Σ^1_3 Absoluteness and the Second Uniform Indiscernible

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Abstract

We show that that if every real has a sharp and there are Δ^1_2-definable prewellorderings of $\mathbb{R}$ of ordinal ranks unbounded in ω_2, then there is an inner model for a strong cardinal. Similarly, assuming the same sharps, the Core Model K is Σ^1_3-absolute unless there is an inner model for a strong cardinal.

1 Introduction, and Preliminaries

1.1 Background

In this paper we give proofs, assuming the reals are closed under the sharp function, that if $\boldsymbol{\delta}^1_2 = \omega_2$ then there are inner models with strong cardinals (*cf* Def.1.1), and that if the core model K is not Σ^1_3-absolute (Def.1.2 below) then again there are such models. We are interested in constructing these so-called core models, and our motivation is that such hypotheses allow these constructions to take place in such a way that the resulting models contain large cardinals. Very broadly speaking, one can view these models as built using a generalisation of Gödel's notion of constructibility. The first such core model, K_{DJ} , was built by Dodd and Jensen, (see [1]), and Jensen then showed, again assuming the reals were closed under sharps, that if K_{DJ} is not Σ^1_3-correct, then there is an inner model with a measurable cardinal. (And in fact that "$O^{\dagger}$" existed, that is, there exists a class of indiscernibles for such a model.) The wellorder of the reals in K_{DJ} is seen to be Σ^1_3. This fact can be contrasted with the Σ^1_2 wellorder of $\mathbb{R} \cap L$ (where L is the constructible universe), and with the theorem of Shoenfield that L is Σ^1_2-absolute.

The assumption that the reals have sharps is essential for this result, as it is for the result on $\boldsymbol{\delta}^1_2$. It is a result of Harrington (see [6]) that it is consistent (by forcing over a model of $V = L$, where of course no reals have sharps) that $\boldsymbol{\delta}^1_2$ can be arbitrarily large, so the presence of sharps for reals is necessary to give us any knowledge about absolute upper bounds here.

Both results depend on certain computations concerning indiscernibles for models $L[a]$ (for $a \in \mathbb{R}$, $L[a]$ is the constructible hierarchy relative to the predicate a). Jensen proved a result indicating how the indiscernibles are related to certain structures in K_{DJ}, a "Patterns of Indiscernibles" type of result whose ancestor is due to Paris ([14]) concerning reals a with $O^{\#} \notin L[a]$. Mitchell has shown by generalising this argument and "counting indiscernibles" again that his core model which allows measurable cardinals up to $o(\kappa) = \kappa^{++}$ is Σ^1_3-absolute, if rigid [13].

The proof of absoluteness here is based in an essential way on an alternative proof due to Magidor for K_{DJ}. His method is simpler than that of of Jensen and Mitchell, both of whose proofs exploit a periodicity phenomenon in the indiscernibles' behaviour. Magidor's argument does need a slight strengthening of hypothesis. (Jensen could show that assuming only the existence of $a^{\#}$, for a a solution of a Π^1_2 predicate, then there is a solution in K_{DJ}; Magidor's argument requires that there are infinitely many sharps above a.) But the gain in simplicity smooths the way to obtaining the result here and in any case the arguments of Jensen and Mitchell do not seem easily generalisable to larger models.

$\boldsymbol{\delta}^1_2$ is seen to be connected with indiscernibles in the following way. Let $a \subseteq \omega$. Then let I^a denote the closed and unbounded, (cub), beneath each uncountable regular cardinal class of Silver indiscernibles for $L[a]$. Let $\langle \iota^a_\tau | \tau \in On \rangle$ enumerate I^a. Let us assume for the rest of this paper, that for every real a, $a^{\#}$ exists. Then $I = \cap_{a \in \mathbb{R}} I^a$ is also cub, and if it is enumerated as $\langle u_\tau | \ 1 \leq \tau < \infty \rangle$, it is easily seen that $u_1 = \omega_1$ and $u_2 \leq \omega_2$.

The following are well known:

Definition 1.1 $\boldsymbol{\delta}^1_2 = \sup\{rk(\leq) \mid \leq$ *is a* $\boldsymbol{\Delta}^1_2$ *definable prewellordering of a subset of* $\mathbb{R}$ $\}$

Fact 1.2 $\omega_1 < (\omega_1^+)^{L[a]} < \iota^a_{\omega_1+1} < (\omega_1^+)^{L[a^{\#}]}$ *for any real* a.

Fact 1.3 *Any* $\Delta^1_2(a)$ *prewellorder of (a subset of)* $\mathbb{R}$ *has ordinal rank* $< (\omega_1^+)^{L[a]}$.

Let $WO = \{a \in \mathbb{R} | a$ codes a wellorder of $\omega\}$. For $a \in WO$ let $|a|$ be its rank. For $A \subseteq \omega_1$ let $Code(A) =_{df} \{a \in WO \mid |a| \in A\}$.

Fact 1.4 *For $a \in \mathbb{R}$ there is $A \subseteq \omega_1$, a wellorder of length $> (\omega_1^+)^{L[a]}$,so that, setting $a \preceq b$ if $a, b \in Code(A)$ and $|a| <_A |b|$, then $\preceq$ is a prewellorder of $Code(A)$ of length $> \omega_1^{+L[a]}$ and is $\Pi^1_1(a^\#)$.*

Facts 1-3 then yield immediately:

Fact 1.5 $\boldsymbol{\delta}^1_2 = u_2$

The first upper bound on the consistency strength of $ZFC + \forall r \in \mathbb{R}$ ("$r^\#$ exists") $+ u_2 = \omega_2$ was obtained in [17], where the authors obtain a model for $ZFC + \forall r \in \mathbb{R}$ ("$r^\#$ exists") $+ u_2 = \omega_2 +$ "the nonstationary ideal on ω_1 is ω_2 saturated" by forcing over a ground model satisfying $ZF + AD +$ the axiom of choice for families indexed by the reals. Woodin then showed how one could avoid the choice hypothesis on the ground model here (*cf.* [19]). More recently, Shelah has shown by quite different methods that one can force over a ground model satisfying $ZFC+$ "there is a Woodin cardinal" to obtain a model in which the nonstationary ideal on ω_1 is ω_2 saturated. Still more recently, Woodin has shown that the ω_2 saturation of the nonstationary ideal on ω_1, together with the existence of a measurable cardinal, implies that $u_2 = \omega_2$. The best upper bound (also due to Woodin) on the consistency strength of $ZFC + \forall r \in \mathbb{R}$ ("$r^\#$ exists") $+ u_2 = \omega_2$ now known is slightly weaker than $ZFC +$ "there is a Woodin cardinal with a measurable above it": it is that of $ZFC +$ "there exists a Woodin cardinal δ and $\delta^* < \delta$ with: (i) δ^* is Woodin in $L(V_{\delta^*})$ and (ii) $V_{\delta^*} \prec V_\delta$ (see 3.21 in [20]). The earliest lower bound on the consistency strength of $ZFC + \forall r \in \mathbb{R}$ ("$r^\#$ exists") $+ u_2 = \omega_2$ was obtained independently by Welch and Martin (*cf.* [18]), who used Jensen's Σ^1_3 correctness argument to obtain $\forall r \in \mathbb{R}$ ($r^\dagger$ exists) from $ZFC + \forall r \in \mathbb{R}$ ("$r^\#$ exists") $+ u_2 = \omega_2$. We know of no further work improving the lower bound prior to our construction here of an inner model with a strong cardinal. We conjecture that $ZFC + \forall r \in \mathbb{R}$ ("$r^\#$ exists") $+ u_2 = \omega_2$ proves that there is an inner model with a Woodin cardinal.

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The rest of this section lays out the material needed from fine-structure theory. In §2 we study universal iterations, and prove some results about them which are both useful later in the paper, and perhaps are also of independent interest. §3 proves $u_2 < \omega_2$ assuming no indiscernibles for an inner model with a strong cardinal and §4 gives the result on Σ^1_3-absoluteness. There are some generalisations of the method in §5. Throughout the paper ω_1 or ω_2 denotes those initial ordinals calculated in V.

Definition 1.6 *An inner model W is* Σ^1_n-absolute, *or* correct, *if for any Σ^1_n formula of analysis Φ and any $\vec{a}$ (a sequence of parameters from $\mathbb{R}^W$), then $\Phi(\vec{a})$ holds if and only if $\Phi(\vec{a})^W$ holds.*

1.2 Preliminaries

The new material in this paper is concerned with 'computing' the lengths of mice which iterate past the core model as defined inside models of the form $L[a]$ for $a \in \mathbb{R}$. We assume the reader is familiar with the notation of [12]. We recall here some of this material.

We use freely the notation and ideas of [12], and assume the reader has familiarity with the fine-structural notions of premouse, mouse, core model, etc. from that account. Actually, many of the complexities of [12] will not arise here, as they are caused by the possibility of mice satisfying large cardinal hypotheses stronger than " there is a strong cardinal". In particular, the mice studied here do not really require the full mechanism of iteration trees for their comparison. The comparison process given in [12] , applied to the mice with which we are concerned here, does actually give rise to iteration trees, but in an inessential way, as it is easy to "linearize" the trees arising in comparisons at this level. So although we prefer to use the language of [12], the mathematical prerequisite for reading this paper is just core model theory below a strong cardinal, and any reader familiar with that theory as developed by Dodd, Jensen, Koepke, and Mitchell in [2] should have no difficulty in translating our work here into the language of that paper.

A *premouse* is a structure $M = \mathcal{J}_\alpha^{E^M} = \langle J_\alpha^{E^M}, E^M, \widetilde{E}_\alpha^M \rangle$ where E^M is a predicate coding a *good extender sequence* over M. (We have dropped the vector arrow of [12] over E^M.) Further, if $\widetilde{E}_\alpha^M \neq \emptyset$ it codes a (κ, α) extender E_α^M over M; we set $crit(E_\alpha) = \kappa, lh(E_\alpha) = \alpha$; we set also $\nu(E_\alpha)$ to be the strict supremum of the generators of E_α. (We are setting $E_\alpha = 0$ in those cases citeMiSt would leave E_α undefined. Also, for notational ease we shall index the J-hierarchy using only limit ordinals, so that $On \cap J_\alpha^E = \alpha$.) By the *$M$-sequence* we mean the class of extenders indexed in this way by the predicates E^M or $\widetilde{E}_\alpha^M$. By the *extended M-sequence* we mean the class of those extenders that are trivial completions of $E_\gamma \precneqq \eta$ where E_γ is on the M-sequence and η is the strict supremum of a set of generators of E_γ. Thus the extended M-sequence is closed under taking initial segments as specified in Clause 5(b) of [12] Def.1.0.4. We shall use the following notation for the truncations of premice.

Definition 1.7 *For M as above, $M||\beta = \mathcal{J}_\beta^M = \langle J_\beta^{E^M}, E^M, \widetilde{E}_\beta^M \rangle$; $M|\beta = \langle J_\beta^{E^M}, E^M, \emptyset \rangle$. For premice M and N, N is an* initial segment of M *iff there is $\beta \leq On \cap M$ with $N = \mathcal{J}_\beta^M$.*

Let $n \leq \omega$. A premouse M is n-sound just in case $M = \mathfrak{C}_n(M)$; that is, M is its own n'th core. Roughly speaking, this means that M is the closure under Σ_n^M Skolem functions of its Σ_n projectum $\rho_n^M \cup p_n(M)$ where the latter is M's n-th standard parameter. We call $\pi : M \longrightarrow N$ an *n-embedding* just in case M and N are n-sound, π is generalized $r\Sigma_{n+1}$ elementary, $\pi(p_i(M)) = p_i(N)$ for all $i \leq n$, $\pi(\rho_i^M) = \rho_i^N$ for all $i < n$, and $\sup \pi``\rho_n^M = \rho_n^N$. Such embeddings arise primarily in the Σ_n ultrapower construction: if E is an extender over M and $crit(E) < \rho_n^M$, then we can form an ultrapower $Ult_n(M, E)$ using (roughly speaking) Σ_n^M functions, and if M is n-sound, then the canonical $\pi : M \longrightarrow Ult_n(M, E)$ is an n-embedding. Let $n \leq \omega$, and M be n-sound. An *n-maximal iteration tree* of length θ on M is a system $\mathcal{T} = \langle T, D, deg, \langle E_i : i+1 < \theta \rangle \rangle$ with associated premice M_i and embeddings $\pi_{i,j}$ such that $M_0 = M$ and for all $i + 1 < \theta$:
(i) E_i is on the M_i sequence, and $lh(E_i) > lh(E_j)$ for $i > j$,
(ii) T is a tree order, and $i + 1$ is the T-successor of the least j such that $crit(E_i) < \nu(E_j)$, and
(iii) $M_{i+1} = Ult_k(M_{i+1}^*, E_i)$, where, setting $j = T - pred(i+1)$, M_{i+1}^* is the longest initial segment of M_j over which E_i is an extender, and $k = deg(i+1)$ is the largest k such that $crit(E_i) < \rho_k^{M_{i+1}^*}$ and $k \leq n$ if $D \cap [0, i+1] = \emptyset$; moreover $i + 1 \in D$ iff $M_{i+1}^* \neq M_j$;
(iv) if i is a limit ordinal, then $D \cap [0, i)_T$ is finite, and M_i is the direct limit of the M_j, for jTi sufficiently large, under the maps $\pi_{j,k}$. We call D the set of places where $\mathcal{T}$ "*drops (in model)*", and $deg(i+1)$ the "*degree*" of the ultrapower producing M_{i+1}. Note that if iTj and for all $e \in (i, j]_T$, $e \notin D$ and $deg(e) \geq k$, then $\pi_{i,j}$ is a k-embedding.

An n-sound premouse is called an n-mouse just in case it is n-iterable in the sense of 5.1.4 of [12]; roughly, this means that it is well behaved with respect to linear compositions of n-maximal iteration trees. We shall drop the reference to n in "n-mouse" when context permits. This notion of mousehood is only appropriate for 1-small premice, but in this paper we shall only be considering still smaller premice, namely those "below $O^\P$". For these premice, "almost linear" iterability guarantees mousehood.

Definition 1.8 *(i) A premouse M is* below $O^\P$ *iff whenever E is a (κ,β) extender on the M-sequence and $\lambda<\kappa$, then $\sup\{\gamma \mid crit(E^M_\gamma)=\lambda\}<\kappa$.*
(ii) $O^\P$ is the unique ω-sound mouse M such that M is not below $O^\P$, but every proper initial seg. of M is below $O^\P$.

It is easy to see that $O^\P$ is active (*i.e.* its last extender predicate is nonempty), and that if μ is the critical point of its last extender, then there is a $\kappa<\mu$ such that for cofinally many $\lambda<\mu$, there is a (κ,λ) extender on the $O^\P$ sequence. It is also easy to see that $\rho_1^{O^\P}=\omega$, and thus $O^\P$ is countable.

Definition 1.9 *A cardinal κ is* strong *if $\forall\alpha\exists j_\alpha(crit(j_\alpha)=\kappa\wedge j_\alpha: V\longrightarrow_e M_\alpha\wedge V_\alpha\subseteq V_\alpha^{M_\alpha})$.*

If we iterate the top measure of $O^\P$ ON times, we leave behind a proper class inner model with a strong cardinal. The iteration points of the top measure give a cub class of indiscernibles for W, and $O^\P$ is constructibly (in fact Turing) equivalent to the type of these indiscernibles. It is easy to see that a premouse M being below $O^\P$ is equivalent to the condition that if $crit(E^M_\beta)=\kappa$ then $\mathcal{J}^M_\kappa\models$ " there are no strong cardinals ". If we are restricting ourselves to premice below $O^\P$, then the iterations arising are of the following simple kind:

Definition 1.10 *An iteration tree $\mathcal{T}$ of length θ is* almost linear *if for any $\alpha<\beta<\theta$ T-$pred(\beta+1)=\alpha\Rightarrow\beta=\alpha+n$ for some $n<\omega\wedge\forall k\leq n(crit(E^{\mathcal{T}}_\beta)=crit(E^{\mathcal{T}}_{\alpha+k}))$.*

It is easy to argue from the initial segment condition that for premice below $O^\P$ that every iteration tree on M is almost linear (cf. §8 of [16]).

If $\mathcal{T}$ is an n-maximal iteration tree on M with last model P, and $\mathcal{T}$ does not drop, then the canonical embedding $i: M\longrightarrow P$ is an n-embedding. We need to consider slightly weaker embeddings as well, as these arise in the copying construction.

Definition 1.11 *Let M,N be premice. Suppose $i:M\longrightarrow N$. i is a* near n-embedding *iff*
(i) M,N are n-sound; (ii) i is $r\Sigma_{n+1}$ elementary; (iii) $i(\rho_k(M))=\rho_k(N)$ for all $k<n$, and $\rho_n(N)\geq\sup i``\rho_n(M)$.

Such a definition is obtained by weakening the definition of n-embedding by not requiring that $\rho_n(N)=\sup i``\rho_n(M)$.

Fundamental is the following version of the Dodd-Jensen Lemma. (*cf.* Lemma 5.3 of [12]).

Fact 1.12 *(Dodd-Jensen) If $\mathcal{T}=\langle T, deg, D,\langle E_i, M_i^*\mid i+1\leq\theta\rangle\rangle$ is an n-maximal iteration on an n-iterable M and $\sigma: M\longrightarrow M^{\mathcal{T}}_\theta||\gamma$ is a near n-embedding where $n\leq\omega$, then*
(i) $M^{\mathcal{T}}_\theta||\gamma=M^{\mathcal{T}}_\theta$: (ii) T does not drop in model or degree: (iii) $i^{\mathcal{T}}_{0\theta}(\nu)\leq\sigma(\nu)$ for all $\nu\in M$.

The *comparison* of suitable premice is as in §7 of [12]. That process results (in this setting) in:

Fact 1.13 *(The Comparison Lemma) Let M,N be below $O^\P$ and be n-sound, n-iterable premice (for $n\leq\omega$). Then there are n-maximal padded almost linear iteration trees $\mathcal{T}$ on M and $\mathcal{U}$ on N, of the same length $\theta+1$, with $\theta<\max\{card(M),card(N)\}^+$, so that either (a) M_θ is an initial segment of N_θ, $deg(i+1)=n$ for all $i+1\in[0,\theta]_T$, and $D^{\mathcal{T}}=\emptyset$, or (b)* (vice versa) *$N_\theta$ is an initial segment of M_θ, $deg(i+1)=n$ for all $i+1\in[0,\theta]_U$, and $D^{\mathcal{U}}=\emptyset$.*

Note that for almost linear iteration trees, if $D^{\mathcal{T}} = \emptyset$, then the property that there is no dropping in degree along the main branch $[0, \theta]_T$ is actually equivalent to $deg^{\mathcal{T}}(i+1) = n$ for all $i + 1 < \theta$.

The comparison lemma implies that any two mice have a common iterate, namely, the shorter of the two final models their coiteration produces.

Definition 1.14 *(i) A* weasel *is a class sized mouse:* $W = \langle J_{\infty}^{E^W}, \in, E^W \rangle$ *where for all* $\alpha \in On$ $\langle J_{\alpha}^{E^W}, E^W, \widetilde{E}_{\alpha}^W \rangle = W||\alpha$ *is a mouse.*

(ii) A weasel W *is* universal *if in the coiteration of* W *with any other mouse or weasel* P *to models* (W_θ, P_θ) $(\theta \leq \infty)$, P_θ *is an initial segment of* W_θ.

A universal weasel iterates past all non-universal weasels and of course past all mice of set size.

As is well known, a weasel W satisfies that κ is strong if and only if $\{\alpha \mid E_{\alpha}^W \neq \emptyset \wedge crit(E_{\alpha}^W) = \kappa\}$ is unbounded in On.

If some (equivalently all) universal weasels satisfy that there are no strong cardinals, then the universal weasels are precisely those weasels which iterate past all set sized mice. As we remarked earlier, this is the situation of most interest to us here. (On the other hand, if universal weasels satisfy "there is a strong cardinal", then there is a non-universal weasel which iterates past all set sized mice.)

A construction of K, assuming $\neg O^{\P}$, and a development of the facts we need here (all due to Dodd, Jensen, and Mitchell) can be found in §8 of [16]. (Actually there it is assumed that there is a measurable cardinal Ω in the universe, but this assumption is not required for the construction of K below $O^{\P}$.)

Fact 1.15 (The Weak Covering Theorem) *Assume* $\neg O^{\P}$, *then if* β *is a singular cardinal,* $\beta^+ = \beta^{+^K}$. *If* β *is inaccessible then* $cf(\beta^{+^K}) \geq \beta$.

In fact, correctly computing successors of singulars characterises universal weasels.

Fact 1.16 (cf. *[16] §3) W is a universal weasel if and only if on a stationary class of* β, $\beta^+ = \beta^{+^W}$. *Further, if assume there is no inner model for a strong cardinal, then* W *is a universal weasel if and only if* $\beta^+ = \beta^{+^W}$ *for arbitrarily large cardinals* β.

Fact 1.17 *If* $\neg O^{\P}$, *then for every universal weasel* W, *there is an almost linear iteration tree* T *on* K *with last model* W, *and such that* T *does not drop in model or degree.*

Definition 1.18 K_x *for* $x \subseteq \omega$ *is the core model built up from* x*-mice, that is mice with an extra predicate symbol for the set* x.

It is easy to see that the above results relativise to K_x in the appropriate way.

1.2.1 The Mouse Order

We shall need some basic facts about the mouse order. These results are due to Jensen and Mitchell, and are fairly well known, but there seems to be no careful discussion of them in print. The basic facts about the mouse order are direct consequences of the Comparison Lemma and the Dodd-Jensen Lemma, and so should extend as far as inner model theory goes. Nevertheless, in order to avoid some awkwardness, we retain our assumption that all mice are below $O^{\P}$ in this discussion. Let us say that (M, n) is iterable just in case M is n-sound and n-iterable in the sense of [12] Definition 5.1.4, except that in clause 2 of 5.1.4 we allow arbitrary linear compositions of n-maximal iteration trees rather than only compositions of length $\leq \omega$. Let us call such a composition in which the α-th component tree is k-maximal, where $k = n$ if $\alpha = 0$ and k is largest such that the base model of the α-th component tree is k-sound otherwise, an n-maximal iteration of (M, n). (If one drops the requirement that iteration trees use extenders in increasing length order, then these compositions are n-maximal iteration trees themselves, which is probably more natural.) Fact 1.12 the Dodd-Jensen Lemma, easily generalises from n-maximal iteration trees to n-maximal iterations, and we shall refer to 1.12 when we really mean this generalisation of it.

If (M, n) and (P, k) are iterable, then they can be compared, producing almost linear n-maximal and k-maximal iteration trees $\mathcal{T}$ and $\mathcal{U}$ on M and P. We modify our terminology slightly so that an iteration is allowed to drop in model or degree at its last step (without going on to use an extender); with this convention, we can arrange that $\mathcal{T}$ and $\mathcal{U}$ have the same last model, and that the degrees of their last nodes are the same. We call $(\mathcal{T}, \mathcal{U})$ the (n, k) coiteration of M with N.

Definition 1.19 $\mathcal{T}$ drops (in model or degree) *if $D^{\mathcal{T}} \neq \emptyset$, or if $deg^{\mathcal{T}}(\alpha) < deg^{\mathcal{T}}(0)$ for some α.*

(Again this is not the right notion for arbitrary iteration trees, as one should restrict attention to the main branch, but for almost linear trees the two notions coincide.) Then if $(\mathcal{T}, \mathcal{U})$ is the (n, k) coiteration of M with N, then at most one of $\mathcal{T}$ and $\mathcal{U}$ drops. If $\mathcal{T}$ is an n-maximal iteration tree on M with last model P, and $\mathcal{T}$ does not drop, then the canonical embedding $i : M \longrightarrow P$ is an n-embedding. We need to consider near n-embeddings as well, as these arise in the copying construction. If $\pi : M \longrightarrow N$ is a near n-embedding, (N, n) is iterable, and $\mathcal{T}$ is a n-maximal iteration tree on M, then $\mathcal{T}$ "copies" to a n-maximal iteration tree πT on N, with the copy map $\pi_\alpha : M_\alpha^{\mathcal{T}} \longrightarrow N_\alpha^{\mathcal{T}}$ being a near $deg^{\mathcal{T}}(\alpha)$-embedding. (See [15] Lemma 1.3.)

Definition 1.20 *The field $\mathbb{M}$ of the mouse order is*

$$\mathbb{M} = \{(M, n) : M \textit{ is an } n\textit{-sound mouse, is below } O^{\P}, \textit{ and } (M, n) \textit{ is iterable}\}.$$

Definition 1.21 *For (M, n) and (P, k) in $\mathbb{M}$ put:*
$(M, n) \leq_* (P, k)$ iff there is a k-maximal iteration $\mathcal{U}$ of P with last model $P_\alpha^{\mathcal{U}}$ such that $deg^{\mathcal{U}}(\alpha) = n$, and a near n-embedding of M into $P_\alpha^{\mathcal{U}}$.

Note, trivially, that if $(M, n), (P, k) \in \mathbb{M}$ and $\sigma : M \longrightarrow P$ is an m-embedding, where $m, k \geq n$ then $(M, n) \leq_* (P, k)$.

Lemma 1.22 *$\leq_*$ is connected and transitive.*

Proof: Connectedness is an immediate consequence of the comparison lemma. Now let $(M, n) \leq_* (P, k)$ as witnessed by the iteration $\mathcal{U}$ of P and the near n-embedding $\pi : M \longrightarrow P_\alpha^{\mathcal{U}}$. Let $(Q, e) \leq_* (M, n)$ as witnessed by the iteration $\mathcal{W}$ of M and the near e-embedding $\tau : Q \longrightarrow M_\beta^{\mathcal{W}}$. Since $\mathcal{W}$ is n-maximal we can form $\pi\mathcal{W}$, and we have a near $deg^{\mathcal{W}}(\beta)$-embedding $\sigma : M_\beta^{\mathcal{W}} \longrightarrow (P_\alpha^{\mathcal{U}})_\beta^{\pi\mathcal{W}}$, as in the diagram below.

$$
\begin{tikzcd}
(P,k) \arrow[r, "\mathcal{U}"] & (P^{\mathcal{U}}_\alpha, n) \arrow[r, "\pi\mathcal{W}"] & ((P^{\mathcal{U}}_\alpha)^{\pi\mathcal{W}}_\beta, e) \\
(M,n) \arrow[ur, "\pi"] \arrow[r, "\mathcal{W}"] & (M^{\mathcal{W}}_\beta, e) \arrow[ur, "\sigma"] & \\
(Q,e) \arrow[ur, "\tau"] & &
\end{tikzcd}
$$

But $deg^{\mathcal{W}}(\beta) = e$, and so $\pi\mathcal{W} \circ \mathcal{U}$ and $\sigma \circ \tau$ witness $(M,n) \leq_* (Q,e)$. **Q.E.D.**

Now set:

Definition 1.23 *$(M,n) =_* (P,k)$ iff $(M,n) \leq_* (P,k)$ and $(P,k) \leq_* (M,n)$ and*

$(M,n) <_* (P,k)$ iff $(M,n) \leq_* (P,k)$ and $(M,n) \neq_* (P,k)$.

Lemma 1.24 *For (M,n) and (P,k) in $\mathbb{M}$, $(M,n) <_* (P,k)$ iff there is an iteration $\mathcal{U}$ of (P,k) which drops (in model or degree), and a near n-embedding $\pi : M \longrightarrow P^{\mathcal{U}}_\alpha$, the last model of $\mathcal{U}$.*

Proof: ($\Rightarrow$) Let $(\mathcal{T},\mathcal{U})$ be the (n,k) coiteration of M with P. Recall our convention is that $\mathcal{T}$ and $\mathcal{U}$ have the same last model, and their last nodes have the same degree. $\mathcal{U}$ must drop since otherwise $\mathcal{T}$ and $i_{\mathcal{U}}$ witness that $(P,k) \leq_* (M,n)$. Since $\mathcal{U}$ drops, $\mathcal{T}$ does not, and $i_{\mathcal{T}}$ is the desired near n-embedding. ($\Leftarrow$) Let $\mathcal{U}$ and π be as supposed. Suppose the n-maximal iteration $\mathcal{W}$ (with last model $M^{\mathcal{W}}_\infty$) and near k-embedding $\tau : P \longrightarrow M^{\mathcal{W}}_\infty$ witness $(P,k) \leq_* (M,n)$. As π is a near n-embedding and $\mathcal{W}$ is n-bounded, let $\pi\mathcal{W}$ be the copied tree on the last model of $\mathcal{U}$, $P^{\mathcal{U}}_\alpha$, and let ψ be the copy map $\psi : M^{\mathcal{W}}_\infty \longrightarrow (P^{\mathcal{U}}_\alpha)^{\pi\mathcal{W}}_\infty$ as shown.

$$
\begin{tikzcd}
(M,n) \arrow[rr, "\pi"] \arrow[dd, "\mathcal{W}"'] & & (P^{\mathcal{U}}_\alpha) \arrow[dd, "\pi\mathcal{W}"] \\
& (P,k) \arrow[ur, "\mathcal{U}"] \arrow[dl, "\tau"'] & \\
(M^{\mathcal{W}}_\infty, k) \arrow[rr, "\psi"] & & (P^{\mathcal{U}}_\alpha)^{\pi\mathcal{W}}_\infty
\end{tikzcd}
$$

By virtue of Lemma 1.3 of [15], ψ is a near k-embedding, and thus so is $\psi \circ \tau$. But $\pi\mathcal{W} \circ \mathcal{U}$ is a k-maximal iteration of P, which on a final segment of its tree ordering has $deg^{\pi\mathcal{W}\circ\mathcal{U}}(\alpha) = k$ (since this is true for $\mathcal{W}$). By Dodd-Jensen there is no dropping on $\pi\mathcal{W} \circ \mathcal{U}$, contradicting the existence of a drop on $\mathcal{U}$! **Q.E.D.**

We remark that if $n < \omega$ and $(M, n+1) \in \mathbb{M}$, then $(M,n) <_* (M,n+1)$. The iteration of $(M,n+1)$ verifying Lemma 1.24 is the trivial one which does nothing but drop once in degree. Straightforward applications of Lemma 1.24 yield:

Corollary 1.25 *$(M,n) =_* (P,k)$ iff $n = k$ and there are n-maximal iterations $\mathcal{T}$ of M and $\mathcal{U}$ of P which do not drop and which have the same last model.*

Using this we see at once that the result of coiteration determines the mouse order:

Corollary 1.26 *Let (M,n) and (P,k) be in $\mathbb{M}$, and $(\mathcal{T},\mathcal{U})$ be their (n,k) coiteration. Then*

$$(M,n) \leq_* (P,k) \text{ iff } \mathcal{T} \text{ does not drop (in model or degree).}$$

Finally, what makes the mouse order most important is its wellfoundedness.

Lemma 1.27 *$\leq_*$ is a prewellorder.*

Proof: All we have left to verify is wellfoundedness. Given X a nonempty subset of $\mathbb{M}$, we can find a $\leq_*$-minimal element of X by simultaneously comparing all the mice in X. One element of X must iterate without dropping to an initial segment of all the others. We leave the details to the reader. **Q.E.D.**

2 Universal Iterations

In this section we restrict ourselves to premice below $O^{\P}$. We now introduce some concepts and results due to Jensen and Mitchell.

Definition 2.1 *Let M, N be n-sound, n-iterable premice. $\pi : M \longrightarrow N$ is an n-*derived iteration *of M, if there are n-maximal iterations $\mathcal{T},\mathcal{U}$ on M, N with $D^{\mathcal{T}}, D^{\mathcal{U}} = \emptyset$, with a common last model $P = M^{\mathcal{T}}_\infty = N^{\mathcal{U}}_\infty$, satisfying $deg^{\mathcal{T}}(\alpha) = deg^{\mathcal{U}}(\alpha) = n$ for all α, and iteration maps π_{MP}, π_{NP} so that $ran(\pi_{MP}) \subseteq ran(\pi_{NP})$, and $\pi = \pi^{-1}_{NP} \circ \pi_{MP}$.*

Lemma 2.2 *If $\pi : M \longrightarrow N$ is an n-derived iteration map, and $\sigma : M \longrightarrow N$ any near n-embedding, then $\pi(\eta) \leq \sigma(\eta)$ for all $\eta \in On \cap M$. Hence the derived n-iteration map is unique, and we write it π_{MN}.*

Proof: Let $\mathcal{T}, \pi_{MP}, \pi_{NP}$ be as in the definition of n-derived iteration. Then $\pi_{NP} \circ \sigma$ is a near n-embedding into P. Apply the Dodd-Jensen Lemma (Fact 1.12). **Q.E.D.**

Lemma 2.3 *(Jensen) Let $\pi : M \longrightarrow N$ be an n-derived iteration. Then π is an iteration map. That is, there is an almost linear n-maximal iteration tree $\mathcal{T}$ on M with $D^{\mathcal{T}} = \emptyset$, $deg^{\mathcal{T}}(\alpha) = n$, for all α, and with last model $M^{\mathcal{T}}_\infty = N$ such that $\pi = i^{\mathcal{T}}_{0\infty}$.*

Proof: Let $(\mathcal{T},\mathcal{U})$ be the (n,n) coiteration of M with N. Let P be the common iterate of M and N guaranteed by 2.1. Since $(M,n) =_* (P,n) =_* (N,n)$, neither $\mathcal{T}$ nor $\mathcal{U}$ drops, and we have n-embeddings $\pi^{\mathcal{T}}_{0,\theta}$ and $\pi^{\mathcal{U}}_{0,\theta}$ from M and N respectively into the common last model $M_\theta = N_\theta$. (We have padded the coiteration to save notation.)
Claim 1. $\pi^{\mathcal{T}}_{0,\theta} = \pi^{\mathcal{U}}_{0,\theta} \circ \pi_{M,N}$
Proof: We consider the (n,n) coiteration $(\mathcal{V},\bar{\mathcal{U}})$ of P with $N_\theta =_{df} \bar{N}$.

$$\begin{tikzcd}
 & P \arrow[drr, "\mathcal{V}"] & & \\
M \arrow[ur] \arrow[rr, "\pi_{MN}"] \arrow[dd, "\mathcal{T}"'] & & N \arrow[ul] \arrow[dd, "\mathcal{U}"] & P_\mu \\
 & & & \bar{N}_\mu \\
M_\theta & & N_\theta = \bar{N} \arrow[ur, "\bar{\mathcal{U}}"'] &
\end{tikzcd}$$

Since $(P, n) =_* (\bar{N}, n)$, $\mathcal{V}$ and $\bar{\mathcal{U}}$ do not drop, and we have n-embeddings σ and τ of P and $\bar{N}$ into the common last model $P_\mu = \bar{N}_\mu$ of $\mathcal{V}$ and $\bar{\mathcal{U}}$. Note that $\sigma \circ \pi_{NP} = \tau \circ \pi^{\mathcal{U}}_{0,\theta}$ by Dodd-Jensen. Now let η be an ordinal in M. We have

$$\tau(\pi^{\mathcal{U}}_{0,\theta}(\pi_{MN}(\eta))) = \sigma(\pi_{NP}(\pi_{MN}(\eta))) = \sigma(\pi_{MP}(\eta)) = \tau(\pi^{\mathcal{T}}_{0,\theta}(\eta)),$$

the last equality by Dodd-Jensen applied to the two iteration maps $\sigma \circ \pi_{MP}$ and $\tau \circ \pi^{\mathcal{T}}_{0,\theta}$. Peeling off τ, we have claim 1.

We now show by induction on $i < \theta$:
(i) $N_i = N$, and
(ii) If $i \in [0, \theta]_{\mathcal{T}}$, then $\pi^{\mathcal{T}}_{i,\theta}``M_i \subseteq \pi^{\mathcal{U}}_{0,\theta}``N$ (and hence $\pi_{M_i N} = (\pi^{\mathcal{U}}_{0,\theta})^{-1} \circ \pi^{\mathcal{T}}_{i,\theta}$ is the derived iteration map).

Suppose $i = j + 1$ and (i) and (ii) hold for j. ($i = 0$ is the content of Claim 1.) Let $E^{\mathcal{U}}_j, E^{\mathcal{T}}_j$ be the extenders chosen at the j'th stage of the coiteration.

Claim 2 $E^{\mathcal{U}}_j = \emptyset$.

Proof: Suppose otherwise. Let $\kappa = crit(E^{\mathcal{U}}_j)$. Let $(l + 1)U\theta$ and $U\text{-}pred(l + 1) = 0$, so that $crit(E^{\mathcal{U}}_l) = crit(\pi^{\mathcal{U}}_{0,\theta}) = \kappa$ by almost linearity. Let $h + 1 \geq i$ be least so that $h + 1 \in [0, \theta]_T$; setting $k = T\text{-}pred(h + 1)$, we have $k \leq j$ and $k \in [0, \theta]_T$.

Subclaim $crit(E^{\mathcal{T}}_h) = \kappa$.

Proof: By the inductive hypothesis (iii) at $k \leq j$ $\pi^{\mathcal{T}}_{k,\theta}``M_k \subseteq \pi^{\mathcal{U}}_{0,\theta}``N$. As $\kappa \notin ran\pi^{\mathcal{U}}_{0,\theta}$, $crit(\pi^{\mathcal{T}}_{k,\theta}) =_{df} \kappa' \leq \kappa$. Then $crit(E^{\mathcal{T}}_h) = \kappa'$. But by $\neg O^{\P}$ and the initial segment condition, $\kappa' \geq \kappa$ (as otherwise for unboundedly many $\eta < \kappa$ $E^{\mathcal{T}}_h \precneqq \eta$ is on the $M^{\mathcal{U}}_j$ sequence). **Q.E.D.(Subclaim).**

Then $crit(E^{\mathcal{T}}_h) = \kappa = crit(E^{\mathcal{U}}_l)$. We use the familiar argument that these extenders agree below the minimum of the two suprema of their respective sets of generators, and this is a contradiction to coiteration (*cf.* [12] Lemma 7.2).

If $a \in [\nu(E^{\mathcal{T}}_h) \cap \nu(E^{\mathcal{U}}_l)]^{<\omega}$:

$$\begin{aligned}
A \in (E^{\mathcal{T}}_h)_a &\iff a \in \pi^{\mathcal{T}}_{k,\theta}(A) = \pi^{\mathcal{U}}_{0,\theta} \circ \pi_{M_k N}(A) \ \ \text{(using (iii) at } k \leq j) \\
&\iff a \in \pi^{\mathcal{U}}_{0,\theta}(\pi_{M_k N}(A) \cap [\kappa]^{|a|}) \\
&\iff a \in \pi^{\mathcal{U}}_{0,\theta}(A) \iff A \in (E^{\mathcal{U}}_l)_a
\end{aligned}$$

Q.E.D.(Claim 2).

We thus have $N_{j+1} = N_j = N$. Suppose now $i \in [0, \theta]_{\mathcal{T}}$. (Thus $h = j$ and $E^{\mathcal{T}}_j \neq \emptyset$.) Observe that $crit(\pi^{\mathcal{T}}_{i,\theta}) > lh(E^{\mathcal{T}}_j)$ (as we are below $O^{\P}$). Let us assume that $M_i = Ult_0(M_k, E^{\mathcal{T}}_j)$, the case that M_i is a n-ultrapower for $n > 0$ being quite similar. So $M_i = \{\pi^{\mathcal{T}}_{k,i}(f)(a) \mid f \in M_k \text{ and } a \in [\nu(E^{\mathcal{T}}_j]^{<\omega}\}$. Then

$$\begin{aligned}
\pi^{\mathcal{T}}_{i,\theta}(\pi^{\mathcal{T}}_{k,i}(f)(a)) &= \pi^{\mathcal{T}}_{k,\theta}(f)(a) \quad (\text{as } crit(\pi^{\mathcal{T}}_{i,\theta}) > lh(E^{\mathcal{T}}_j)) \\
&= \pi^{\mathcal{U}}_{0,\theta}(\pi_{M_k N}(f))(a) \quad (\text{by (iii) at } k) \\
&= \pi^{\mathcal{U}}_{0,\theta}(\pi_{M_k N}(f)(a)) \quad (\text{as } crit(\pi^{\mathcal{U}}_{0,\theta}) \text{ cannot be } < lh(E^{\mathcal{T}}_j)).
\end{aligned}$$

Hence $\pi^{\mathcal{T}}_{i,\theta}$"$M_i \subseteq \pi^{\mathcal{U}}_{0,\theta}$"$N$ as required.

Suppose now that i is a limit and (ii), (iii) hold for $j < i$. (ii) is then trivial at i. Note that $i \in [0,\theta]_T$, and so for any $x \in M_i$, if $x = \pi^{\mathcal{T}}_{j,i}(\bar{x})$ with $j <_T i$, then $\pi^{\mathcal{T}}_{i,\theta}(x) = \pi^{\mathcal{T}}_{j,\theta}(\bar{x}) \in \pi^{\mathcal{U}}_{0,\theta}$"$N$ by the inductive hypothesis. This yields (iii) immediately. **Q.E.D.**

Remark: The proof shows that if $\pi : M \longrightarrow N$ is an n-derived iteration map, then the comparison of (M, N) results in M iterating (without any drops) to N without the latter moving. In particular, any iteration map associated to an n-maximal linear compositions of iteration trees can in fact be generated using a single n-maximal iteration tree.

Definition 2.4 *Let E be an extender over some premouse M, and let $Y \subseteq lh(E)$ be such that $crit(E) \in Y$. We let $E \precneqq Y = \{(a, x) \mid a \in [Y]^{<\omega}$ and $x \in E_a\}$, and call $E \precneqq Y$ the* subextender *of E with support Y.*

If Y is not transitive, then $E \precneqq Y$ is not literally an extender, but it is equivalent to one. More precisely, let $j : M \longrightarrow Ult_0(M, E)$ be the canonical embedding, and $Z = \{[a, f]^M_E \mid a \in Y^{<\omega} \text{ and } f \in M\}$. We write $Ult_0(M, E \precneqq Y))$ for the transitive collapse of the substructure of $Ult_0(M, E)$ with universe Z, and let $\pi : Ult_0(M, E \precneqq Y) \longrightarrow Ult_0(M, E)$ be the inverse of the collapse map. It is easy to see that π is a cofinal $r\Sigma_0$ embedding, so that $Ult_0(M, E \precneqq Y)$ is a premouse. Since $ran(j) \subseteq Z$, we may define $i : M \longrightarrow Ult_0(M, E \precneqq Y)$ by : $i(x) = \pi^{-1}(j(x))$. It is easy to check that i is a 0-embedding, that $crit(i) = crit(E)$, and that $Ult_0(M, E \precneqq Y) = \{i(f)(a) \mid f \in M \text{ and } a \in \bar{Y}^{<\omega}\}$, where π"$\bar{Y} = Y$. Let G be the trivial completion of the $(crit(i), \sup \bar{Y}))$ extender derived from i, so that $Ult_0(M, E \precneqq Y) = Ult_0(M, G)$ and $i = i^M_G$. We call G the extender *equivalent* to $E \precneqq Y$, or the *completion* of $E \precneqq Y$, and write $G = (E \precneqq Y)^c$. It is easy to see that for any ξ, ξ is a generator of $(E \precneqq Y)^c$ iff $\pi(\xi)$ is a generator of E.

According to the following result of Jensen and Mitchell, if M is a mouse below $O^{\P}$, then every subextender of an extender on the M-sequence is equivalent to an extender which is at most finitely many ultrapowers away from the M-sequence. This is sometimes called the derived extender lemma; it strengthens the axiom/theorem that the M-sequence is closed under initial segment up to taking one ultrapower. (However, unlike closure under initial segment, which is true for all the mice we know about, the derived extender lemma is false of mice just beyond $O^{\P}$.)

Definition 2.5 *An extender E with $crit(E) = \kappa$ is at* depth k from *M, if there is an n and a finite n-maximal iteration tree $\mathcal{I}$ on M with $\mathcal{I} = \langle I, deg, D, \langle F_i, M^*_{i+1}\rangle_{i+1 \leq k}\rangle$, where each F_i is on the $M^{\mathcal{I}}_i$ sequence, with $crit(F_i) = \kappa$ for $i < k$, and E is on the $M^{\mathcal{I}}_k$ sequence.*

Note that we can always take $\mathcal{I}$ to be 0-maximal, since a 0-ultrapower and an n-ultrapower agree to the image of the critical point, and hence past the lengths of the extenders in question.

Lemma 2.6 *(Jensen-Mitchell) Let M be a 0-iterable premouse below $O^{\P}$, and let E be an extender at finite depth from M; then for any $Y \subseteq lh(E)$ such that $crit(E) \in Y$, $(E \precneqq Y)^c$ is at finite depth from M.*

Proof: Clearly, we may as well assume that E is on the M-sequence. Let $N = Ult_0(M, E)$ and $\bar{N} = Ult_0(M, E \precneqq Y)$. Let $i : M \longrightarrow \bar{N}$ and $j : M \longrightarrow N$ be the canonical embeddings, and $\pi : \bar{N} \longrightarrow N$ the factor map described above. Let $\pi``\bar{Y} = Y$. We have

$$(M, 0) \leq_* (\bar{N}, 0) \leq_* (N, 0) \leq_* (M, 0),$$

so $(M, 0) =_* (\bar{N}, 0)$. Letting $(\mathcal{T}, \mathcal{U})$ be the $(0, 0)$ coiteration of M with $\bar{N}$, we see that neither $\mathcal{T}$ nor $\mathcal{U}$ drops in model or degree, and they produce a common last model P and 0-embeddings $k : M \longrightarrow P$ and $l : \bar{N} \longrightarrow P$. We claim that $\bar{N}$ is a 0-derived iterate of M, with derived iteration map i; for that it suffices to see that $k = l \circ i$. Dodd-Jensen gives at once that $k(\xi) \leq l(i(\xi))$ for all ξ. To see the reverse inequality, we copy $\mathcal{U}$ to a tree $\pi\mathcal{U}$ on N. We have the diagram

$$\begin{tikzcd}
& P \arrow[rr, "\tilde{\pi}"] & & R \\
M \arrow[ur, "k"] \arrow[rr, "i"] \arrow[rrrr, bend right, "j"'] & & \bar{N} \arrow[ul, "l"'] \arrow[rr, "\pi"] & & N \arrow[ul, "t"']
\end{tikzcd}$$

Thus for any ordinal ξ of M, we have

$$\tilde{\pi}(k(\xi)) \geq t(j(\xi)) = t(\pi(i(\xi))) = \tilde{pi}(l(i(\xi))$$

because $t \circ j$ is an iteration map. Peeling off $\tilde{\pi}$, we have $k(\xi) \geq l(i(\xi))$. Thus for all $x \in M$, $k(x) = l(i(x))$, and $\bar{N}$ is indeed a derived iterate of M with iteration map i. By Lemma 2.3, $\bar{N}$ is an iterate of M, and in fact $P = \bar{N}$, $i = k$, and $l = id$. It remains to show that i is the map for a 1-step ultrapower of M by some F at finite depth from M.

Let $\bar{N} = M_\theta^{\mathcal{T}}$, and notice that $\theta > 0$. Let $\kappa = crit(E) = crit(i)$; then $\kappa = crit(E_m^{\mathcal{T}})$, where $m + 1 \in [0, \theta]_T$ and $T\text{-}pred(m + 1) = 0$. We claim that $E_m^{\mathcal{T}}$ is our required F; that is, that no further extenders are used in $\mathcal{T}$. Suppose otherwise. We then have $n + 1 \in [0, \theta]_T$ such that T-$pred(n + 1) = m + 1$. Notice that if ξ is a generator of $E_n^{\mathcal{T}}$, then $\xi \neq \pi_{0,\theta}^{\mathcal{T}}(f)(a)$ for all $a \in [\xi]^{<\omega}$ and $f \in M$, and since $i = \pi_{0,\theta}^{\mathcal{T}}$, this means ξ is a generator of $(E \precneqq Y)^c$. In particular, letting $\bar{\eta} = crit(E_n^{\mathcal{T}})$, $\bar{\eta}$ is a generator of $(E \precneqq Y)^c$. Note also $\kappa < \bar{\eta}$, and $\eta = \pi(\bar{\eta})$ is a generator of E. Now it follows easily from the initial segment and coherence conditions on good extender sequences (p.7,[12]) that if Q is a premouse, H an extender on the Q-sequence, and γ a generator of H such that $crit(H) < \gamma$, then there is an extender J on the sequence of $Ult_0(Q, H)$ such that $crit(J) < \gamma$ and $lh(J) \geq \gamma$. Applying this fact with $Q = M, H = E$, and $\gamma = \eta$, we obtain an extender J on the N-sequence such that $crit(J) < \eta \leq lh(J)$. Since we are below $O^{\P}$, $crit(J) = \kappa$. Pulling back under π, we get the existence of a $\bar{J}$ on the $\bar{N}$ sequence such that $crit(\bar{J}) = \kappa < crit(E_n^{\mathcal{T}}) \leq lh(\bar{J})$. But then the initial segment condition implies there are unboundedly many $\beta < crit(E_n^{\mathcal{T}})$ such that $crit(E_\beta^{\bar{N}}) = \kappa$. Since $M_n^{\mathcal{T}}$ agrees with $\bar{N}$ through $crit(E_n^{\mathcal{T}})$, $M_n^{\mathcal{T}}$ is not below $O^{\P}$, a contradiction.

Q.E.D.

Let $i : M \longrightarrow N$ be an n-embedding, and suppose F is at finite depth from M and $crit(F) < \rho_n^M$. There is a clear sense in which we can apply i to F, even though F may not be in M, or on the M-sequence. For let $\mathcal{J}$ be the finite n-maximal tree on M with last model P such that F is on the P-sequence (and all extenders used have the same critical point as F). We can copy $\mathcal{J}$ to a tree $i\mathcal{J}$ on N (*cf.* §5 of [12]), and we have an n-embedding $\pi : P \longrightarrow Q$, where Q is the last model of $i\mathcal{J}$. We shall take $\pi(F)$ to be the result of moving F by i. It is clear that $\pi(F)$ is at the same depth from N as F is from M. Also, the agreement among the copy maps implies that for $x \subseteq crit(F)$ such that $x \in M$, $i(x) = \pi(x)$. Thus we have that for all $a \in [\nu(F)]^{<\omega}$ and $x \in M$, $x \in F_a$ iff $\pi(x) \in \pi(F)_{\pi(a)}$ iff $i(x) \in \pi(F)_{\pi(a)}$. Let us call π the (i, F) copy map. We need a lemma on interchanging the order of ultrapowers by extenders at finite depth from M.

Lemma 2.7 *Let $(M, n) \in \mathbb{M}$, and let E and F be at finite depth from M. Suppose $crit(E) \leq crit(F) < \rho_n^M$. Let π be the (i_E^M, F) copy map, $X = \pi``\nu(F)$, and $G = (\pi(F)\precneqq X)^c$. Then $Ult_n(Ult_n(M, E), G) = Ult_n(Ult_n(M, F), E)$. Moreover, setting $R = Ult_n(M, E)$ and $N = Ult_n(M, F)$, the following diagram commutes:*

$$\begin{array}{ccc} Ult_n(M,E) = R & \xrightarrow{\;i_G^R\;} & Ult_n(R,G) = Ult_n(N,E) \\ \uparrow{\scriptstyle i_E^M} & & \uparrow{\scriptstyle i_E^N} \\ M & \xrightarrow{\;i_F^M\;} & Ult_n(M,F) = N \end{array}$$

Proof: For notational simplicity, we do the case $n = 0$, and write Ult for Ult_0. We shall use the following familiar notation: if $b = \{\alpha_0, \ldots, \alpha_{k-1}\}$ and $v = \{v_0, \ldots, v_{k-1}\}$ are sets of ordinals in increasing order, and $a = \{\alpha_{i_0}, \ldots, \alpha_{i_n}\}$ where $i_0 < \cdots < i_n$, then $v_{ab} = v_{i_0}, \ldots, v_{i_n}$. When the context permits, we shall write v_a for v_{ab}. We shall identify $Ult(R, G)$ with the submodel of $Ult(R, \pi(F))$ to which it is isomorphic, and from which we defined it; that is, we shall consider members of $Ult(R, G)$ as being of the form $[\pi(c), h]^R_{\pi(F)}$, where $c \in [\nu(F)]^{<\omega}$ and $h \in R$. We define an isomorphism $\sigma : Ult(R, G) \longrightarrow Ult(N, E)$ as follows: for $c \in [\nu(F)]^{<\omega}, b \in [\nu(E)]^{<\omega}, f \in M$ a function with $dom(f) = [\kappa]^{|b|}$ and $\forall u \in [\kappa]^{|b|}(f(u)$ is a function with $dom(f(u)) = [\mu]^{|c|})$, where $\kappa = crit(E)$ and $\mu = crit(F)$, we set

$$\sigma\left(\left[\pi(c), [b, f]_E^M\right]^R_{\pi(F)}\right) = \left[b, \lambda u.[c, f(u)]_F^M\right]^N_E.$$

Notice first in this context, $[c, f(u)]_F^M = i_F^M(f(u))(c) = i_F^M(f)(u)(c)$ because $u \in [\kappa]^{<\omega}$ and $\kappa \leq \mu$, and thus the function $\lambda u.[c, f(u)]_F^M$ is in N. So the expression on the right defines an element of $Ult(N, E)$.

Next we show σ is well-defined on equivalence classes, and $\in$-preserving. For we have:

$$\left[\pi(c),[b,f]_E^M\right]_{\pi(F)}^R \stackrel{\dot{\in}}{=} \left[\pi(d),[a,g]_E^M\right]_{\pi(F)}^R \iff$$

by Loś for $Ult(R,\pi(F))$, for $\pi(F)_{\pi(c\cup d)}$ a.e. v:

$$[b,f]_E^M(v_{\pi(c)}) \stackrel{\dot{\in}}{=} [a,g]_E^M(v_{\pi(d)}) \iff$$

by Loś for $Ult(M,E)$, for $E_{a\cup b}$ a.e. u, for $F_{c\cup d}$ a.e. v:

$$f(u_b)(v_c) \stackrel{\dot{\in}}{=} g(u_a)(v_d) \iff$$

by Loś for $Ult(M,F)$, for $E_{a\cup b}$ a.e. u:

$$[c,f(u_b)]_F^M \stackrel{\dot{\in}}{=} [d,g(u_a)]_F^M \iff$$

by Loś for $Ult(N,E)$:

$$\left[b,\lambda u.[c,f(u)]_F^M\right]_E^N \stackrel{\dot{\in}}{=} \left[a,\lambda u.[d,g(u)]_F^M\right]_E^N$$

[Here are some details on the second equivalence. Let $h \in M$ be given by: $h(u) = \{v \mid f(u_b)(v_c) \stackrel{\dot{\in}}{=} g(u_a)(v_d)\}$. Notice that $F_{c\cup d} \cap ran(h)$ is in M, by weak amenability of F and the agreement between M and the model where F appears. Now $h(u) \in F_{c\cup d}$ for $E_{a\cup b}$ a.e. u iff $[a\cup b,h]_E^M \in i_E^M(F_{c\cup d}\cap ran(h))$ iff $[a\cup b,h]_E^M \in \pi(F_{c\cup d}\cap ran(h))$ (since π and i_E^M agree on subsets of μ) iff $[a\cup b,h]_E^M \in \pi(F)_{c\cup d}$. But of course, $[a\cup b,h]_E^M = \{v \mid [b,f]_E^M(v_\pi(c) \stackrel{\dot{\in}}{=} [a,g]_E^M(v_\pi(d))\}$.]

Finally we show that σ is a surjection. Let $[b,g]_E^N$ be an arbitrary element of $Ult(N,E)$. Let $g = i_F^M(h)(c)$ where $c \in [\nu(F)]^{<\omega}$. Then for $u \in [\kappa]^{|b|}$ we have

$$g(u) = i_F^M(h)(c)(u) = i_F^M(f(u))(c) = [c,f(u)]_F^M,$$

where $f \in M$ is such that $dom(f) = [\kappa]^{|b|}$ and for $u \in [\kappa]^{|b|}$, $f(u)$ is a function defined by : $f(u)(v) = h(v)(u)$ for all $v \in dom(h)$ such that $u \in dom(h(v))$. Thus $[b,g]_E^N = \left[b,\lambda u.[c,f(u)]_F^M\right]_E^N$, as desired. **Q.E.D.**

Let (M,n) be iterable, and $\theta > On \cap M$ be any sufficiently closed ordinal, for example, an M-admissible ordinal or more usually a cardinal.

Definition 2.8 *An n-maximal iteration $\mathcal{U}$ of (M,n) of length $\theta+1$ is* n-universal *iff*

(i) U does not drop ((i.e. $D^{\mathcal{U}} = \emptyset$ *and $deg^{\mathcal{U}}(\theta) = n$), and*

(ii) for any $\alpha \in [0,\theta]_U$ and F at finite depth from $M_\alpha^{\mathcal{U}}$ such that $crit(F) < \rho_n^{M_\alpha^{\mathcal{U}}}$, there is a β in $(\alpha,\theta]_U$ such that, letting π be the $(i_{\alpha,\beta}^{\mathcal{U}},F)$ copy map and $\gamma+1$ be least in $(\beta,\theta]_U$, we have $\pi(F) = E_\gamma^{\mathcal{U}}$.

For any $(M,n) \in \mathbb{M}$ and M-admissible ordinal $\theta > On \cap M$, one can easily construct an n-universal iteration of (M,n) of length $\theta+1$ by dovetailing. There are of course many such iterations, but they all produce the same last model, as we shall now show.

Lemma 2.9 *Let $(M,n) \in \mathbb{M}, \theta > On\cap M$ be M-admissible, and $\mathcal{T}$ and $\mathcal{U}$ be n-universal iterations of (M,n) of length $\theta+1$; then $M_\theta^{\mathcal{T}} = M_\theta^{\mathcal{U}}$.*

The proof of 2.9 rests on the fact that for every $\alpha < \theta$, $M_\theta^{\mathcal{U}}$ is an iterate of $M_\alpha^{\mathcal{T}}$. We prove this fact first.

Lemma 2.10 *Let $\mathcal{U}$ be an n-universal iteration of (M, n) of length $\theta + 1$, where θ is M-admissible, and let $\mathcal{V}$ be an iteration of (M, n) of length $\mu + 1 < \theta$ which does not drop in model or degree; then there is an iteration $\mathcal{T}$ of $(M^{\mathcal{V}}_{\mu}, n)$ of length $\zeta + 1 < \theta$ which does not drop in model or degree, and such that $M^{\mathcal{T}}_{\zeta} = M^{\mathcal{U}}_{\alpha}$ for some $\alpha \in [0, \theta]_U$.*

Proof: The proof is by induction on the order type of $[0, \mu]_V$. It is a notational nuisance that $\mathcal{U}, \mathcal{V}$ and the tree $\mathcal{T}$ we are to construct are only almost linear. In order to avoid some awkwardness, we shall for the duration of this proof re-index $\mathcal{U}, \mathcal{V}$ and $\mathcal{T}$ so that only the models and extenders along the "main branches", *i.e.* those leading to the final model, get indices. Thus $M^{\mathcal{U}}_{\alpha}$ is the α'th model on $[0, \theta]_U$, and $M^{\mathcal{U}}_{\alpha+1} = Ult_n(M^{\mathcal{U}}_{\alpha}, E^{\mathcal{U}}_{\alpha})$, and similarly for $\mathcal{V}$ and $\mathcal{T}$. In this way, almost linear iterations are equivalent to linear iterations in which the extender used to get to the next model is always at finite depth from the current one. So re-phrased, our proof is by induction on $\mu + 1$, the length of $\mathcal{V}$. The case $\mu = 0$ is trivial, so we proceed to the case $\mu = 1$. Let $F_0 = E^{\mathcal{V}}_0$ be the first extender used in $\mathcal{V}$. Notice that if E and F are at finite depth from P, and $crit(E) = crit(F)$, then exactly one of the following holds: (a) E is at finite depth from $Ult_0(P, F)$,(b) $E = F$, (c) F is at finite depth from $Ult_0(P, E)$. The proof is an easy application of the coherence condition: consider the first place where the sequences of ultrapowers leading to E and F diverge, and ask which uses the longer extender at that point. The trichotomy also holds with Ult_0 replaced by Ult_k whenever $crit(E) < \rho^P_k$. Notice also that if E and F are at finite depth from P and $crit(E) < crit(F)$, then E is at finite depth from $Ult_0(P, F)$, but F is not at finite depth from $Ult_0(P, E)$. Thus for any E and F at finite depth from P, exactly one of (a),(b),and (c) above holds, and in (a) and (c) we can replace 0 by any appropriate k. We now define inductively an iteration $\mathcal{T}$ of $Ult_n(M, F_0)$, together with an extender F_{γ} at finite depth from $M^{\mathcal{U}}_{\gamma}$, in such a way that the following diagram commutes, with all maps n-embeddings:

$$\begin{tikzcd}
M \arrow[r, "E^{\mathcal{U}}_0"] \arrow[d, "E^{\mathcal{V}}_0 = F_0"] & M^{\mathcal{U}}_1 \arrow[r, "E^{\mathcal{U}}_1"] \arrow[d, "F_1"] & M^{\mathcal{U}}_2 \arrow[r, "E^{\mathcal{U}}_2"] \arrow[d, "F_2"] & {} & \cdots & M^{\mathcal{U}}_{\alpha} \arrow[d, "E^{\mathcal{U}}_{\alpha}"] \\
Ult_n(M, F_0) \arrow[r, "E^{\mathcal{T}}_1"'] & M^{\mathcal{T}}_1 \arrow[r, "E^{\mathcal{T}}_1"'] & M^{\mathcal{T}}_2 \arrow[r] & {} & \cdots & M^{\mathcal{T}}_{\alpha} = M^{\mathcal{U}}_{\alpha+1}
\end{tikzcd}$$

For the successor step of the induction assume we are given F_{γ} at some finite depth from $M^{\mathcal{U}}_{\gamma}$, with $crit(F_{\gamma}) < \rho_n^{M^{\mathcal{U}}_{\gamma}}$, such that $i_{F_{\gamma}} : M^{\mathcal{U}}_{\gamma} \longrightarrow M^{\mathcal{T}}_{\gamma} = Ult_n(M^{\mathcal{U}}_{\gamma}, F_{\gamma})$ is an n-embedding; and $\mathcal{T} \precneqq \gamma + 1$ has been constructed with $\forall \delta \leq \gamma \, deg^{\mathcal{T} \precneqq \gamma+1}(\delta) = n$. If $F_{\gamma} \neq E^{\mathcal{U}}_{\gamma}$:
(1) If $E^{\mathcal{U}}_{\gamma}$ is at finite depth from $Ult_n(M^{\mathcal{U}}_{\gamma}, F_{\gamma})$, set $E^{\mathcal{T}}_{\gamma} = E^{\mathcal{U}}_{\gamma}$ and $F_{\gamma+1} = (\pi(F_{\gamma}) \precneqq X)^c$, where π is the $(i^{\mathcal{U}}_{\gamma,\gamma+1}, F_{\gamma})$ copy map and $X = \pi``\nu(F_{\gamma})$;
(2) if F_{γ} is at finite depth from $Ult_n(M^{\mathcal{U}}_{\gamma}, E^{\mathcal{U}}_{\gamma})$, set $F_{\gamma+1} = F_{\gamma}$ and $E^{\mathcal{T}}_{\gamma} = (\pi(E^{\mathcal{U}}_{\gamma}) \precneqq X)^c$, where π is the $(i_{F_{\gamma}}, E^{\mathcal{U}}_{\gamma})$ copy map and $X = \pi``\nu(E^{\mathcal{U}}_{\gamma})$.

As we have observed, if $E^{\mathcal{U}}_{\gamma} \neq F_{\gamma}$, then exactly one of (1) and (2) must hold. It is clear that in either case, $F_{\gamma+1}$ is at finite depth from $M^{\mathcal{U}}_{\gamma+1}$, and $crit(F_{\gamma+1}) \leq i^{\mathcal{U}}_{\gamma,\gamma+1}(crit(F_{\gamma})) < \rho_n^{M^{\mathcal{U}}_{\gamma+1}}$. Moreover, by 2.7, $M^{\mathcal{T}}_{\gamma+1} = Ult_n(M^{\mathcal{U}}_{\gamma+1}, F_{\gamma+1})$, and the diagram continues to commute. This finishes the successor step in the construction of T.

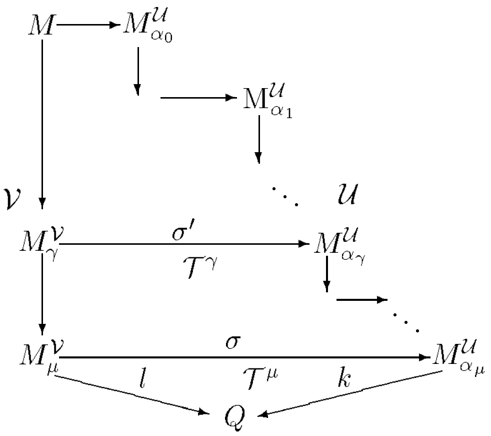
Now let λ be a limit ordinal, and suppose that for all $\alpha < \lambda$ we have defined F_{α} at finite depth from $M^{\mathcal{U}}_{\alpha}$ and $E^{\mathcal{T}}_{\alpha}$, using the method above at successor steps, and the method we are about to use at earlier limit steps. Note there are only finitely many $\gamma < \lambda$ such that $crit(F_\gamma) = crit(E^{\mathcal{U}}_\gamma)$ and (2) holds in the definition of $F_{\gamma+1}$, since $crit(F_{\gamma+1}) < i^{\mathcal{U}}_{\gamma,\gamma+1}(crit(F_\gamma))$ in this case. Let $\alpha_0 < \lambda$ bound such γ. For $\alpha < \beta \leq \lambda$, let $\pi_{\alpha,\beta}$ be the $(i^{\mathcal{U}}_{\alpha,\beta}, F_\alpha)$ copy map. It is easy to see that for $\alpha_0 < \alpha < \beta < \lambda, F_\beta = \pi_{\alpha,\beta}(F_\alpha)$ or F_β is at finite depth from $Ult_n(M^{\mathcal{U}}_\beta, E^{\mathcal{U}}_\beta)$. So we can find $\alpha_1 < \lambda$, with $\alpha_0 < \alpha_1$, such that $F_\beta = \pi_{\alpha,\beta}(F_\alpha)$ whenever $\alpha_1 < \alpha < \beta < \lambda$. This also implies that the $\pi_{\alpha,\beta}$'s commute when $\alpha_1 < \alpha < \beta \leq \lambda$. We can therefore set F_λ to be the common value of $\pi_{\alpha,\lambda}(F_\alpha)$, for all $\alpha < \lambda$ sufficiently large. Clearly, F_λ is at finite depth from $M^{\mathcal{U}}_\lambda$ and $crit(F_\lambda) < \rho_n^{M^{\mathcal{U}}_\lambda}$. We must check that $M^{\mathcal{T}}_\lambda = Ult_n(M^{\mathcal{U}}_\lambda, F_\lambda)$ and that the canonical embedding i_{F_λ} continues to make the diagram commute. But it is easy to see by induction that for $\alpha_1 < \alpha < \lambda$ (taking $n = 0$ for notational simplicity):

$$(3) \qquad i^{\mathcal{T}}_{\alpha,\alpha+1}([a, f]^{M^{\mathcal{U}}_\alpha}_{F_\alpha}) = [i^{\mathcal{U}}_{\alpha,\alpha+1}(a), i^{\mathcal{U}}_{\alpha,\alpha+1}(f)]^{M^{\mathcal{U}}_{\alpha+1}}_{F_{\alpha+1}}.$$

This holds by the proof of the Lemma 2.7, by reading the corners in the diagram there in two different directions: with M, $Ult_n(N, E)$ there as $M^{\mathcal{U}}_\alpha, M^{\mathcal{T}}_{\alpha+1}$ here, and N, R as $M^{\mathcal{T}}_\alpha, M^{\mathcal{U}}_{\alpha+1}$ (in case (1)), and as $M^{\mathcal{U}}_{\alpha+1}, M^{\mathcal{T}}_\alpha$ (in case (2)), and the maps read appropriately. (3) then clearly holds in the direct limit, that is, with λ replacing $\alpha + 1$.

The construction of $\mathcal{T}$ continues until we reach a γ so that $F_\gamma = E^{\mathcal{U}}_\gamma$. Such exists, since $\mathcal{U}$ is n-universal. At this stage we set α to be γ and we have then completed the diagram above, and the proof in the case $\mu = 1$.

At successor steps $\mu = \nu + 1$, having constructed inductively $\mathcal{T} = \mathcal{T}^\nu$ an iteration of $M^{\mathcal{T}}_0 = M^{\mathcal{V}}_\nu$ with $M^{\mathcal{T}}_\infty = M^{\mathcal{U}}_\beta$ say, we build a similar diagram using starting models $M^{\mathcal{V}}_\nu$ and $Ult_n(M^{\mathcal{V}}_\nu, E^{\mathcal{V}}_\nu) = M^{\mathcal{V}}_{\nu+1}$, and extender $E^{\mathcal{V}}_\nu$, in place of $M, Ult_n(M, E^{\mathcal{V}}_0)$ and $E^{\mathcal{V}}_0$, and use as iteration along the top bar T^ν continued by $\mathcal{U}$ from $\beta + 1$ onwards.



Proceeding to the limit we have the above picture.

We need to show there is σ a welldefined n-maximal iteration map of $M^{\mathcal{V}}_\mu$ to some $M^{\mathcal{U}}_{\alpha_\mu} = \underrightarrow{Lim}$ $\langle\langle M^{\mathcal{U}}_{\alpha_\zeta}\rangle, \langle i^{\mathcal{U}}_{\alpha_\zeta,\alpha_\xi}\rangle_{\zeta \leq \xi < \mu}\rangle$ the direct limit along the "zig-zag" (which of course is just the same limit model as in the iteration $\mathcal{U}$). We show that σ is a derived n-iteration map and then by Lemma 2.3 we have $M^{\mathcal{U}}_{\alpha_\mu}$ is an iteration *via* some $\mathcal{T}^\mu$ say, of $M^{\mathcal{V}}_\mu$. Let $x \in M^{\mathcal{V}}_\mu$ and let $\bar{x} \in M^{\mathcal{V}}_\gamma$ with $i^{\mathcal{V}}_{\gamma\mu}(\bar{x}) = x$. All the prior maps in the diagram are commuting iteration maps; set $\sigma' =_{df} i^{\mathcal{T}^\gamma}_{0,\infty}$. So we may define $\sigma(x) = i^{\mathcal{U}}_{\alpha_\gamma,\alpha_\mu}(\sigma'(x))$. By commutativity σ is well-defined, and is an n-embedding. Compare

$M^{\mathcal{U}}_{\alpha_\mu}$ and $M^{\mathcal{V}}_{\mu}$ as shown to a common Q with resulting comparison maps k and l. By Dodd-Jensen $k \circ i^{\mathcal{U}}_{\alpha_\gamma,\alpha_\mu} \circ \sigma' = l \circ i^{\mathcal{V}}_{\gamma,\mu}$ (as both sides are n-embeddings without drops). We thus have:

$$k(\sigma(x)) = k(i^{\mathcal{U}}_{\alpha_\gamma,\alpha_\mu}(\sigma'(\bar{x})) = l(i^{\mathcal{V}}_{\gamma,\mu}(\bar{x})).$$

In particular $ran\, l \subseteq ran\, k$ and Lemma 2.3 applies.

This completes the limit case and the induction. **Q.E.D.**

Proof of Lemma 2.9 Let $M^{\mathcal{U}}_\theta, M^{\mathcal{V}}_\theta$ be the last models of two n-universal iterations of (M, n). Suppose $x = i^{\mathcal{U}}_{\gamma\theta}(\bar{x})$. There is a $\delta < \theta$ so that $M^{\mathcal{V}}_\delta$ is an n-maximal iterate (without drops) of $M^{\mathcal{U}}_\gamma$, by application of the last lemma. We may thus define $\tilde{\imath} : M^{\mathcal{U}}_\theta \longrightarrow M^{\mathcal{V}}_\theta$ by $\tilde{\imath}(x) = i^{\mathcal{V}}_{\delta\theta}(i_{M^{\mathcal{U}}_\gamma M^{\mathcal{V}}_\delta}(\bar{x}))$. By identifying isomorphic direct limits, it is easy to check that $\tilde{\imath}$ is the required identity. **Q.E.D.**

Corollary 2.11 *Let (M, m) and (N, m) be in $\mathbb{M}$, let θ be a cardinal $>$ card(M), card(N), and let $M^{\mathcal{U}}_\theta$ and $N^{\mathcal{V}}_\theta$ be m-universal iterations of length θ of M and N respectively. Then*

(i) $(M, m) =_* (N, m)$ iff $M^{\mathcal{U}}_\theta = N^{\mathcal{V}}_\theta$, and
(ii) if $(M, m) =_* (N, m)$, then $M^{\mathcal{U}}_\theta \in L[M] \cap L[N]$

Proof: In $L[M]$ (respectively $L[N]$) let $\mathcal{U}$ (respectively $\mathcal{V}$) be any m-universal iterations of (M, m) (respectively (N, m)) of length $\theta + 1$. Coiterate $(M, m), (N, m)$ to a common (Q, m). These iterations do not drop and are m-maximal. By Lemma 2.10 there is an m-maximal iteration $\mathcal{W}_0$ of (Q, m) with last model $M^{\mathcal{U}}_{\gamma_0}$ for some $\gamma_0 < \theta$. Similarly there is an m-maximal iteration $\mathcal{W}_1$ without drops of $(M^{\mathcal{U}}_{\gamma_0}, m)$ to a model $(N^{\mathcal{V}}_{\delta_0}, m)$ for some $\delta_0 < \theta$, and an m-maximal iteration $\mathcal{W}_2$ of $(N^{\mathcal{V}}_{\delta_0}, m)$ to $(M^{\mathcal{U}}_{\gamma_1}, m)$. Proceeding back and forth in this manner there are m-maximal iterations $\mathcal{W}_{2n}$ of $(N^{\mathcal{V}}_{\delta_{n-1}}, m)$ to $(M^{\mathcal{U}}_{\gamma_n}, m))$ and $\mathcal{W}_{2n+1}$ of $(N^{\mathcal{V}}_{\delta_n}, m)$ to $(M^{\mathcal{U}}_{\gamma_n}, m))$. Letting $\tilde{\gamma} =_{df} \sup_n\{\gamma_n\}$ and $\tilde{\delta} =_{df} \sup_n\{\delta_n\}$ we see that $M^{\mathcal{U}}_{\tilde{\gamma}}$ and $N^{\mathcal{V}}_{\tilde{\delta}}$ are common models to both $\mathcal{U}$, $\mathcal{V}$. Clearly this happens unboundedly often below θ and hence $M^{\mathcal{U}}_\theta = N^{\mathcal{V}}_\theta \in L[M] \cap L[N]$. **Q.E.D.**

For θ a cardinal, $\theta >$ card(M) we may also view the last model $M^{\mathcal{U}}_\theta$ of such an n-universal θ iteration $\mathcal{U}$ of (M, n) as the direct limit under comparison of all mice $(N, n) =_* (M, n)$ with card(N) $< \theta$, as follows. Let $\mathcal{N}$ be the set of such mice. Suppose $N, P \in \mathcal{N}$. Set $N \preceq P$ if there is $\pi : N \longrightarrow P$ an n-maximal iteration (without drops) with (unique) map $\pi = \pi_{NP}$. Thus $\langle \mathcal{N}, \preceq \rangle$ is a directed system. We form a direct limit as follows: let (Q, n) be the resulting last model from the (n, n) coiteration of of (N, n) and (P, n). Let π_{NQ}, π_{PQ} be the iteration maps, and so $P \preceq Q$, $N \preceq Q$. For any $N, P \in \mathcal{N}$, any $x \in N$ $y \in P$ we define $x \stackrel{\dot{\in}}{\sim} y$ if $\pi_{NQ}(x) \stackrel{\in}{=} \pi_{PQ}(y)$. Then set $\mathcal{D} = \{[x]_\sim \mid \exists N \in \mathcal{N}(x \in N)\}$ and it is easily seen that $\sim$ is welldefined and that $\mathcal{D}$ is as set of equivalence classes. Set $\pi_N(x) = [x]_\sim$ for $(N, n) \in \mathcal{N}$; then π_N is welldefined and is an $\in$-embedding of $\langle N, \in \rangle$ into $\langle \mathcal{D}, \dot{\in} \rangle$. Defining predicates $\dot{E}$, $\dot{F}$ in the usual manner we can construe $\langle\langle \mathcal{D}, \dot{\in}, \dot{E}, \dot{F} \rangle, \langle \pi_N \rangle_{N \in \mathcal{N}} \rangle = \underrightarrow{Lim}\, \langle\langle N, \in, E^N, F^N \rangle, \langle \pi_{NP} \rangle_{N \preceq P \in \mathcal{N}} \rangle$. We shall have:

Corollary 2.12 $\langle \mathcal{D}, \dot{\in}, \dot{E}, \dot{F} \rangle \cong \langle M^{\mathcal{T}}_\theta, \in, E^{M^{\mathcal{T}}_\theta}, F^{M^{\mathcal{T}}_\theta} \rangle$, *where $M^{\mathcal{T}}_\theta$ is the last model on $\mathcal{T}$, an n-universal iteration of length θ of (M, n).*

Proof: Let $N \in \mathcal{N}$, and let P be a common iterate of N and M witnessing this. By 2.10, there is an $\alpha < \theta$ such that $M^{\mathcal{T}}_\alpha$ is an iterate of P. It follows that $N \preceq M^{\mathcal{T}}_\alpha$. Thus $\{M^{\mathcal{T}}_\alpha \ : \ \alpha < \theta\}$ is

$\preceq$ cofinal in $\mathcal{N}$. By the Dodd-Jensen lemma, the $i^{\mathcal{T}}_{\alpha,\beta}$ are just the maps $\pi_{M^{\mathcal{T}}_\alpha, M^{\mathcal{T}}_\beta}$ used to define $\mathcal{D}$. Thus $M^{\mathcal{T}}_\theta = \mathcal{D}$. **Q.E.D.**

Definition 2.13 *Let $\theta > card(M)$, $k \leq \omega$, and let $M^{\mathcal{U}}_\theta$ be the k-universal iteration of (M,k). Then set $o(M,k,\theta) =_{df} On \cap M^{\mathcal{U}}_\theta$.*

Using this we get:

Lemma 2.14 *If $\theta >$ card(M) and $\mathcal{T}$ is a k-maximal iteration of (M,k) without any dropping, and with last model $M^{\mathcal{T}}_\theta$, then $M^{\mathcal{T}}_\theta \cap On \leq o(M,k,\theta)$.*

Proof: Then $M^{\mathcal{T}}_\theta$ is a direct limit of smaller $M^{\mathcal{T}}_\gamma =_* M$. But by means of the last corollary, we may embed the direct limit model $M^{\mathcal{T}}_\theta$ into $M^{\mathcal{U}}_\theta$ and the result follows. **Q.E.D.**

If we are only dealing with premice below $O^{\P}$ then the mouse order below any particular mouse N has no holes in $L[N]$:

Lemma 2.15 *Let $(M,k) \leq_* (N,m)$. Then there is $\widetilde{M} \in L[N]$ with $(\widetilde{M},k) =_* (M,k)$.*

Proof: Suppose not and let $(N,m) \in \mathbb{M}$ be least (in $\leq_*$) for which this fails for some (M,k). Then $(M,k) <_* (N,m)$ and if the (k,m) coiteration of $(M,k),(N,m)$ results in trees $\mathcal{T},\mathcal{U}$, there is a least stage j so that $i^{\mathcal{U}}_{j,j+1} : N^{\mathcal{U}*}_j \longrightarrow N^{\mathcal{U}}_{j+1}$ involves a drop in model or degree. By Lemma 2.10, if θ is a suitably large cardinal and $\mathcal{W}$ is an m-universal iteration of N, with $\mathcal{W} \in L[N]$, then there is $\gamma < \theta$ with $N^{\mathcal{W}}_\gamma$ an m-maximal iterate of $N^{\mathcal{U}}_j$ and $\sigma : N^{\mathcal{U}}_j \longrightarrow N^{\mathcal{W}}_\gamma$ an m-embedding. Hence $(M,k) <_* (N^{\mathcal{U}}_j, m) =_* (N^{\mathcal{W}}_\gamma, m)$. Let $deg^{\mathcal{U}}(j+1) = \bar{m}$. If $i^{\mathcal{U}}_{j,j+1}$ does not drop in model but $\bar{m} < m$, then we shall have the following (the strict $<_*$ by virtue of the m-embedding σ):

$$(M,k) \leq_* (N^{\mathcal{U}}_j, \bar{m}) =_* (N^{\mathcal{W}}_\gamma, \bar{m}) <_* (N^{\mathcal{W}}_\gamma, m).$$

This contradicts the leastness of (N,m), since $(N^{\mathcal{W}}_\gamma, m) \in L[N]$. Hence there is a drop in model and $N^{\mathcal{U}*}_j = \mathcal{J}^{N^{\mathcal{U}}_j}_\alpha$ for some $\alpha < On \cap N^{\mathcal{U}}_j$. If $\sigma(\alpha) = \tilde{\alpha}$, then $\sigma \upharpoonright N^*_j : N^*_j \longrightarrow N^{\mathcal{W}}_\gamma || \tilde{\alpha}$ is a fully elementary embedding and we have:

$$(M,k) \leq_* (N^*_j, \bar{m}) \leq_* (N^{\mathcal{W}}_\gamma || \tilde{\alpha}, \bar{m}) <_* (N^{\mathcal{W}}_\gamma, m).$$

Again $N^{\mathcal{W}}_\gamma || \tilde{\alpha} \in L[N]$, and this contradicts our choice of (N,m) as $\leq_*$-least. **Q.E.D.**

3 The second uniform indiscernible

We assume in this section $\neg O^{\P}$.

Definition 3.1 *$K^a =_{df} K^{L[a]}$ for a any set of ordinals.*

K^a is thus a weasel of the form $\langle L^E, \in, E \rangle \models$ "$V = K$".

Lemma 3.2 *Let $c \in I^a$, then $(c^+)^{L[a]} = (c^+)^{K^a}$.*

Proof: Let $j : L[a] \to L[a]$ be elementary with the first ordinal moved by j being c. Suppose $c' =_{df} (c^+)^{K^a} < (c^+)^{L[a]}$. Let E_j be the extender derived from the embedding j, that is for $b \in [j(c)]^{<\omega}$ set $X \in (E_j)_b \iff b \in j(X)$. By an old argument of Kunen, the fragment of the extender on K^a is in $L[a]$: let $F = E_j \bigcap([j(c)]^{<\omega} \times K^a)$. We are claiming $F \in L[a]$: let $\langle X_\alpha \mid \alpha < c\rangle$ enumerate $(\mathcal{P}([c]^{<\omega}))^{K^a}$in $L[a]$; for $b \in [j(c)]^{<\omega}$ we have

$$(E_j)_b = \{j(X_\alpha) \cap c \mid \alpha < c \wedge b \in j(X_\alpha)\}$$

Since $j(\langle X_\alpha \mid \alpha < c\rangle) \in L[a]$, we have that $F \in L[a]$ as required. Now work in $L[a]$, and let $i : K \longrightarrow Ult(K, F)$ be the canonical embedding. Because $O^\P$ does not exist, $i = i_{\mathcal{T}}$ where $\mathcal{T}$ is some almost linear iteration tree on K. Let E be the first extender used on the main branch of $\mathcal{T}$, and ν the natural length of E. Notice that $\nu \geq j(c)$, since K agrees with $Ult(K, F)$ up to $j(c)$, which is a cardinal of V (that is, $L[a]$). But $j(c) = i(c) = i^{\mathcal{T}}(c) \geq i_E(c) > \nu$, with the last inequality holding because E is not of superstrong type. Contradiction! **Q.E.D.**

Remark: One can prove 3.2 under the weaker hypothesis that $L[a]$ satisfies that there is no inner model with a Woodin cardinal. The proof just given that $F \in L[a]$ works under this weaker hypothesis. One can then show that F is on the K^a sequence by using the fact that $L[a]$ has background certificates sufficient to guarantee the iterability of the phalanx $((K, Ult(K, F{\precneqq}\nu)), \nu)$ for all $\nu \leq j(c)$. See [16] 8.6 and 8.7.

Lemma 3.3 *For any $a \in \mathbb{R}$ there is a countable $M = M_a, n = n_a$ so that $(M, n) \in \mathbb{M} \cap L[a^\#]$ and iterates past K^a.*

Proof: In $L[a^\#]$ we may define the comparison of $K^{a^\#}$ with K^a. Let $\mathcal{T},\mathcal{U}$ be the resultant On-length iteration trees. As $L[a]$ computes all successor cardinals incorrectly, we have for all $c \in I^a$ $c^+ > (c^+)^{L[a]} = (c^+)^{K^a}$; hence K^a is not universal. As $K(= K^{a^\#})$ is universal but K^a is not, a simple reflection argument shows $D^{\mathcal{U}} = \emptyset$. As $M^{\mathcal{T}}_\infty \neq M^{\mathcal{U}}_\infty$ we have by Def. 1.14 that some ordinal γ is sent out to On on the $\mathcal{T}$ side. We claim

(i) $\exists C \subseteq On, C$ cub, so that $i, j \in C,\ i < j \longrightarrow i = \kappa_i < \kappa_j = j$ and $\pi^{\mathcal{T}}_{i,j}(\kappa_i) = \kappa_j$.

(ii) $i \in C \longrightarrow \pi^{\mathcal{U}}_{0,i}``i \subseteq i$.

A standard regressive function argument shows there is a stationary set C satisfying (i), and it is easy to see that such a C can be closed. Suppose (ii) fails. Then some ordinal on the K^a side also must be sent up to On. By the standard closure argument (*cf.* the Claim on p.72 of [12]), there is a cub $D \subseteq On$ such that:

(1) $\alpha \in D \Rightarrow \alpha = \kappa_\alpha = crit(\pi^{\mathcal{T}}_{\alpha,\theta}) = crit(\pi^{\mathcal{U}}_{\alpha,\theta})$

(2) $\alpha < \beta \in D \Rightarrow \pi^{\mathcal{T}}_{\alpha,\beta}(\alpha) = \beta \wedge \pi^{\mathcal{U}}_{\alpha,\beta}(\alpha) = \beta$

(3) $\alpha = \kappa_\alpha \in D \wedge A \subseteq [\kappa_\alpha]^n \wedge A \in |E^{\mathcal{T}}_\alpha| \cap |E^{\mathcal{U}}_\alpha| \Rightarrow \pi^{\mathcal{T}}_{\alpha,\theta}(A) = \pi^{\mathcal{U}}_{\alpha,\theta}(A)$

We should then conclude $E^{\mathcal{T}}_\alpha{\precneqq}\nu = E^{\mathcal{U}}_\alpha{\precneqq}\nu$ ($\nu = \min\{\nu(E^{\mathcal{T}}_\alpha), \nu(E^{\mathcal{U}}_\alpha)\}$, the minimum of the two suprema of the generators of the extenders) and there would be no further need to coiterate. And this would be a contradiction.

Hence $\pi^{\mathcal{U}}_{0,On}``On \subseteq On$. Again by a closure and reflection argument we may assume that C fulfills the requirement (ii). So let j be the least element of C and $\kappa = \kappa_j$; as we are below $O^\P$, there is a least ordinal α bounding the length of the extenders in $M^{\mathcal{T}}_j$ with critical point κ. It is then easy to see that the coiteration of $M^{\mathcal{T}}_j|\alpha$ and $M^{\mathcal{U}}_j$ uses the same extenders as $\mathcal{T}$ and $\mathcal{U}$ for $i \geq j$. Set M equal to this $M^{\mathcal{T}}_j$. Clearly M iterates past K^a and is definable in $L[a^\#]$. Thus it is countable in $L[a^{\#\#}]$. **Q.E.D.**

Corollary 3.4 *If M is countable and $(M,n) \in \mathbb{M}$ and (M,n) iterates past K^a, then for $c = \omega_1^V$, $(c^+)^{L[M]} > o(M,n,c) \geq (c^+)^{L[a]}$.*

Proof: With $j, \kappa, C, \mathcal{T}, \mathcal{U}$ as in the last proof, we shall also have that C satisfies:

(iii)$sup\, \pi^{\mathcal{T}}_{j,c}``(\kappa^+)^{M_j^{\mathcal{T}}} = \pi^{\mathcal{T}}_{j,c}((\kappa^+)^{M_j^{\mathcal{T}}}) = (c^+)^{M_c^{\mathcal{T}}} = (c^+)^{M_c^{\mathcal{U}}}$

As $K^a = M_0^{\mathcal{U}}$, $(c^+)^{M_c^{\mathcal{U}}} \geq (c^+)^{K^a}$. By Lemma 2.14 $o(M,n,c) \geq On \cap M_c^{\mathcal{T}} \geq (c^+)^{M_c^{\mathcal{T}}} = (c^+)^{M_c^{\mathcal{U}}} \geq (c^+)^{K^a} = (c^+)^{L[a]}$ with the last equality by Lemma 3.2. Clearly $(c^+)^{L[M]} > o(M,n,c)$, as $card^{L[M]}(M) < c$. **Q.E.D.**

Lemma 3.5 $\boldsymbol{\delta^1_2} = sup\{(c^+)^{L[a]} \mid a \in \mathbb{R}\} = sup\{o(M,n,c) \mid M \in \mathbb{M} \cap HC\}$.

Proof: Setting, for $a \in \mathbb{R}$, (M^a, n^a) the mouse $(M,n) \in L[a^\#]$ occurring in the proof of 3.3, we have $(c^+)^{L[a]} \leq o(M^a, n^a, c) \leq (c^+)^{L[a^\#]}$. The result follows. **Q.E.D.**

Lemma 3.6 $\boldsymbol{\delta^1_2} = \omega_2 \Rightarrow \exists\, (M,n) \in HC\ (K|\omega_1^V, \omega) <_* (M,n)$.

Proof: Let $c = \omega_1^V$. Consider any $(M,n) \in HC$ and let it be compared with $(K|c,\omega)$ with resulting trees $\mathcal{T}, \mathcal{U}$.

Case 1 $lh(\mathcal{T}), lh(\mathcal{U}) = \gamma < c$.

Then $(M,n) <_* (K|c,\omega)$ as clearly $D^{\mathcal{U}} \neq \emptyset$. Note that there is then $\alpha < c$ with $(M,n) <_* (K|\alpha,\omega)$ as there is a bound below c on the length of any $lh(E_i^{\mathcal{U}})$ used on the $\mathcal{U}$-side. Thus if Case 1 held for all such (M,n) we should have $o(M,n,c) < \delta =_{df} sup\{o(K|\alpha,\omega,c) \mid \alpha < c\}$. The latter supremum δ is definable in K, and hence $\delta < \omega_2^K \leq \omega_2$. Thus $\boldsymbol{\delta^1_2} < \omega_2$ by Lemma 3.5. Hence Case 2 must hold for some $(M,n) \in HC$ where:

Case 2 $lh(\mathcal{T}) = lh(\mathcal{U}) \geq c$.

The following claim then proves the lemma:

Claim $lh(\mathcal{T}) = lh(\mathcal{U}) = c+1$, $\pi^{\mathcal{U}}_{0,c}``c \subseteq c$, and $\exists \kappa, j < c\ \pi^{\mathcal{U}}_{j,c}(\kappa) = c$, and $D^{\mathcal{T}} \neq \emptyset$.

Proof: The existence of j, κ follows immediately from the case hypothesis. Suppose $\pi^{\mathcal{U}}_{0,c}``c \not\subseteq c$. Then also on the $\mathcal{U}$ side there must be λ, i with $\pi^{\mathcal{U}}_{i,c}(\lambda) = c$. But now we may rerun the argument of Lemma 3.3 to get the same contradiction on a cub set $D \subseteq c$ now. We thus have $M_c^{\mathcal{U}}$ a proper initial segment of $M_c^{\mathcal{T}}$, and the final iteration is the trivial one of a drop in model on the $\mathcal{T}$-side. **Q.E.D.**

Corollary 3.7 $\boldsymbol{\delta^1_2} = \omega_2 \Rightarrow \forall x \in \mathbb{R}(\omega_1$ *is inaccessible in* $K_x)$.

Proof: Let $c = \omega_1^V$. The last lemma also showed the comparison of some (M,n) with (K,ω) resulted in trees $(\mathcal{T},\mathcal{U})$ with $M_c^{\mathcal{T}}$ a proper initial segment of $M_c^{\mathcal{U}}$, with $\{\kappa \mid \kappa = crit(E_i^{\mathcal{T}}), i < c\}$ unbounded in c. Hence $M_c^{\mathcal{U}} \models$ "there are arbitrarily large cardinals". As $\pi^{\mathcal{U}}_{0c} : K|c \longrightarrow M_c^{\mathcal{U}}$ is elementary, we have that $K \models$ "c is inaccessible". But note that the whole argument of 3.2 - 3.7 relativizes to any real $x \in \mathbb{R}$. **Q.E.D.**

The following lemma shows that $K|\omega_2$ is universal for mice of cardinality less than ω_2 (in that any such mouse satisfies $(M,\omega) <_* (K|\omega_2,\omega)$). Jensen in [5] shows it is consistent relative to the existence of a measurable cardinal that $K|\omega_1$ is not universal for countable mice, in this sense.

Lemma 3.8 *Let $(M,\omega) \in \mathbb{M}$ with $On \cap M < \omega_2$ Then the (ω,ω)-coiteration of (M,ω) with (K,ω) via trees $(\mathcal{T},\mathcal{U})$ has length $\theta < \omega_2$. Hence there is $(N,\omega) \in K_{\omega_2}$ with $(M,\omega) =_* (N,\omega)$.*

Proof: Suppose $lh(\mathcal{T}) = lh(\mathcal{U})$ is $\theta \geq \omega_2$. Let $P = M^{\mathcal{T}}_{\omega_2}$, $Q = M^{\mathcal{U}}_{\omega_2}$, the models produced at the ω_2'nd stage of the coiteration. Then by repeating the argument of the 3.3 and 3.4, (with ω_2 replacing On), we should have a $C \subseteq \omega_2$, cub, and satisfying (i)-(iii) of 3.3 and 3.4, and for some $i < \omega_2$, some $\kappa = crit(E^{\mathcal{T}}_i)$, that $\pi^{\mathcal{T}}_{i,\omega_2}(\kappa) = \omega_2$; $\pi^{\mathcal{T}}_{i,\omega_2}$"$(\kappa^+)^{M^{\mathcal{T}}_i}$ is cofinal in $(\omega_2^+)^P$, and hence the latter has cofinality $\leq \omega_1$. On the other hand, $D^{\mathcal{U}} \cap \omega_2 = \emptyset$ with $\pi^{\mathcal{U}}_{0,\omega_2}$"$\omega_2 \subseteq \omega_2$, and $\pi^{\mathcal{U}}_{0,\omega_2}((\omega_2^+)^K) = (\omega_2^+)^Q$. By the Weak Covering Theorem for K (Fact 1.15) the latter has cofinality $\geq \omega_2$. Thus $(\omega_2^+)^P \neq (\omega_2^+)^Q$.
(1) $(\omega_2^+)^P < (\omega_2^+)^Q$, and $\omega_2 + 1 \in D^{\mathcal{U}}$.
Proof: By universality of K, $D^{\mathcal{T}} = \emptyset$. Then at least one of $E^{\mathcal{T}}_{\omega_2}, E^{\mathcal{U}}_{\omega_2}$ is non-empty. Suppose $E^{\mathcal{T}}_{\omega_2} \neq \emptyset$ with critical point λ. As $D^{\mathcal{T}} = \emptyset$, $\lambda \leq \omega_2$. But $\lambda < \omega_2$ is impossible, as this would imply that unboundedly many critical points below ω_2 have been used on the $\mathcal{T}$ side, and this contradicts the fact that we are below $O^{\P}$. Hence $\lambda = \omega_2$. But $(\omega_2^+)^P > (\omega_2^+)^Q$ would require $\omega_2 + 1 \in D^{\mathcal{T}}$ which is a contradiction. Hence the reverse inequality holds, and this means $\omega_2 + 1 \in D^{\mathcal{U}}$.

If $E^{\mathcal{T}}_{\omega_2} = \emptyset$, $E^{\mathcal{U}}_{\omega_2}$ is not. Similarly $crit(E^{\mathcal{U}}_{\omega_2}) < \omega_2$ would be a contradiction using the initial segment condition and $\neg O^{\P}$, as there would then be arbitrarily long initial segments of $E^{\mathcal{U}}_{\omega_2}$ on the $M^{\mathcal{T}}_{\omega_2}$ sequence. Hence $crit(E^{\mathcal{U}}_{\omega_2}) = \omega_2$, and again $(\omega_2^+)^P \not> (\omega_2^+)^Q$. Thus $\omega_2 + 1 \in D^{\mathcal{U}}$. **Q.E.D.(1)**

But now we shall reflect the facts of (1) below ω_2, to get $D^{\mathcal{U}} \cap \omega_2 \neq \emptyset$ – a contradiction. Let $X \prec V_\eta$ be an elementary substructure of a suitable large V_η. Suppose $\omega_1 \cup \{M, K|\omega_3, \mathcal{T}, \mathcal{U}, D\} \subseteq X$, with $X \cap \omega_2 \in \omega_2$, and $card(X) = \omega_1$. Let $j : N \longrightarrow V_\eta$ be the inverse of the transitive collapse, with $j(\langle \kappa, \widetilde{K}, \widetilde{\mathcal{T}}, \widetilde{\mathcal{U}} \rangle) = \langle \omega_2, K|\omega_3, \mathcal{T}, \mathcal{U} \rangle$, and note that as $X \cap \omega_2$ is transitive, that $\widetilde{\mathcal{T}} \restriction \kappa = \mathcal{T} \restriction \kappa$, $\widetilde{\mathcal{U}} \restriction \kappa = \mathcal{U} \restriction \kappa$, and $\widetilde{C} = C \cap \kappa$. Let $\widetilde{P} = M^{\widetilde{\mathcal{T}}}_\kappa, \widetilde{Q} = M^{\widetilde{\mathcal{U}}}_\kappa$. Then note that $j(\langle \widetilde{P}, \widetilde{Q} \rangle) = \langle P, Q \rangle$. Let $R = M^{\mathcal{T}}_\kappa$, $S = M^{\mathcal{U}}_\kappa$.
(2) $(\kappa^+)^R = (\kappa^+)^{\widetilde{P}} \wedge \mathcal{J}^R_\alpha = \mathcal{J}^{\widetilde{P}}_\alpha$ for any $\alpha < (\kappa^+)^R$. Thus $\widetilde{P}|(\kappa^+)^{\widetilde{P}} = R|(\kappa^+)^{\widetilde{P}}$.
Proof: For $\alpha < (\kappa^+)^R$ note that $\mathcal{J}^R_\alpha = \varinjlim \langle \mathcal{J}^{M^{\mathcal{T}}_i}_{\alpha_i}, \pi^{\mathcal{T}}_{i,j} \rangle_{i<j \in C \cap \kappa} = \varinjlim \langle \mathcal{J}^{M^{\widetilde{\mathcal{T}}}_i}_{\alpha_i}, \pi^{\widetilde{\mathcal{T}}}_{i,j} \rangle_{i<j \in \widetilde{C}}$ and the result follows. **Q.E.D.(2)**
Let $\tilde{\kappa} =_{df} (\kappa^+)^{\widetilde{K}}$.
(3) $\widetilde{K}|\tilde{\kappa} = K|\tilde{\kappa}$.
Proof: By condensation (*cf.* [16] 8.2), using the elementary maps $j \restriction \widetilde{K}|\alpha \longrightarrow K|j(\alpha)$ for $\alpha < \tilde{\kappa}$. **Q.E.D.(3)**
By arguing as in (2) and using (3) we have:
(4) $\widetilde{Q}|\tilde{\kappa} = S|\tilde{\kappa}$.
(5) $(\kappa^+)^{\widetilde{P}} < \tilde{\kappa} = (\kappa^+)^{\widetilde{Q}}$ and $\kappa + 1 \in D^{\widetilde{\mathcal{U}}}$.
Proof: By (1) and then elementarity of j. **Q.E.D.(5)**

By (2) and (4), the hierarchies of $R, \widetilde{P}$ and of $S, \widetilde{Q}$ agree up to their respective κ^+'s. Hence $\kappa + 1 \in D^{\mathcal{U}}$ - contradicting $lh(\mathcal{T}) = lh(\mathcal{U}) \geq \omega_2$! Hence $lh(\mathcal{T}) < \omega_2$.

Since, by Lemma 2.15, the mouse ordering in K has no holes, there is $N \in K$ with $(N, \omega) =_* (M, \omega)$. By Löwenheim-Skolem we may take $N \in K_{\omega_2}$. **Q.E.D.**

We now prove the main theorem of this section.

Theorem 3.9 *Assume* $\neg O^{\P}$ *and* $\forall r \in \mathbb{R}$ ($r^\#$ *exists*). *Then* $\boldsymbol{\delta}^1_2 < \omega_2$.

Proof: Let $c = \omega_1^V$. Suppose not, then we have seen some countable $(M, n) >_* (K|c, \omega)$. Let $x \in \mathbb{R}$ code M. Let $W = K^{K_x}$.

Claim 1 W is universal.

Proof: Suppose not, then by Def. 1.14(ii) there is a mouse or a weasel P, so that if (P, W) are compared *via* trees $(\mathcal{T}, \mathcal{U})$, $M^{\mathcal{T}}_{\infty}$ is not an initial segment of $M^{\mathcal{U}}_{\infty}$.

As $D^{\mathcal{U}} = \emptyset$, by the usual arguments, there is a cub $D \subseteq On$ so that $i < j \in D \Rightarrow \pi^{\mathcal{T}}_{i,j}(\kappa_i) = \kappa_j = j$ and $crit(E^{\mathcal{T}}_j) = \kappa_j$, whilst $\pi^{\mathcal{U}}_{0,j}``\kappa_j \subseteq \kappa_j$ and $crit(\pi^{\mathcal{U}}_{j,\infty}) \geq \kappa_j$. Hence if γ is any cardinal of V, with $\gamma \in D$, then γ is inaccessible in both of the final models $M^{\mathcal{T}}_{\infty}, M^{\mathcal{U}}_{\infty}$. Since W is assumed not to be universal, by Fact 1.16 we may shrink D to a smaller cub class if need be so that:

(1) For $\beta \in D\, (\beta^+)^W < \beta^+$.

Hence W computes all sufficiently large successors of singulars in D incorrectly.

(2) For all sufficiently large singular $\beta \in D$, β is regular in K_x.

Otherwise, if β were singular in K_x, by the Weak Covering Lemma applied inside K_x, $(\beta^+)^{K_x} = (\beta^+)^W$; but by $\neg x^{\P}$, the Weak Covering theorem holds over K_x, and so $\beta^+ = (\beta^+)^{K_x}$ - thus contradicting (1). **Q.E.D.(2)**

Again by $\neg x^{\P}$, applying Weak Covering over K_x:

(3) For all sufficiently large singular cardinals $\beta \in D$, $cf^{K_x}((\beta^+)^W) \geq \beta$.

Now take i least in D with $\kappa = \kappa_i$ and $D^{\mathcal{T}} \backslash \kappa = \emptyset$. Then $\pi^{\mathcal{T}}_{i,j}``(\kappa^+)^{M^{\mathcal{T}}_i}$ is cofinal into

$$\pi^{\mathcal{T}}_{i,j}((\kappa^+)^{M^{\mathcal{T}}_i}) = (\kappa^+_j)^{M^{\mathcal{T}}_j} = (\kappa^+_j)^{M^{\mathcal{U}}_j}$$

Set $\gamma = cf((\kappa^+)^{M^{\mathcal{T}}_i})$, then:

(4) $\gamma \in D \longrightarrow cf((\kappa^+_j)^W) = \gamma$.

Pick then, some sufficiently large singular cardinal $\lambda = \kappa_\lambda \in D$ with $cf(\lambda) = \bar{\gamma} \neq \gamma$. By (3) we have $cf^{K_x}((\lambda^+)^W) \geq \lambda$ whilst $cf^V(\lambda) = \bar{\gamma} \neq \gamma$. Hence:

(5) $cf^V((\lambda^+)^W) = \bar{\gamma}$.

By (2) λ is regular in W, and as $\pi^{\mathcal{U}}_{j,\lambda}``\lambda \subseteq \lambda$, we have, by induction on $j < \lambda$, that $\pi^{\mathcal{U}}_{j,j+1}(\langle \lambda, (\lambda^+)^W \rangle) = \langle \lambda, (\lambda^+)^W \rangle$, and so $(\lambda^+)^W = (\lambda^+)^{M^{\mathcal{U}}_j} = (\lambda^+)^{M^{\mathcal{U}}_\lambda}$. As $D^{\mathcal{U}} = \emptyset$ and $crit(\pi^{\mathcal{U}}_{j,\infty}) \geq \kappa_j$, $(\lambda^+)^{M^{\mathcal{U}}_\lambda} = (\lambda^+)^{M^{\mathcal{T}}_\lambda}$. Hence $cf((\lambda^+)^{M^{\mathcal{U}}_\lambda}) = \gamma \neq \bar{\gamma} = cf((\lambda^+)^W)$ (the first equality being (4) and the second being (5)) - a contradiction! **Q.E.D.(Claim 1)**

Claim 2 $(M, n) <_* (W|c, \omega)$.

Proof: Apply the last lemma inside K_x: if $\tau = \omega_2^{K_x}$ then $(W|\tau, \omega) >_* (M, n)$. But c is inaccessible in K_x. So $c > \tau$! **Q.E.D.(Claim 2)**

As W is universal, and as we are below $O^{\P}$, W is an iterate of K (Thm. 8.13 of [16]). Consequently the iteration map shows $(W|c, \omega) \leq_* (K|c, \omega)$. But then $(M, n) <_* (K|c, \omega)$ by Claim 2 – the final contradiction. **Q.E.D.**

4 Σ^1_3-absoluteness.

In this section we prove:

Theorem 4.1 *Suppose $\forall a \in \mathbb{R}\ (a^{\#}$ exists $)$ but $\neg O^{\P}$. Then K is Σ^1_3 correct.*

For X a class of ordinals we let $[X]^{<\omega}$ be the class of increasing n-tuples from X.

Definition 4.2 $f \sim_C g \iff$ *there exists a regular cardinal μ, and $n < \omega$ so that $f : [\mu]^n \longrightarrow \mu$, $\quad g : [\mu]^n \longrightarrow \mu$, $C \subseteq \mu$ is closed and unbounded, and $\forall \vec{\gamma} \in [C]^n\ f(\vec{\gamma}) = g(\vec{\gamma})$.*

Theorem 4.1 follows from the following theorem by results of Martin and Solovay [11], (or *cf.*[9]), since the conclusion is enough to show that there is a tree $T \in K$ such that $p[T]$ is a universal Π^1_2 set of reals, and thus K is Σ^1_3-correct.

Theorem 4.3 *Assume $\forall a \in \mathbb{R}$ ($a^\#$ exists) but $\neg O^\P$. Let $\mu > \omega_2$ be regular, and suppose $f : [\mu]^n \longrightarrow \mu$ where $f \in L[x]$ for some real x. Then there are $g, C \in K$ with $f \sim_C g$.*

Proof: We define reals x^i and mice $M^i \in L[x^i]$ as follows: $x^0 = x$, $M^0 = \emptyset$; $x^{i+1} = x^{i\#}$; $(M^{i+1}, \omega) \in \mathbb{M}$ is chosen $<_*$-least so that (M^{i+1}, ω) is an $<_*$ upper bound for $\mathbb{M} \cap L[x^i]$. Note the M^i exist (and may be chosen to be countable in $L[x^{i+1}]$ by Lemma 3.3. By appealing to Lemma 3.8, find $\theta < \omega_2$ so that for all i the ω-universal θ-iterates, N_i, of M^i exists, and so that $(M^i, \omega) =_* (N_i, \omega)$ and the latter is in $L[x^i] \cap K$. Let $U = \langle u_i \mid 1 \le i < \infty \rangle$ enumerate $\bigcap_{i<\omega} I^{x^i} \backslash \omega_2$. Thus $u_1 = \omega_2$, and U is the cub class of "uniform indiscernibles" for the x^i above ω_2.
Claim 1 U is a K-definable class.
Proof: As U is closed, it suffices to show for all α that $u_{\alpha+1}$ is so definable from u_α. 3.2–3.4 showed that

$$\iota^{x^{i-1}}_{u_1+1} < (u_1^+)^{L[x^i]} \le o(M^i, \omega, u_1) = o(N_i, \omega, u_1) < (u_1^+)^{L[x^{i+1}]} < \iota^{x^{i+1}}_{u_1+1} \tag{1}$$

Hence:

$$u_2 = sup_{i<\omega}\{\iota^{x^i}_{u_1+1}\} = sup_{i<\omega}\{(u_1^+)^{L[x^i]}\} = sup_{i<\omega}\{o(N_i, \omega, u_1)\} \tag{2}$$

Just by indiscernibility Equation (1) holds with u_α replacing u_1. Hence:

$$u_{\alpha+1} = sup_{i<\omega}\{(u_\alpha^+)^{L[x^i]}\} = sup_{i<\omega}\{o(M^i, \omega, u_\alpha)\} = sup_{i<\omega}\{o(N_i, \omega, u_\alpha)\} \tag{3}$$

Let $(Q, n) \in \mathbb{M} \cap K$ be a $<_*$ least upper bound for $\{(N_i, \omega)\}_{i<\omega}$. As all the $N_i \in H_{\omega_2}$, by Löwenheim-Skolem we may assume that $(Q, n) \in H_{\omega_2}$ also. Then we claim:

$$u_{\alpha+1} = sup\{o(P, e, u_\alpha) \mid (P, e) <_* (Q, n) \text{ and } P \in K \text{ and } card^K(P) < \omega_2\} \tag{4}$$

since given any such (P, e), we have $(P, e) <_* (M^i, \omega)$ for some i, and since the mouse order of $L[x^i]$ has no holes (Lemma 2.15), $(P, e) =_* (R, e)$ for some $R \in L[x^i]$, and, again by Löwenheim- Skolem, also with $card^{L[x^i]}(R) < \omega_2$. By uniform indiscernibility then $o(P, e, u_\alpha) = o(R, e, u_\alpha) < u_{\alpha+1}$. That $u_{\alpha+1}$ is not greater than the given supremum, is witnessed by the N_i of Equation (3). Hence U is definable in K. **Q.E.D.(Claim 1).**

By standard arguments about Silver indiscernibles, there is a term τ_0 so that $\forall \vec{\gamma} \in [\mu]^n \quad f(\vec{\gamma}) = \tau_0^{L[x^\#]}[d, \vec{\gamma}]$ where d is a fixed finite sequence from μ. (We have taken τ_0 in $L[x^\#]$ rather than in $L[x]$ since then by an argument of Solovay, (*cf.*[8] §8,Lemma C) we shall not need any indiscernibles from $I^{x^\#} \backslash \mu$ to define f.) Let $v_1, \ldots, v_{n+1}$ be $\mu = u_\mu, u_{\mu+1}, \ldots, u_{\mu+n}$. Define $P_i =_{df} \langle N_0, \ldots, N_i \rangle$ for $i < \omega$. Note that as N_i dominates all the mice of $L[x^{i-1}]$ that $(N_{i-1})^\#$ and $(P_{i-1})^\#$ (coded as sets of ordinals) belong to $L[P_i]$.
Claim 2 For any $\xi < v_{n+1}$ there are $i < \omega$, $\vec{d} \in [\mu]^{<\omega}$, and a term τ, so that $\xi = \tau^{L[P_i]}[\vec{d}, v_1, \ldots, v_n]$.
Proof: (This is a standard argument similar to the result for the class of uniform indiscernibles for all reals.) By induction on n; then we assume that $v_n < \xi < v_{n+1}$. By the usual properties of Silver indiscernibles

$$\xi = \sigma_0^{L[P_0]}[\vec{c}, v_1, \vec{v}_1, v_2, \ldots, \vec{v}_{n-1}, v_n, \vec{v}]$$

where $\vec{c}, \vec{v}_j, \vec{v} \in [I^{P_0}]^{<\omega}$ and $\max \vec{c} < v_1 < \min \vec{v}_1 \le \max \vec{v}_1 < v_2 < \cdots < v_n < \xi \le \min \vec{v}$. If $lh(\vec{v}) = k$, then we may assume $\vec{v}$ are the next k cardinals of $L[P_1]$ above ξ, as these are in I^{P_0}. By the inductive hypothesis, there are terms σ_j, $\vec{d}_j \in [\mu]^{<\omega}$, and $i_j < \omega$ for $1 \le j < n$, so that σ_j defines $\vec{v}_j$ using ordinals $\vec{d}_j < v_1$, and $v_1, \ldots, v_j$ in $L[P_{i_j}]$. Putting these facts together for $1 \le j < n$, we see that setting $i = \max\{1, i_1, \ldots, i_{n-1}\}$, $\vec{d} = \langle \vec{c}, \vec{d}_0, \ldots, \vec{d}_{n-1} \rangle$, there is a term τ defining ξ in $L[P_i]$ as in the statement of the Claim. **Q.E.D.(Claim 2)**

Now define $\xi = \tau_0^{L[x^\#]}[d, v_1, \ldots, v_n]$. Then $\xi < v_{n+1}$. Apply the last claim and find $\vec{d}, \tau, i < \omega$ as there. Now set $g(\vec{\gamma}) = \tau^{L[P_i]}[\vec{d}, \vec{\gamma}]$. Then $g \in K \cap L[x^{i+1}]$ and g agrees with f on n-tuples from $I^{x^{i+2}} \cap \mu \setminus \max\{\sup d, \sup \vec{d}\}$, and hence from $C =_{df} U \cap \mu \setminus \max\{\sup d, \sup \vec{d}\}$. Thus $g \sim_C f$. **Q.E.D.**

Remark: By a slight modification the theorem can be seen to hold for $\mu = \omega_2$ also.

5 Extensions

We conclude by discussing some extensions of the previous sections and considering some open questions.

Beyond $O^{\P}$ the nature of universality for weasels changes (see the remark after Def. 1.14). We showed in 3.8 that $K|\omega_2$ was universal to an extent. However the proof of 3.8 did not use the assumption that $card(M) < \omega_2$, and a slight notational modification only yields immediately:

Lemma 5.1 *Assume $\neg O^{\P}$. Then $K|\omega_2$ is universal for mice W with $On \cap W \le \omega_2$, that is, for any such mouse $(W, \omega) \le_* (K|\omega_2, \omega)$.*

Proof: If the conclusion is false then the coiteration of (W, ω) with $(K|\omega_2, \omega)$ has length $\ge \omega_2$. As we are below $O^{\P}$, it is easy to see that there is $\tau < \omega_2$ so that the coiteration of (W_τ, ω) with $(K|\omega_2, \omega)$ uses the same extenders on both sides and has length $\ge \omega_2$. This would contradict 3.8. **Q.E.D.**

Question 1 Are 3.8 and 5.1 true under the weaker assumption that there is no inner model of a Woodin cardinal?

We used the assumption $\neg O^{\P}$ to see that there were no "holes" in the mouse ordering in any inner model (Lemma 2.15).

Question 2 Assume there is no inner model with a Woodin cardinal. Does Lemma 2.15 still hold?

In Lemma 3.3 $\neg O^{\P}$ was used to show that in $L[a^\#]$ there was a proper initial segment of $K^{a^\#}$ that iterated past the weasel K^a. Beyond $O^{\P}$ it is possible to have weasels P, Q with $(P|\alpha, \omega) <_* (Q, \omega)$ and $(P, \omega) >_* (Q|\alpha, \omega)$ for all $\alpha \in On$, whilst comparing (P, Q) results in $P_\infty^{\mathcal{T}}$ a "proper initial segment" of $Q_\infty^{\mathcal{U}}$. It is thus conceivable that no proper initial segment of $K^{a^\#}$ iterates past K^a. Explicitly this means there is no ordinal bound on the extenders needed from $K^{a^\#}$ to show that $K^{a^\#}$ is universal in $L[a^\#]$. As also pointed out by W.Mitchell, this shows that there must be a strong cardinal in $K^{a^\#}$. If no proper initial segment of $K^{a^{\#\#}}$ goes past K^a then there is an inner model with two strong cardinals. Pushing this further yields:

Lemma 5.2 *Suppose $\forall X (X^{\#^\#})$ exists, but there is no inner model with a strong cardinal that is a limit of strong cardinals. Then K is Σ^1_3-correct.*

By $X^{\#^\#}$ we mean the a sharp for an inner model M_X containing X, and which is itself closed under the sharp function.

Question 3 Can there exist reals a so that $\forall \alpha \in On\ K^a$ iterates past $K^{a^\#}|\alpha$?

The first author has shown subsequently the following, using different arguments:

Theorem 5.3 *Suppose there are two measurable cardinals, but that there is no inner model with a Woodin cardinal. Then the K of [16] is Σ^1_3-absolute.*

Question 4 In this theorem the larger of the two measurables is used to construct K whilst the smaller is used to prove absoluteness. Can this lower measurable be eliminated?

Question 5 If a is a Π^1_2 singleton, and we assume only $a^\#$ exists, can we still prove that $a \in K$? As remarked in the introduction, Jensen's argument proving absoluteness over K_{DJ} below a measurable cardinal, assuming $\neg O^\dagger$ and using the Patterns of Indiscernibles type of result gives this, but Magidor's argument does not.

A lower bound for the consistency strength of $\boldsymbol{\delta^1_2} = \omega_2$ remains still at the level of $O^\P$. However we may get a little more using an argument of Hauser and Hjorth[7].

Lemma 5.4 *[7] Suppose there is no inner model with a Woodin cardinal and that ω_1 is inaccessible in K. Then if $N = W||\alpha$ where W is a universal weasel and $\alpha < \omega_1$ is regular in W, then either $(N, \omega) <_* (K|\omega_1, \omega)$ or $\exists \kappa < \omega_1\ K \models$ "κ is $< \omega_1^V$-strong".*

Theorem 5.5 *Suppose $\forall a \in \mathbb{R}(a^\#$ exists) but there is no inner model of a Woodin cardinal. Then*

$$\boldsymbol{\delta^1_2} = \omega_2 \Rightarrow \exists \kappa < \omega_1 (\ K \models \text{“}\kappa \textit{ is } < \omega_1^V\textit{-strong”}).$$

Proof: $\boldsymbol{\delta^1_2} = \omega_2$ already implies that ω_1 is inaccessible in K_x for any real x as in 3.7. Lemma 3.5 shows that $\boldsymbol{\delta^1_2} = \sup\{o(M, \omega) \mid (M, \omega)$ is iterable, $M \in HC\}$, and so our hypothesis implied that for some $(M, \omega) \in HC$, $(M, \omega) >_* (K|\omega_1, \omega)$. Relativizing to K_x where $x \in \mathbb{R}$ coded M, we saw that a countable initial segment N of the universal weasel $W = K^{K_x}$ satisfied $(N, \omega) >_* (K|\omega_1, \omega)$. As ω_1 is inaccessible in K_x it is so in W. We may thus additionally assume $On \cap N$ is regular in W. Hence $(N, \omega) >_* (K|\omega_1, \omega)$ and the lemma cited above now applies. **Q.E.D.**

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