n formulae in such a language in the usual way. Following [8] for $\Phi(v_0, v_1, \ldots, v_n)$ for a Σ_n formula, set $h_\Phi^{\mathcal{A}}(x_1, \ldots, x_n) =$ the $<_{\mathcal{A}}$-least x such that $\Phi(x, x_1, \ldots, x_n)$ (if such exists) and is 0 otherwise. Then h_Φ is a canonical skolem function, and we let $\tau_\Phi(v_1, \ldots, v_n)$ be the canonical skolem term denoting this function. We note that the canonical wellorder $<_{\mathcal{A}}$ is $\Sigma_1^{\mathcal{A}}$, but that $h_\Phi^{\mathcal{A}}$ is not necessarily Σ_n. Let $j : \mathcal{A} \to \mathcal{B}$ be a Σ_n preserving embedding $(1 \leq n \leq \omega)$ between two such structures, then for all $x_1, \ldots x_n \in \mathcal{A}$ and every Σ_n skolem term τ we have $\tau^{\mathcal{B}}(j(x_1), \ldots, j(x_n)) = j(\tau^{\mathcal{A}}(x_1, \ldots, x_n))$.

Definition 17 If $X \subseteq \mathcal{A}$, then $H_n^{\mathcal{A}}(X)$ is the minimal substructure of $\mathcal{A}$, containing X, and closed under the Σ_n skolem functions of $\mathcal{A}$ (for $1 \leq n \leq \omega$).

1.2 The Indiscernibles Lemma

We formulate Jensen's Indiscernibles Lemma (see [4] Ch. 16).

Definition 18 If $\mathcal{A} = \langle J_\alpha^{\vec{A}}, \in, \vec{A}, \vec{B} \rangle$ is a transitive amenable model, then $I \subseteq \omega\alpha$ is a *good set of indiscernibles* for $\mathcal{A}$ if (i) $I \subseteq \omega\alpha$; (ii) $I \backslash \gamma$ is a set of indiscernibles for $\langle \mathcal{A}, \langle \dot{\xi} \rangle_{\xi < \gamma} \rangle$
(iii) $\gamma \in I \Rightarrow \mathcal{A} \mid \gamma =_{df} \langle J_\gamma^{\vec{A}} \in, \vec{A}, \vec{B} \restriction \gamma \rangle \prec \mathcal{A}$.

It is easy to see that if $\mathcal{A}$ has an infinite good set of indiscernibles, then each $\gamma \in I$ is strongly inaccessible in $\mathcal{A}$, and thus $\mathcal{A} \models$ ZFC. Jensen's Indiscernibles Lemma (stated slightly more generally than in [4]) is

Lemma 21 *Let $\bar{K} = \langle \bar{K}_\alpha, \in, \bar{E}, \vec{B} \rangle \models$ "$V = K$", and let $I \subseteq \omega\alpha$ be good indiscernibles for $\bar{K}$, with cf* (ot $(I)) > \omega$. Then there is $I' \supseteq I$, $I' \in K$, I' good indiscernibles for $\bar{K}$.

We shall use a couple of observations on the proof of this lemma. The proof in [4] although given for the "old-style" K is easily transformed into the current setting.

Fact 6. Let $\bar{K}$, I be as above. Let $\gamma^* = \sup I$. Then $cf(\gamma^*) \geq \omega_1$ and the proof shows there is a mouse $M_{\gamma^*} = \langle J^{E^{M_{\gamma^*}}}_{\beta_{\gamma^*}}, \in, E^{M_{\gamma^*}}_{\beta_{\gamma^*}} \rangle \models$ "$E_{\beta_{\gamma^*}}$ is a normal measure on γ^*" $E^{M_{\gamma^*}} \restriction \gamma^* = E^{\bar{K}} \restriction \gamma^*$; $E^{M_{\gamma^*}}_{\beta_{\gamma^*}}$ is ω-closed, being obtained as the final segment filter on I.

Fact 7. The set $I^* \supseteq I$ of good indiscernibles is obtained as $\bigcap_{n<\omega} X_n$ where each X_n is $\Sigma_1(M_{\gamma^*})$.

Lemma 22 *Let $M = \langle J^{E^M}_\alpha, E^M, E^M_\alpha \rangle \models$ "E^M_α is a measure on κ", where M is a mouse. Let $K^M_\kappa = \langle J^{E^M}_\kappa, E^M \rangle = M \mid \kappa$. Then, in $L[M]$, the following are equivalent*: (i) There is a mouse $N = \langle J^{E^N}_\beta, \in, E^N_\beta \rangle \models$ "E^N_β is a measure on κ" with $N \mid \kappa = M \mid \kappa$, and E^N_β is ω-closed.
(ii) $\exists I (I$ good for $M \mid \kappa$, cf (ot(I)) $> \omega)$.

Proof. (i) $\Rightarrow$ (ii) follows by the argument of [4] p. 131, which shows for any mouse N as above there are $X_n \in E^N_\beta$ such that $\bigcap_n X_n$ (if non-empty) are good indiscernibles for $M \mid \kappa$. (ii) $\Rightarrow$ (i) is the lemma and accompanying facts above. Q.E.D.

Lemma 23 *Let M be active with $M \models$ "E^M_α is a normal measure on κ", M a mouse.* (i) *Suppose $cf^{L[M]}(\alpha) > \omega$, then there is $N = \langle J^{E^M}_\beta, E^M \restriction \beta, E^M_\alpha \restriction \beta \rangle$ which is an ω-closed mouse in $L[M]$.* (ii) *If M is sound and $L[M] \models$ "$cf^{L[M]}(\alpha) = \omega$ and E^M_α is ω-closed", then $\rho_M = \kappa$.*

Proof. If $cf(\alpha) > \omega$, then we may easily find a $\beta < \alpha$ such that N as above is amenable. It is easy to verify that N satisfies the clauses for being a premouse and hence a mouse. But $\mathcal{P}(\kappa)^{L[N]} \subseteq M$, and as $M \models$ "E^M_α is κ-complete" this suffices. For (ii): by our definition of active mouse $\rho_M \leq \kappa$. In $L[M]$, let $\langle \beta_n \mid n < \omega \rangle$ be a sequence cofinal in α so that $E_n =_{df} E^M_\alpha \cap J^{E^M}_{\beta_n} \in J^{E^M}_{\beta_{n+1}}$. . Clearly $cf(\kappa) \neq \omega$ (by ω-closure). Suppose $\rho_M < \kappa$. By Soundness h_M"$(\omega \times \{\rho_M \cup \{p_M\}\}) = |M|$. Let $\gamma_n = \sup h_{\langle J^M_{\beta_{n+1}}, E_n \rangle}$"$(\omega \times \{\rho_M \cup \{p_M\}\}) \cap \kappa$. Then $\langle \gamma_n \rangle$ is cofinal in κ - a contradiction. (See the argument in Case 3 (i) of the Main theorem below.) Q.E.D.

Lemma 24 *Let $M = \langle J^{E^M}_\alpha, E^M_\alpha \rangle$ be active, with E^M_α on κ. Let M satisfy all the conditions for being a Σ_1-mouse, except that $\rho_M = \alpha$. Then there is N an active initial segment of M, $N = \langle J^{E^M}_\beta, E^M \restriction \beta, E^M_\beta \rangle \models$ "E^M_β is a measure on κ", $E^M_\beta \subseteq E^M_\alpha$, and $L[N] \models$ "E^M_β is ω-closed."*

Proof. Since $\rho_M = \alpha$, $M \models$ "$KP + \Sigma_1$-separation". We may find in $E^M_\alpha \cap M$ the sets X_n (in a Σ^M_1 way so that the sequence $\langle X_n \mid n \in \omega \rangle \in M$) whose intersection gives $I \subseteq \kappa$ a good set of indiscernibles for $J_\kappa^{E^M}$, with $I \in M$ and hence $ot(I) = \kappa$. We may now construct the requisite initial segment N inside M using the (ii) $\Rightarrow$ (i) direction of Lemma 22; E_β^M is the final segment filter on I, and $\mathrm{cf}^M(\mathrm{ot}\ (I)) = \kappa$. Q.E.D.

Thus sound ω-closed mice occur (well below) "Σ_2-mice" in the $<_*$ ordering of all mice. On the other hand in $L^\sharp$ (The smallest inner model of $\forall X \subseteq \mathrm{On}(X^\sharp$ exists)) all active mice M have $\rho_M < \kappa_M$.

2 Decomposing ordinals and subsets of K

The following is the main definition of ordinal collapsing from [8].

Definition 1 *The ordinal α* decomposes at a mouse *M if $\alpha \subseteq H_\omega^M(X \cup \rho)$ for some $\rho < \alpha$, and some countable X (not necessarily in K).*

Trivially, every α that is not a K-cardinal decomposes at some mouse. And of "decomposing mice" for α, there will be some M which is in the $\leq_*$-least equivalence class of such mice, and if such an M is active there will be a least iterate of $\mathrm{core}(M)$ at which α decomposes.

Definition 2 *If α decomposes, we set $M(\alpha)$ to be the least simple iterate of the $\leq_*$-least sound mouse at which α decomposes. We set $n(\alpha)$ to be the least n, $1 \leq n \leq \omega$, so that $\alpha \subseteq H_n^M(X \cup \rho)$ for some $\rho < \alpha$ and countable X.*

We proceed to prove

Theorem 1 *Suppose there are no ω-closed mice on E^K. Let $\tilde{\mathcal{G}}$ be the algebra of functions consisting of the rudimentary functions, and the function $\alpha \mapsto K_\alpha$, and closed under composition. For $\tilde{g} \in \tilde{\mathcal{G}}$ define $g(\alpha_1, \ldots, \alpha_{k_n}) = \tilde{g}(\alpha_1, \ldots, \alpha_{k_n})$ if the latter is defined and is an ordinal $< \beta$; and let it be zero otherwise. Let $\mathcal{G}$ be the algebra of such g's. Let $X \subseteq \beta$ be closed under the functions of $\mathcal{G}$. Then X is the countable union of sets $\langle Y_n \mid n < \omega \rangle$ with each $Y_n \in K$.*

Let $X \subseteq \beta$ be such a set. Following the strategy of [8] we prove by induction on $\gamma \leq \beta$ that $\gamma \in X \cup \{\beta\} \Rightarrow X \cap \gamma$ is such a countable union of sets in K. We assume that X is cofinal in β. Such an X is then closed under the canonical Σ_1-skolem function, h_{K_β}, for K_β. The induction steps are trivial except for the case that $\mathrm{cf}\ (\mathrm{ot}(X \cap \gamma)) > \omega$ so we consider such a case. Let $Y = h_{K_\beta}(X)$. By assumption $Y \cap On = X$.

Let $j : \bar{K} \overset{\sim}{\Leftrightarrow} Y$ be the inverse of the transitive collapse. Then $j : \bar{K} \rightarrow_{\Sigma_1} K_\beta$. Let $\kappa = \mathrm{crit}(j)$, the first point moved by j. (If there is no such κ, then j is the identity and $X = \beta \in K$!) Let $j(\alpha) = \gamma$ if $\gamma < \beta$, and otherwise set $\alpha = On \cap \bar{K}$. Note by our assumption on γ, $\alpha > \kappa$. The following corresponds to Lemma 2.4 of [8].

Lemma 1 *If $(K_\beta, \bar{K})$ coiterate to $(K'_\theta, \bar{K}_\theta)$ then $\theta < \omega_1$.*

Proof. Suppose $\theta \geq \omega_1$. In the comparison $\bar{K}$ does not move (so $\bar{K}_\theta = \bar{K}$). Let j be the last truncation stage less than ω_1, the iteration on the K_β side is some mouse iterating past $\bar{K}$ using indices $\langle \nu_i \rangle_{j \leq i < \theta}$ on critical points $\langle \kappa_i \rangle_{j \leq i < \theta}$. Then $\langle \kappa_i \rangle_{j \leq i < \omega_1}$ is a sequence of Σ_1 indiscernible for $\langle K'_{\omega_1},\ \mathrm{ran}\ \pi_{j\omega_1} \rangle$ and it is easily seen to be an ω_1-sequence of good indiscernibles for $K'_{\omega_1} \mid \kappa_{\omega_1} = \bar{K} \mid \kappa_{\omega_1}$ (where we have set $\kappa_{\omega_1} = \sup_{i<\omega_1} \kappa_i$). (The "remarkability" property of the $\langle \kappa_i \rangle$ sequence follows from the fact that for any j $\langle \kappa_k \rangle_{j<k\leq\omega_1}$ is a sequence of Σ_1 indiscernibles for $\langle K'_{\omega_1},\ \mathrm{ran}\ \pi_{j\omega_1},\ \langle \xi \rangle_{\xi < \kappa_j} \rangle$; the "goodness" property from $\bar{K} \mid \kappa_i = K'_{\kappa_i} \prec K'_{\kappa_k} = \bar{K} \mid \kappa_k$ for $j \leq i < k \leq \omega_1$).) Let $I = \langle j(\kappa_i) \mid j \leq i < \omega_1 \rangle$. Then I is an ω_1-sequence of good indiscernibles for K_β, and we can employ the indiscernibles lemma to find an ω-closed sound mouse on $\sup I$, and hence on the K sequence, contrary to hypothesis. Q.E.D.

The argument in fact shows

Corollary 2 *Let $M \geq_* \bar{K}$. Then if $(M, \bar{K})$ coiterate to $(M_\theta, \bar{K})$, $\theta < \omega_1$. The same holds if we replace $\bar{K}$ by any $\bar{K} \mid \tau$.*

Lemma 3 *Suppose α does not decompose at any mouse $M \in L[E^{\bar{K}} \restriction \delta]$ for any $\delta < \alpha$. Let M be a mouse,* $\mathrm{On} \cap M < \alpha$, *<u>either</u> α decomposes at a countable iterate of M, <u>or</u> $M \in \bar{K} \mid \alpha$.*

Proof. If $M \geq_* \bar{K} \mid \alpha$ then by Corollary 2 and the argument of Lemma 1, we have a countable iterate of M decomposing α, which we get by comparing M with $\bar{K}|\alpha$. So suppose $M <_* \bar{K} \mid \alpha$. As $cf(\alpha) > \omega, E^{\bar{K}}_\alpha = \emptyset$, so $M \leq_* \bar{K}|\nu$ for some least $\nu < \alpha$. If $(M, \bar{K}|\nu)$ compare to $(M, (\bar{K}|\nu)_\tau)$ then as $On \cap M < \alpha$, $\tau < \alpha$. If $\bar{K}|\alpha$ is a ZF^- model or a union of such, then $(\bar{K}|\nu)_\tau \in \bar{K}|\alpha$ and hence so is M. Otherwise $\{\nu < \alpha | \bar{E}_\nu \neq \emptyset\}$ is bounded below α by some δ, and $\bar{K}|\alpha \models$ "there is a largest cardinal λ" and some instance of replacement fails over $\bar{K}|\alpha$. But then α decomposes at $\bar{K}|\alpha \in L[E \restriction \delta]$ contrary to hypothesis. Q.E.D.

Lemma 4 *Let $\alpha = j^{-1}(\gamma)$ be as above. Then α decomposes at some mouse.*

Proof. Suppose not. Compare $(M, \bar{K} \mid \alpha)$ to $(M_\theta, \bar{K} \mid \alpha)$ where $M_0 = M = K_\beta$.

(1) $\theta = 0$, that is $\bar{K} \mid \alpha$ is an initial segment of K_β.

Proof. If not let ν_0, κ_0 be the first index and critical point used, on the K side. If $\bar{K} \mid \alpha$ is not an initial segment of K_β then $\kappa_0 < \alpha$, $\nu_0 \leq \alpha$. By Lemma 1 some countable θ'th iterate of $M \mid \nu_0$ has $\bar{K} \mid \alpha$ as an initial segment. If $j < \theta$ is the final truncation point, then as $M_\theta = h_{M_\theta}(\kappa_j \cup \{\kappa_i \mid j \leq i < \theta\} \cup \{p_{M_\theta}\})$ we have $\alpha > \kappa_j$ decomposing at M_θ as witnessed by κ_j and the countable set $X = \{\kappa_i \mid j \leq i < \theta\} \cup \{p_{M_\theta}\}$-contrary to assumption. Q.E.D.(1)

We thus have: $\kappa < \alpha$, α is a K-cardinal, and by (1):

(2) $P(\kappa)^K \subseteq \bar{K}$;

Define a measure $\mathcal{U}$ as usual on $P(\kappa)^K : X \in \mathcal{U} \Leftrightarrow \kappa \in j(X)$.

(3) $Ult(K, \mathcal{U})$ is wellfounded
Proof. Let $\langle [f_k]_{\mathcal{U}} | k < \omega \rangle$ be an $\in_{Ult}$ descending chain of "ordinals" in the ultrapower; suppose λ is such that for all k $\quad f_k \in K_\lambda$; let $X = h_{K_\lambda}(\{f_k\}_{k<\omega} \cup \kappa)$ and let $\pi : M \leftrightarrow X$ be the inverse of the transitive collapse and set $g_k = \pi^{-1}(f_k)$ for $k < \omega$. As α does not decompose at any mouse, $On \cap M < \alpha$. (It cannot equal α as $cf(\alpha)) \neq \omega$.) But then by the previous lemma, $M \in \bar{K}|\alpha$, hence $g_k \in dom(j)$ for all k. But for all $k, \kappa \in j(\{\tau < \kappa | f_{k+1}(\tau) \in f_k(\tau)\}) = j(\{\tau < \kappa | g_{k+1}(\tau) \in g_k(\tau)\}) = j(\{\tau < \kappa | j(g_{k+1})(\tau) \in j(g_k)(\tau)\})$, hence $j(g_{k+1})(\kappa) \in j(g_k)(\kappa)$, as usual a contradiction. Q.E.D.(3)
But (3) gives an embedding of K to K contradicting Lemma 18. Q.E.D. Lemma 4.

Remark In [8], the structure Y was obtained by closing under all Σ_n skolem functions of a ZF^- model. The desired conclusion of Lemma 3 was simply that $M \in \bar{K}$. As we don't have this amount of closure we get an extra assumption which we have to discharge as an (easy) extra case later. But until Case 4 we assume α does not decompose at any mouse $M \in L[E^{\bar{K}} \restriction \beta]$ for any $\beta < \alpha$. We now divide into cases much as in [8]. We do the first case in some detail, although it is most like the corresponding case in [8], to fix some notation.

Case 1. α decomposes at $M(\alpha)$, with $1 < n(\alpha) < \omega$.

Set $n = n(\alpha)$, $n = m+1$, $M = M(\alpha) = \langle J_\beta^{E^M}, E^M, E_\beta^M \rangle$. We form a directed system of mice $M_{\langle p,\mu \rangle}$ where $\overline{\overline{p}} < \omega, \mu < \alpha$

$$M_{\langle p,\mu \rangle} = \pi^{-1}_{\langle p,\mu \rangle} \text{``} H_m^M(p \cup \mu)$$

and $\pi^{-1}_{\langle p,\mu \rangle}$ is the transitive collapse of the hull $H_m^M(p \cup \mu)$. Since $\pi_{\langle p,\mu \rangle} : M_{\langle p,\mu \rangle} \rightarrow_{\Sigma_m} M$, $M_{\langle p,\mu \rangle}$ is a mouse. Set $\langle p, \mu \rangle \leq \langle p', \mu' \rangle$ if $p' \supseteq p$ and $\mu' \geq \mu$. The set I of such indices ordered in this way, enables us to define $\pi_{ij} : M_i \rightarrow_{\Sigma_m} M_j$ where $i \leq j$, $i = \langle p, \mu \rangle, j = \langle p', \mu' \rangle \in I$ and further $M = \varinjlim \langle \langle M_i \rangle, \langle \pi_{i,j} \rangle \rangle_{i \leq j \in I}$.

(1) $\forall \langle p, \mu \rangle \in I$, $M_{\langle p,\mu \rangle} \in \bar{K}$.

Proof. Suppose for a contradiction that $M_{\langle p,\mu \rangle} \notin \bar{K}|\alpha$. Then α decomposes at some countable iterate of $M_{\langle p,\mu \rangle}$, M' say, with $On \cap M' > \omega\alpha$. And so, by the minimality of $M(\alpha)$, $M' =_* M(\alpha) =_* M_{\langle p,\mu \rangle}$. But if j is the last truncation point, at index ν_j and critical point κ_j, $M' = h_{M'}(\kappa_j \cup \{\kappa_i \mid j \leq i < \theta\} \cup \{p_{M'}\})$, and as K_α^M is an initial segment of M', we have α decomposing at M' using Σ_1 hulls. But $n > 1$! Q.E.D.(1).

Let $DS(M, m, \alpha)$ denote the directed system of $M_{\langle p,\mu\rangle}$, for $i = \langle p, \mu\rangle \in I$, and maps $\pi_{ij} : M_i \to_{\Sigma_m} M_j$ for $i \leq j \in I$. We note

(2) $\forall i \leq j \in I \; \pi_{ij} \in \bar{K}$.

Proof. Let $x = \pi^{-1}(y) \in M_i$ where $y \in H_m^M(p, \mu)$ where $i = \langle p, \mu\rangle, j = \langle q, \rho\rangle$. So $y = \tau^M(p, \mu_1, \ldots, \mu_n)$ for some Σ_m skolem term, some $\mu_1 < \cdots < \mu_n < \mu$. Then $\pi_{ij}(x) = \tau^{M_j}(\pi_j^{-1}``p, \mu_1, \ldots, \mu_n)$ and $x = \tau^{M_i}(\pi_i^{-1}``p, \mu_1, \ldots, \mu_n)$. Hence π_{ij} is defined from M_i, M_j, $\pi_i^{-1}``p, \pi_j^{-1}``p$ inside $\bar{K}$ Q.E.D. (2).

Transport the directed system *via* j to a system $TDS(M, m, \alpha)$, where the structures are $j(M_i)$, the maps are $j(\pi_{ik})$ for $i \leq k \in I$, with the same indexing set. Then easily enough $j(\pi_{ij}) : j(M_i) \to_{\Sigma_m} j(M_k)$.

(3) The direct limit of $TDS(M, m, \alpha)$, $\tilde{M}$ say, is a mouse.

Proof. We need to show that $\tilde{M}$ is a wellfounded structure, is a premouse, and has wellfounded iterates. Since we have Σ_1 maps between the M_i, and being a premouse is a Π_1 condition, $\tilde{M}$ is certainly a premouse. In some sense the wellfoundedness of $\tilde{M}$ is a special case of $\tilde{M}$ having wellfounded iterates (as [8] points out) so we prove the latter. It is enough to show every countable iteration of $\tilde{M}$ is continuable. Let $\langle\langle \tilde{M}_i\rangle\langle\tilde{\pi}_{ij}\rangle\rangle_{i \leq j < \theta}$ be an iteration of $\tilde{M}$ that cannot be continued. Let i_0 be the last truncation point of this iteration (so that π_{ij} are total for $i_0 \leq i \leq j < \theta$). By the usual argument there is a countable set of objects $x_i \in \tilde{M}$ which are supports of an infinite descending chain of "ordinals" of $\tilde{M}_\theta : c_i \in_{\tilde{M}_\theta} c_{i-1}$ for $i < \omega$. There is thus a countable subsystem of $TDS(M, m, \alpha)$ with indices $\{\ell_i\}_{i<\omega}$, with $x_i \in j(M_{\ell_i})$, where $\tilde{\pi}_{\ell_i}(\bar{x}_i) = x_i$, $\tilde{\pi}_{\ell_i}$ being the direct limit map of $j(M_{\ell_i}) \to \tilde{M}$. We find ℓ_i without loss of generality so that $i < j \Rightarrow \ell_i <_I \ell_j$ and so that $\bar{M} = \varinjlim \langle j(M_{\ell_i}), \langle j(\pi_{\ell_i \ell_j})\rangle\rangle_{i \leq j < \omega}$ has an illfounded θ'th-iterate. Let $\ell_i = \langle p_{\ell_i}, \mu_{\ell_i}\rangle$. Let $D = \cup_{i<\omega} p_{\ell_i} : \mu = \cup_{i<\omega}\mu_{\ell_i}$. Let $\pi : N \Leftrightarrow Y$ be the inverse of the transitive collapse where $Y = H_m^M(D \cup \mu)$, N is a mouse and $\pi^{-1} \circ \pi_{\ell_i} : M_{\ell_i} \to_{\Sigma_m} N$ for $i < \omega$.

(4) $N <_* M \wedge N \in \bar{K}$

Proof. Note that $\mu < \alpha$ (since $\mu_{\ell_i} < \alpha$ and cf $(\alpha) > \omega$). As D is countable and $m < n(\alpha)$ we must have $N <_* M(\alpha)$, and similarly $On \cap N < \alpha$. As in (2) above. This means $N \in \bar{K}|\alpha$. Q.E.D. (4).

Set $\sigma_i = \pi^{-1} \circ \pi_{\ell_i}$. Then $\sigma_i \in \bar{K}$ by the same argument as for (3). As $\sigma_i : M_{\ell_i} \to_{\Sigma_m} N$, applying j we have $j(\sigma_i) : j(M_{\ell_i}) \to j(N)$, and we have the following commuting diagram.

$$\begin{tikzcd} j(M_{l_i}) \arrow[rr, "j(\sigma_i)"] \arrow[dr, "j(\pi_{ik})"'] & & j(N) \\ & j(M_{l_k}) \arrow[ur, "j(\sigma_k)"'] & \end{tikzcd}$$

$j(N)$ is a mouse, (since $\bar{K} \models N$ a mouse, and $\omega_1 \in K_\beta$ and the latter is sufficiently closed so that being a mouse is absolute) . So $\bar{M}$ is a direct limit that is Σ_m-embeddable into $j(N)$-a θ-iterable premouse. But if $\tau : \bar{M} \to_{\Sigma_k} j(N)$ we can copy the supposed ill-founded iteration of $\bar{M}$ as a length θ iteration of $j(N)$ which is wellfounded! Q.E.D. (3)

Let g_i be the map of $j(M_i)$ into the direct limit $\tilde{M}$. We now finish just as in [8].

(5) There is a "lift-up" map $\tilde{j} : M(\alpha) \to_{\Sigma_n} \tilde{M}; \tilde{j} \restriction \alpha = j \restriction \alpha$.

Proof. Let $x \in M(\alpha)$. Let $i = \langle p, \mu \rangle \in I$ be such that $x \in H_m^M(p \cup \mu)$. Define $\tilde{j}(x) = g_i(j(\pi_i^{-1}(x)))$. One may check that $\tilde{j}$ is welldefined and Σ_m-preserving, and $\tilde{j} \restriction \alpha = j \restriction \alpha$. For the latter, let $\tau < \alpha$. Let $i \in I$ for $\langle \{\tau\}, \tau + 1 \rangle$. Then $\tau + 1 \supseteq H_m^M(\{\tau\} \cup \tau + 1)$ and so $\pi_i^{-1} \restriction \tau + 1 = \mathrm{id} \restriction \tau + 1$, and the same is true $\forall k \in I(k \geq i)$ so $\pi_{ik} \restriction \tau + 1 = \mathrm{id} \restriction \tau + 1$ $\forall k \geq i (k \in I)$ and so $j(\pi_{ik}) \restriction j(\tau) + 1 = \mathrm{id} \restriction j(\tau) + 1$ so $\tilde{j}(\tau) = g_i(j(\tau))$. But $j(\pi_{ik}) \restriction j(\tau) + 1$ being the identity for all $k \geq i$ implies $g_i \restriction j(\tau) + 1 = \mathrm{id} \restriction j(\tau) + 1$.

To see that $\tilde{j}$ is Σ_{m+1} preserving, let $M(\alpha) \models \exists x \varphi(x, x_1, \ldots x_k)$ where φ is Π_m. Let $x_0 \in M(\alpha)$ witness this. Then $\tilde{M} \models \varphi(\tilde{j}(x_0), \tilde{j}(x_1), \ldots, \tilde{j}(x_k)) \Rightarrow \tilde{M} \models \exists x \varphi(x, \tilde{j}(x_1), \ldots, \tilde{j}(x_k))$.

Conversely if $\tilde{M} \models \varphi(x, \tilde{j}(x_1), \ldots, \tilde{j}(x_k))$. Then let $i \in I$ be sufficiently large that $\{x, \tilde{j}(x_1), \ldots, \tilde{j}(x_k)\} \subseteq \mathrm{ran}\, g_i$. Then there is $\{y_1, \ldots, y_k\}$ so that

$$j(M_i) \models \varphi(g_i^{-1}(x), j(\pi_i^{-1}(y_1)), \ldots, j(\pi_i^{-1}(y_k)))\,.$$

Hence $M_i \models \exists x \varphi(x, \pi_i^{-1}(y_1), \ldots, \pi_i^{-1}(y_k))$ and applying the Σ_m map π_i we have $M(\alpha) \models \exists x \varphi(x, y_1, \ldots, y_k)$
Q.E.D. (5)

(6) $X \cap \gamma$ is a countable union of sets in K.

Proof. $H_n^{M(\alpha)}(D \cup \mu) \supseteq \alpha$ for some countable D, and $\mu < \alpha$.

Applying $\tilde{j}$ we get:

$$j``\alpha = H_n^{\tilde{M}}(j``D \cup j``\mu) \cap \gamma\,.$$

But $j``\mu = X \cap j(\mu)$. And as $j(\mu) < \alpha$ by the Inductive Hypothesis, we can represent it as $\bigcup_{n<\omega} C_n$ with each $C_n \in K$ and $C_n \subseteq C_{n+1}$. Let $B_n \subseteq B_{n+1}$ for $n < \omega$ be such that each $B_n$ is finite with $\bigcup_{n<\omega} B_n = j``D$ we get

$$X \cap \gamma = \bigcup_{n<\omega} (H_n^{\tilde{M}}(C_n \cup B_n) \cap \gamma)$$

as required Q.E.D. (6) and Case 1

Remark. The proof is simpler than for [8] at this last point since we have here that $\tilde{M}$ is a mouse and hence is in K.

Case 2. $n = \omega$.

This is identical to the Magidor argument for L: for each $m < \omega$ we consider $DS(M(\alpha), m, \alpha)$, forming a lifted up directed system $TDS_m =_{df} TDS(M(\alpha), m, \alpha)$ with lift-up map $j_m : M \to_{\Sigma_{m+1}} \tilde{M}_m$, each j_m extending $j \restriction \alpha$. If μ and D are such that

$$\alpha \subseteq H^M_\omega(\mu \cup D) = \bigcup_{m<\omega} H^M_m(\mu \cup D)$$

Set $B_k = j``(\alpha \cap H^M_k(\mu \cup D))$. As $j``\alpha = X \cap \gamma$ we have $X \cap \gamma = \bigcup_k B_k$. j_k is Σ_k-preserving and $j_k \restriction \alpha = j \restriction \alpha$. Hence

$$B_k = \gamma \cap H^{\tilde{M}_k}_k(j_k``\mu \cup j_k``D) = \gamma \cap H^{\tilde{M}_k}_k(j``\mu \cup j_k``D) = \gamma \cap H^{\tilde{M}_k}_k((X \cap j(\mu)) \cup j_k``D))$$

By induction hypothesis $X \cap j(\mu) = \bigcup_n Y_n$ with $Y_n \in K$, $Y_n \subseteq Y_{n+1}$. Then $B_k = \bigcup_{n<\omega} H^{\tilde{M}_k}_k(Y_n \cup \{p_0, \ldots, p_n\})$ where $j_k``D = \{p_n \mid n < \omega\}$. As $\tilde{M}_k \in K$, B_k, and so $X \cap \gamma$, is a countable union of sets in K. Q.E.D. Case 2.

Case 3. $n(\alpha) = 1$.

Claim. $\delta(\alpha) = \delta =_{df} On \cap M(\alpha) = \omega\gamma$ for γ a limit ordinal.

Proof. This is trivial if $E^{M(\alpha)}_\delta \neq \emptyset$. Let $M = M(\alpha)$. So suppose $\delta = \omega(\tau + 1)$. Then $\tau \geq \alpha$. Let $\mu < \alpha$ be least so that for some countable D $H^M_1(\mu \cup D) = h_M(\mu \cup D) \supseteq \alpha$. Letting $F = E^M_\tau$ and $M' = \langle J^{E^M}_\tau, \ E^M \restriction \tau, F \rangle$ and letting $\mathrm{rud}_F(U)$ be the rudimentary-in-F closure of the set U under the rudimentary functions, and the function $G(x) = F \cap x$, we have that $|M| = J^{E^M}_{\tau+1} = \mathrm{rud}_F(J^E_\tau)$. So for any $p \in J^{E^M}_{\tau+1}$ there is $q \in J^{E^M}_\tau$ and a rudimentary function f so that $p = f(J^{E^M}_\tau, F, q)$ (noting that $F = F \cap J^{E^M}_\tau$). For each $p \in [D]^{<\omega}$ pick such a $q_p \in J^E_\tau$. Well known arguments (see e.g. [2] pp. 209-210) show that each $\xi \in h_M(\mu \cap p)$ is definable over M' using q (and parameters less than μ). Thus if $\tilde{D} = \{q_p \mid p \in [D]^{<\omega}\}$ we have $H^\omega_{M'}(\mu \cup \tilde{D}) \supseteq \alpha$, contradicting the leastness of M.

Q.E.D. Claim

The most problematic case is the following.

Case 3 (i) $cf(\delta) = \omega \wedge E^M_\delta \neq \emptyset$.

Let $\langle \delta_n \mid n < \omega \rangle$ be chosen cofinal in δ. We cannot infer that $H^M_1(\mu \cup D) = \bigcup_{n<\omega} H^{M|\delta_n}_1(\mu \cup D)$ as $E^M_\delta \neq \emptyset$. As M is amenable, if we set $E = E^M_\delta$, $\delta_n < \delta_{n+1}$ may be chosen so that $E \cap J^{E^M}_{\delta_n} \in J^{E^M}_{\delta_{n+1}}$. Set $E_n = E \cap J^{E^M}_{\delta_n}$. Set $M^n = \langle J^{E^M}_{\delta_n}, E^M \restriction \delta_n, \langle E_k \mid k < n \rangle \rangle$. M^n is then also amenable. Let $\alpha \subseteq H^M_1(D \cup \mu)$.

(1) $\alpha \subseteq \bigcup_{n<\omega} H^{M^n}_1(\mu \cup (D \cap M^n))$

Proof. Let $\eta < \alpha$. Then $\eta = h_M(i, \vec{\xi}, \vec{\zeta})$ where $\vec{\xi} < \mu, \vec{\zeta} \in [D]^{<\omega}$. Since $M \models \varphi_i(\eta, \vec{\xi}, \vec{\zeta})$ there is a transitive $U \in J^{E^M}_\delta$, with $E \cap U \subseteq E_k$ for some k, and so that $\langle U, E^M \restriction U, E \cap U \rangle \models \varphi_i(\eta, \vec{\xi}, \vec{\zeta})$. So for some $m \geq k$

$$\langle J^{E^M}_{\delta_m}, E^M \restriction \delta_m, \langle E_k \rangle_{k<m} \rangle \models \text{“}\exists k < m \exists U \text{transitive} \wedge \langle U, E^M \restriction U, E_k \rangle \models \varphi_i(\eta, \vec{\xi}, \vec{\xi})\text{”}$$

Hence η is $\Sigma_1^{M^k}(\{\vec{\xi}, \vec{\xi}\})$. Q.E.D. (1)

Consider now the directed system $DS(M^k, \omega, \alpha)$. Indices for this are $i \in I$, where $i = \langle p, \mu \rangle$ for finite $p \in M^k$, $\mu < \alpha$, ordered as before. We now take fully elementary skolem hulls of M^k: $H^{M^k}_\omega(p \cup \mu)$, and let $\pi_i : M_i \tilde{\leftrightarrow} H^{M^k}_\omega(p \cup \mu)$ be the inverse of the transitive collapse. Again, for $i \leq j \in I$, set $\pi_{ij} : M_i \to_{\Sigma_\omega} M_j$ equal to $\pi_j^{-1} \circ \pi_i$. For each $k < \omega$ this system $DS(M^k, \omega, \alpha)$ has a direct limit isomorphic to M^k.

(2) For $i \leq j \in I$, $M_i, \pi_{ij} \in \bar{K}$.

Proof. Let $\bar{M}^k = \langle J^{E^M}_{\delta_k}, E^M \restriction \delta_k \rangle$ and $\bar{M}_i = \langle J^{\bar{E}_i}_{\rho_i}, \bar{E}_i \rangle$, where $M_i = \langle J^{\bar{E}_i}_{\rho_i}, \bar{E}_i, \langle F_n \rangle_{n<k} \rangle$. (Thus the final predicates are removed.) Then $\bar{M}^k$ and the $\bar{M}_i$ for $i \in I$ are mice (as $\pi_i : \bar{M}_i \to_{\Sigma_\omega} \bar{M}^k \overset{\text{id}}{\to}_{\Sigma_0} \langle J^{E^M}_\delta, E^M \rangle$. But then $\bar{M}_i \leq_* \bar{M}^k <_* M$. Hence α does not decompose at $\bar{M}_i$. Since $|\bar{M}_i| = H^{\bar{M}_i}_\omega(\pi_i^{-1}(p) \cup \mu)$ this means $On \cap \bar{M}_i = \rho_i < \alpha$. By Lemma 3 $\bar{M}_i \in \bar{K} \mid \alpha$. Since the last predicate of M_i is just a finite set of elements of $J^{\bar{E}_i}_{\rho_i}$. $M_i \in \bar{K} \mid \alpha$. The argument that $\pi_{ij} \in \bar{K}$ is then just as before. Q.E.D. (2)

We may thus form the lift-up directed system $TDS(M^k, \omega, \alpha)$ arguing as in Case 1, that this system has a well founded direct limit $\tilde{M}^k = \langle J^{\tilde{E}}_{\tilde{\delta}}, \tilde{E}, \langle \tilde{F}_\ell \rangle_{\ell<k} \rangle$ so that $\langle J^{\tilde{E}}_{\tilde{\delta}}, \tilde{E} \rangle$ is a mouse, and each $\tilde{F}_\ell \in J^{\tilde{E}}_{\tilde{\delta}}$. And so that the map $\tilde{j}_k : M^k \to_{\Sigma_\omega} \tilde{M}^k$ extends $j \restriction \alpha$. Let

$$X_k = j_k\text{“}(\alpha \cap H_1^{M^k}(D \cap M^k \cup \mu)) = \gamma \cap H_1^{\tilde{M}^k}(j_k\text{“}(D \cap M^k) \cup j\text{“}\mu)$$

$\mu < \alpha$ so $j\text{“}\mu = X \cap j(\mu)$ which by the inductive hypothesis is $\bigcup_{n<\omega} Y_n$ where each $Y_n \in K$. Assume $Y_n \subseteq Y_{n+1}$ for $n < \omega$. Hence

$$X_k = \bigcup_{\substack{\ell<\omega \\ p \in [D \cap M^k]^{<\omega}}} \gamma \cap H_1^{\tilde{M}^k}(j_k\text{“}p \cup Y_\ell)\,.$$

As $\tilde{M}^k \in K$ we see X_k is a countable union of sets in K. But $X = \bigcup_{n<\omega} X_n$ by (1).

Q.E.D. Case 3(i).

Case 3(ii) $\text{cf}(\delta) = \omega \wedge E^M_\delta = \emptyset$.

Proceed just as in Case 3(i) defining M as there but simply ignoring the "$\langle E_k \mid k < n \rangle$" apparatus. Q.E.D. Case 3(ii)

Case 3(iii) $\text{cf}(\delta) > \omega$.

Note that $E^M_\delta = \emptyset$, as is easily checked, since we are assuming there are no ω-closed mice. This is the Case that is most nearly like that of [8], so we omit most details. Let the set of indices be now

$I = \{\langle p,\mu,\nu\rangle \mid v < \delta \wedge \quad p \in M \mid \nu,\mu < \alpha\}$. Partially order I by: $\langle p,\mu,\nu\rangle \leq \langle p',\mu',\nu'\rangle$ iff $\mu \leq \mu' \wedge \nu \leq \nu' \wedge p' \supseteq p$. Let, for $i = \langle p,\mu,\nu\rangle \in I$, $\pi_i : M_i \rightarrow_{\Sigma_\omega} M\|\nu$ where ran $\pi_i = H_\omega^{M\|\nu}(\mu \cup p)$. Again each M_i is a mouse, and we may define $\pi_{ij} : M_i \rightarrow_{\Sigma_0} M_j$ for $i \leq j \in I$ again as $\pi_{ij} =_{\mathrm{df}} \pi_j^{-1} \circ \pi_i$. Then M is the direct limit of $DS(M,0,\alpha)$ as is easily checked. Just as before we have

(1) $M_i, \pi_{ij} \in \bar{K} \mid \alpha$ for $i \leq j \in I$.

We carry over *via* j this system to $TDS(M,0,\alpha)$ with embeddings and structures satisfying $j(\pi_{ij}) : j(M_i) \rightarrow_{\Sigma_0} j(M_j)$ for $i \leq j \in I$.

(2) The direct limit of $TDS(M,0,\alpha)$ is a mouse $\tilde{M}$.

(3) There is a cofinal lift-up map $\tilde{j} : M(x) \rightarrow_{\Sigma_0} \tilde{M}$, extending $j \restriction \alpha$.

The proof of (3) is just as in (6) of Case 1- since Σ_0 is easily checked, as is cofinality. In fact the rest of this case concludes just as in (5) Case 1. Q.E.D. Case 3.

We now discharge the assumption under which the previous cases were proven.

Case 4 For some $\delta < \alpha$, α decomposes at some mouse $M \in L[E^{\bar{K}} \restriction \delta]$.

Proof. Let δ be least $< \alpha$ so that α decomposes at some mouse $M \in L[E^{\bar{K}} \restriction \delta]$. Set $\bar{E} = E^{\bar{K}} \restriction \delta$. Let η be the least $\eta' \geq \alpha$ such that $\exists\mu < \alpha\exists$countable $D\exists n(1 \leq n \leq \omega)H_n^{J_{\eta'}^{\bar{E}}}(D \cup \mu) \supseteq \alpha$. Since $\delta < \alpha$ and α is a limit of limits, we may assume $Lim(\mu)$ and $\delta < \mu < \alpha$. But now the predicate $\bar{E}$ is strictly bounded above by μ and we may work as in part 1 of [8], the L part of his paper, by relativising the L-construction there to the $L[\bar{E}]$ hierarchy here, to give us a lift-up map $\tilde{j} : J_\eta^{\bar{E}} \longrightarrow_{\Sigma_n} J_{\tilde{\eta}}^{E\restriction j(\delta)}$. (If $n < \omega$; if $n = \omega$ this is replaced by a collection of embeddings $\tilde{j}_k : J_\eta^{\bar{E}} \longrightarrow_{\Sigma_k} J_{\tilde{\eta}}^{E\restriction j(\delta)}$ for $1 \leq k \leq \omega$.) We shall not say any more about this case as the reader may easily reconstruct it from [8].

Case 4 and Theorem 1.

3 Conclusions

That we cannot improve on this result is contained in the following:

Theorem 1 *If E is an ω-closed filter on the K sequence $\vec{E}^K$, with critical point β, it is consistent that there is $X \subseteq \beta$ closed under the Σ_1-skolem functions for K_β, that is not a countable union of sets in K.*

Proof. This is just Magidor's argument from the introduction to [8] which we only include for completeness. Here we may as well assume $\vec{E}^K$ is a set with last filter the E of the hypothesis. Then $K = L[\vec{E}^K] \models$ "E is ω-closed". Clearly $\omega_1^K \leq \beta$. Now add generically a Cohen subset $G \subseteq \omega_1^K$ using finite conditions $\mathbb{P} \subseteq HC$. Let $\langle \iota_\alpha \mid \alpha < \omega_1^K\rangle$ be a sequence of good indiscernibles obtained from E_β for $K_\beta = \langle K_\beta, \in, \vec{E} \restriction \beta\rangle$. (Note $\mathbb{P}$ is $\Delta_1^{K_{\omega_1}}$). Let $X_0 = \{\iota_\alpha \mid \alpha \in G\}$. Let $X =$ closure of X_0

under the skolem functions for K_β. If X were the countable union of $\{Y_n\}_{n<\omega}$ each $Y_n \in K$, then some Y_n would contain an infinite subset of X_0 and hence an infinite subset of G would be in K - contradicting genericity. Q.E.D.

To get some estimate of the strength of cardinals in K, where there is an ω-closed filter on the K-sequence we use some ideas of [3]. Let $\mathcal{A} = \langle L_\kappa[A], \in, A\rangle$ be a first order structure.

Definition 1 The game $G_{\mathcal{A}}$ is played between I and II, with II always moving first in each round; the game is of potential length any countable ordinal.

II	I_0		$\ldots$	I_ν	
I		$\langle\tau_0, \rho_0\rangle, n_0$	$\ldots$		$\langle\tau_\nu, \rho_\nu\rangle, n_\nu$

Rules. For II: I_ν is a good set of indiscernibles for $\mathcal{A}$ of limit order type; min $\{I_\nu\} > \sup\{\cup_{\mu<\nu} I_\mu\}$, and min $\{I_{\nu+1}\} \geq \tau_\nu$, ot $(I_{\nu+1}) \geq \rho_\nu$. For I: $n_\nu \notin \{n_\mu \mid \mu < \nu\}$; $n_\nu \in \omega$, $\quad \rho_\nu < \omega_1, \tau_\nu < \kappa$.

The first player who cannot move loses.

Remark. This is stronger than any of the games considered in [3].

Definition 2 κ is *virtually ω_1-Erdős* if for every structure $\mathcal{A}$ as above, player I has no winning strategy in $G_{\mathcal{A}}$.

Theorem 2 *Suppose $M = \langle J_\alpha^{E^M}, E^M, E_\alpha^M\rangle$ is an active sound mouse with E_α^M on κ, and E_α^M ω-closed. Then κ is virtually ω_1-Erdős in $L[E^M \restriction \alpha]$.*

Proof. Let $A \subseteq \kappa$ be any predicate in $N = L[E^M \restriction \alpha]$. Let $\mathcal{A} = \langle L_\kappa[A], \in, A\rangle$. Then $\mathcal{A} \in M$, and we may find $\{X_n\} \subseteq \mathcal{P}(\kappa)^M$ with $\bigcap_{n<\omega} X_n = I$ an unbounded sequence in κ, in $L[M]$ of good indiscernibles for $\mathcal{A}$. As $cf(\kappa)^{L[M]} > \omega$ ot$(I) \geq \omega_1^{L[M]}$. Argue now in $L[M]$: clearly if $\sigma \in N$ is a strategy for I in $G_{\mathcal{A}}$ we (as II) may play I_0 = first ω-elements of I; and at subsequent stages play any ρ_ν sequence, $I_{\nu+1}$, from I with the minimum $I_{\nu+1} > \tau_\nu$ and satisfying all other requirements. Clearly a run of play results in which I gets stuck for an integer n_μ.

Claim The run of play $\langle I_0, \langle n_0, \tau_0, \rho_0\rangle, I_1, \ldots, I_\mu\rangle \in N$

Proof. Firstly $\sigma \in H_{\kappa^+}^N = |M|$. Secondly, if $cf^{L[M]}(\alpha) > \omega$, we may take I also in $\mathcal{P}(\kappa) \cap N = \mathcal{P}(\kappa) \cap M$; if $cf^{L[M]}(\alpha) = \omega$, then by Lemma 1.23 $\rho_M^1 = \kappa$ and as I is $\underline{\Pi}_1(M)$, $I \cap \tau \in H_\kappa^M = H_\kappa^N$ for any $\tau < \kappa$. This suffices since $I \cap \sup_{\nu<\mu}\{\cup I_\mu\} < \kappa$.

Hence σ is not winning for I in N. Q.E.D.

Remark. By using arguments of [3], one can see that Levy collapsing a virtually ω_1-Erdős to ω_2 yields a model where the transversal hypothesis TH_{ω_1} fails (TH_κ : "$\exists\langle f_\nu \mid \nu < \kappa^+\rangle$ a sequence of regressive functions with $\text{dom} f_\nu = \kappa$, and for any $\mu < \nu < \kappa^+$ $\{\alpha \mid f_\mu(\alpha) \neq f_\nu(\alpha)\}$ contains a club in κ").

They show that $\neg TH_{\omega_1}$ is equiconsistent with another weak Erdős property (again defined in terms of non-existence of strategies for player I in certain games) which they call "$(< \omega_1, < \omega_1)$ -Erdős". (We do not repeat this definition here.) A quick inspection of our games $G_{\mathcal{A}}$ as defined above show them to be clearly stronger than those of this last definition. Hence

Proposition. $\text{Con}(ZFC + \exists\kappa(\kappa$ virtually ω_1-Erdős$)) \to \text{Con}(ZFC + \exists\kappa(\kappa$is $(< \omega_1, < \omega_1)$-Erdős$)$

They also consider various properties of $\mathcal{P}_{\omega_1}(\omega_2)$. As a sample variant derived from the above definition, one can use their methods to show the following principle $P(\omega_1)$ holds; For every club set $\mathcal{C} \subseteq \mathcal{P}_{\omega_1}(\omega_2)$ there exists $\alpha < \omega_1$ so that for a stationary set of $\beta < \omega_1$

$$\{sup\, X | X \in \mathcal{C}, X \cap \omega_1 = \alpha, ot(X) = \beta\}$$

is stationary in ω_2.

References

[1] J. Baumgartner, *On the Size of Closed Unbounded Sets* APAL, **54**, 1991, 195-227.

[2] J. Devlin, *Constructibility*, Perspectives in Mathematical Logic, Berlin, Springer Verlag, 1984.

[3] H-D. Donder & J-P. Levinski, *Some Principles related to Chang's Conjecture*, Annals of Pure and Applied Logic, **45** (1989) 39-101.

[4] A. J. Dodd, *The Core Model,* London Math. Society Lecture Note Series 61 Cambridge University Press, 1982.

[5] Q. Feng, *A hierarchy of Ramsey Cardinals* APAL, **49** (1990), 257-277.

[6] T. Jech, *Set Theory* Academic Press, New York 1978.

[7] R.B.Jensen, *The Core Model for Measures of Order Zero*, circulated manuscript, 1989.

[8] M. Magidor, *Representing Sets of Ordinals as Countable Unions of Sets in The Core Model*, Trans. A.M.S., vol 317, No. 1, 1990, 91-126.

[9] D.Walker, *On the transversal hypothesis and the weak Kurepa hypothesis*, J.S.L., **53**, No.3, Sept. 88, 854-877.

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