Determinacy and Δ^1_3 - degrees

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Abstract

Assuming that all co-analytic games on integers are determined (or equivalently that all reals have "sharps"), we prove (Theorem 1.3) in ZFC that either $\boldsymbol{\Delta^1_2}$-Determinacy holds, or that there is a real d_0, so that if $D = \{\boldsymbol{d_n}\}$ is a countable collection of Δ^1_3 degrees above that of d_0, then D has a Δ^1_3-minimal upper bound.

1 Introduction

The main theorem of [12] is the following:

Theorem 1.1 *($ZFC + \forall r \in \mathrm{R}$ ($r^{\#}$ exists) $+ \neg 0^{\dagger}$) Every countable set of Δ^1_3 degrees has a minimal upper bound.*

We say that for reals $f, g \in {}^{\omega}\omega$ f *is* Δ^1_3 *in* g, or $f \leq_3 g$, if there are Σ^1_3 relations Φ, Ψ expressible in second order number theory so that $f(n) = m \Leftrightarrow \Phi(n, m, g) \Leftrightarrow \neg\Psi(n, m, g)$. This is a reducibility ordering, and setting $f =_3 g \Leftrightarrow f \leq_3 g$ & $g \leq_3 f$ we have that $=_3$ is an equivalence relation on ${}^{\omega}\omega$. We let $\boldsymbol{f}$ denote the equivalence class $[f]_{=3}$.

In general, when $\leq_r$ is a reducibility ordering (meaning $\leq_r$ is a transitive, reflexive partial ordering, extending $\leq_T$, Turing reducibility, so that if $x \leq_r y$ and $z \leq_r y$ then $x \oplus z \leq_r y$ holds, where $x \oplus z$ is the recursive union of x with y) we shall use $\leq_r$ to denote also the partial ordering amongst r-degrees so induced.

If D is a set of r-degrees, we say that $\boldsymbol{e}$ is an r*-minimal upper bound* for D if

$$\forall \boldsymbol{d} \in D(\boldsymbol{d} <_r \boldsymbol{e}) \wedge \forall \boldsymbol{g}\, (\forall \boldsymbol{d} \in D(\boldsymbol{d} <_r \boldsymbol{g} \wedge \boldsymbol{g} \leq_r \boldsymbol{f}) \rightarrow \boldsymbol{f} =_r \boldsymbol{g})$$

By the expression "$r^{\#}$ exists" we mean that there is a closed and unbounded class of indescernibles, C^r, for $L[r]$, the constructible closure of L with r, and by $O^{\dagger}$, the existence of such a class of indescernibles for $L[\mu]$- an inner model for a measurable cardinal – if such exists. By "$\neg O^{\dagger}$" we mean that such a class does not exist. For information on games see [3]. By contrast with Theorem 1.1 Kechris had earlier shown [4]:

Theorem 1.2 *($ZFC + \boldsymbol{\Delta}^1_2$-Determinacy). If $D = \{\boldsymbol{d}_n\}_{new}$ is a countable collection of Δ^1_3 degrees, that all lie within a single Q-degree, then D has a minimal upper bound.*

At first sight it is tempting to conjecture:

Conjecture 1 ($ZFC + \forall r$ ($r^{\#}$ exists)) Every countable collection of Δ^1_3-degrees has a minimal upper bound.

Whilst Theorem 1.2 lends weight to this assuming $\boldsymbol{\Delta}^1_2$-determinacy, there was a large gap between this assumption and that of Thm 1.1. It is the purpose of this note to close that gap. We prove:

Theorem 1.3 *Assume $ZFC + \forall r$ ($r^{\#}$ exists). Either $\boldsymbol{\Delta}^1_2$-Determinacy holds, or there is a $d_0 \in {}^{\omega}\omega$ so that for every countable set $D = \{d_n\}$ of Δ^1_3-degrees, with $[d_0]_3 \in D$, D has a Δ^1_3-minimal upper bound.*

The real d_0 may be taken as a real so that $d_0^\dagger$ does not exist (if there is such), or such that $\Delta^1_2(d_0)$-determinancy fails, otherwise. (This dichotomy is somewhat artificial, and only arises through our need to use Steel's Σ^1_3-correctness theorem for his core model, which requires the use of two measurable cardinals; it should not be necesary.) The inner model machinery we use is due to Steel [11] building on the fine-structure of iterations trees, due to Mitchell & Steel [8]. We assume the reader is familiar with [8], [11], and [12]. The basic structure of the proof of Theorem 3 is still that of [12]. This note details how [12] may be amended or "read" to show how to relax the extra assumption of [12], and still make the proof of Theorem 1.3 go through. We shall occasionally try to avoid wholescale repetition by assuming the reader has a copy of [12] to refer to.

We first prove fairly directly the following weakening of 1.3.

Theorem 1.4 *Assume there are two measurable cardinals, and that Δ^1_2-Determinacy fails. Then every countable set of Δ^1_3 degrees has a minimal upper bound.*

We indicate at the end how this assumption can be weakened to that of sharps for reals to get 1.3

2

The basic tools of Theorem 1.1 were the Σ^1_3-correctness of the Dodd-Jensen Core Model, K_{DJ}, together with the analysis of Σ^1_3 sets as unions of $\aleph_1$-Borel sets of [13], where the codes of the Borel sets could be taken from reals coding wellorders and mice, from K_{DJ}. Essentially this analysis held because the class of uniform indiscernibles C for reals could be computed in K_{DJ} as

$$\bigcap_{r \subseteq \gamma < \omega_1^V} C^r$$

from sharps for bounded subsets of ω_1^V in K. Thus "$u_n = u_n^{K_{DJ}}$" for $n < \omega$.

Here we shall use:

Theorem 2.1 *(Steel)[11] 7.9 Assume there are two measurable cardinals $\kappa < \Omega$, but that there is no inner model of a Woodin cardinal. Then the core model K of [11] is Σ^1_3-correct.*

We say that a model M is Σ^1_3-correct, if for any $x \in M$, and any non-empty $\Pi^1_2(x)$ set B of reals, then B has a member $y \in M$. It is still unknown whether $u_2 < \omega_2$ follows from the hypothesis of Theorem 2.1, but its proof does show that computing the uniform indiscernibles for bounded subsets of the lower measurable cardinal κ, yields the same class, both in V and the Steel K.

Let $D = \{\boldsymbol{d}_n\}_{n\in\omega}$ be a fixed countable collection of Δ^1_3-degrees. Without loss of generality we assume $n < m \to \boldsymbol{d}_n <_3 \boldsymbol{d}_m$. Let $d_n \in \boldsymbol{d}_n \cap {}^\omega\omega$ be a representative of each class, and let $z = \oplus\langle d_n | n < \omega\rangle$ some recursive coding of the d_n.

We assume for some n that lightface $\Delta^1_2(d_n)$-Determinacy fails. Clearly we may assume that $n = 0$, and in fact we shall take $d_0 = 0$ as the proof will relativize uniformly. Initially we shall work under the assumptions of Thm. 2.1, and at the end of the paper indicate that they may be removed, if we continue to assume the existence of #'s for reals. For z a set of ordinals, we let K^z be the Steel core model relativized to the predicate z, and $(K^z)^c$ the associated preliminary model (see [11] §§1 - 5). All references to "mice" will mean a 1-small, ω-iterable mouse as defined in [8], Defs. 5.1.4& 5.1.5 (unless otherwise stated).

As Δ^1_2-determinacy fails $K^c \models$ "there are no Woodin cardinals" (*cf.* [11] 7.14) and all iteration trees then are *simple* (*cf.* [8] 5.1.2), and furthermore we have a Π^1_2 iterability condition on mice.

Definition 2.2 *Let $f \in {}^\omega\omega$. Let τ be such that*

$$\langle K^f_\tau, \in, f, E^{K^f}\rangle \prec_{\Sigma_1} \langle K^f_{\omega_1}, \in, f, E^{K^f}\rangle.$$

Then τ is K^f-stable. We set $\tau(f)$ to be the least K^f-stable.

Definition 2.3 $\delta^1_3(f) = \sup\{\|X\| : X \in WO \wedge X \leq_3 f\}$.

Lemma 2.4 *If K^f is Σ^1_3-correct, $\tau(f) = \delta^1_3(f)$ for any $f \subseteq \omega$.*

Proof: Just as for δ^1_2, the first stable ordinal and L, using the $\Sigma^1_3(f)$-correctness of K^f. **Q.E.D.**

In [12] we used mainly the fact that below a measurable cardinal the constructible closure $K_{\omega_1}[f]$ had the same domain as $K^f_{\omega_1}$. We see no *a priori* reason for this to be true for larger core models (we replace it with Lemma 2.18(iii) below). However:

Lemma 2.5 *If $\lambda \leq \omega_1^K$ and $y \in {}^\omega\omega$, then $K_\lambda =_{df} \langle J^{E^K}_\lambda, \in, E^{K}{\restriction}\lambda\rangle = (K_\lambda)^{K^y}$.*

Proof: Let $\overline{K} = K^{K^y}$.
Claim $\overline{K}$ is a universal weasel.
Proof: By the Weak Covering Lemma for K^y (see [7]) for a cub class $D \subseteq \Omega$ of cardinals, we shall have

$$\beta \in D \to cf(\beta^{+K^y}) \geq \beta.$$

Similarily building K inside K^y, (note that enough of the measure on Ω survives into K^y for the construction to take place), we appeal to the Weak Covering Lemma for K inside K^y to get the same conclusion on a cub set $D_1 \subseteq \Omega$ of K^y-cardinals

$$\beta \in D_1 \to cf^{K^y}(\beta^{+\overline{K}}) \geq \beta.$$

Consider the comparison of K with $\overline{K}$ with resultant trees $\mathcal{T}, \mathcal{U}$. (Again we follow [8] §7 in our definition of comparison, excepting that we shall always consider comparisons of mice as terminating in a common model). If $\overline{K}$ is not universal then, by a standard argument,([11],§8) there is a cub $E \subseteq \Omega$, with $i < j \in E \to \pi^{\mathcal{T}}_{ij}(\kappa_i) = \kappa_j = j$ (where $\kappa_i = crit(E^{\mathcal{U}}_i)$) with $\pi^{\mathcal{U}}_{ij}{}''\kappa_j \subseteq \kappa_j$, whilst $\pi^{\mathcal{T}}_{ij}{}''(\kappa^+_i)^{M^{\mathcal{T}}_i}$ is cofinal in $(\kappa^+_j)^{M^{\mathcal{T}}_j}$. Letting $i_0 = \min E$, and choosing a regular $j \in E \cap D \cap D_1$, we have $\pi^{\mathcal{U}}_{0,j}(\kappa_j) = \kappa_j$ and $cf(\pi^{\mathcal{U}}_{0,i}((\kappa^+_j)^{\bar{K}})) \geq \kappa_j$ for all $i \leq j$. But the latter is $(\kappa^+_j)^{M^{\mathcal{U}}_j}$ and this has cofinality $(\kappa^+_{i_0})^{M^{\mathcal{T}}_{i_0}} < \kappa_j$! **Q.E.D.**(Claim)

Consequently on neither side of the coiteration is there any truncation on the main branch. This can only mean $K_\lambda = \overline{K}_\lambda$ for any λ less than the first measurable of K and of $\overline{K}$. **Q.E.D.**

Definition 2.6 *For $x \in {}^\omega\omega$ $\Delta^1_3(x) = \{M \mid M$ a mouse, $M \in K^x_{\delta^1_3(x)}\}$.*

We shall use a relativized form of the above argument:

Lemma 2.7 *(i) For $x \in K^y$ $x, y \in {}^\omega\omega$ $\Delta^1_3(x) = (\Delta^1_3(x))^{K^y}$. (ii) If $x \leq_3 y$ then $\Delta^1_3(x) \subseteq \Delta^1_3(y)$*

Proof: (i): 2.5, relativized to K^x, shows that, if $\tau = \omega_1^{K^x}$, then $K^x_\tau = (K^x_\tau)^{K^y}$. For (ii): by the same argument, noting that $x \in K^y$ and that $\delta^1_3(x) \leq \delta^1_3(y)$ (both by the $\Sigma^1_3(y)$-correctness of K^y). The result is immediate. **Q.E.D.**

Corollary 2.8 *If $x =_3 y$ then $\Delta^1_3(x) = \Delta^1_3(y)$.*

Lemma 2.9 *Let $\gamma = \omega_1^{K^f}$, then $K^f_\gamma \models \forall x \exists \alpha [K^f_\alpha \models \exists M \exists \tau (M$ a mouse and $x \in L_\tau[M, f])]$*

Proof: In K^f define $\bar{K}[f]$ (where $\bar{K} = K^{K^f}$). Let $\widetilde{K}^f = (K^f)^{\bar{K}[f]}$. By the same argument as Lemma 2.5 $|K^f_{\omega_1}| = |\widetilde{K}^f_{\omega_1}| \subseteq |\bar{K}[f]_{\omega_1}| = |K^f_{\omega_1}|$ and so the result is immediate. **Q.E.D.**Let $\delta = \sup_n \{\delta^1_3(d_n) : n < \boldsymbol{\omega}\}$, and $\delta_n = \delta^1_3(d_n)$. We set $\Delta^1_3(D) = \cup_n \Delta^1_3(d_n)$.

The following replaces the notion of "λ-K-degree" from [12].

Definition 2.10 *Let λ be a p.r. closed ordinal, M mouse with $\lambda, On \cap M$. A* λ-M-degree *is an equivalence class of reals under the relation "$f \in L_\lambda[M, g]$" which we write as $f \leq_{\lambda, M} g$.*

($L_\lambda[M, g]$ is the usual constructible closure of M with g). Clearly $\leq_{\lambda, M}$ is a reducibility ordering.

Definition 2.11 *Let $f \in {}^\omega\omega$. Let either $\Delta = \Delta^1_3(f), \bar{\delta} = \delta^1_3(f)$ or $\Delta = \Delta^1_3(D), \bar{\delta} = \delta$ be as defined above. Then we set $x \leq_\Delta y$ if $\exists M \in \Delta$ with $x \leq_{\bar{\delta}, M} y$.*

It is not hard to see that $\leq_\Delta$ is also a reducibility ordering (for transitivity one needs to observe only that if $x \leq_\Delta y \wedge y \leq_\Delta z$ as witnessed by mice $M, N \in \Delta^1_3(f)$ (resp. $\Delta^1_3(D)$) then there is a mouse $P \in \Delta^1_3(f)$ (resp. $\Delta^1_3(D)$) constructibly coding them both by level δ). There are thus notions of minimal λ-M, $\Delta^1_3(f)$, and $\Delta^1_3(D)$-degree *etc.* Apart from the first though, they are of limited utility, as $\Delta^1_3(f)$ is too generalised a set of mice. We replace this notion with that of "$\widetilde{\Delta}^1_3(f)$" defined below.

The proof of [12] used a forcing argument derived from Friedman [1] using perfect trees with various notions of pointedness. For any of the above reducibility notions:

Definition 2.12 *A perfect tree $T \subseteq 2^{<\omega}$ is* r-pointed *if $\forall f(f \in T \to T \leq_r f)$.*

Here as elsewhere we shall write $f \in T$ to mean that f codes the characteristic function of a set of sequence numbers coding a branch through T. Throughout this note T and T^* will refer to perfect trees of sequence numbers. The following is an entirely general fact about pointedness:

Lemma 2.13 (Sacks [10] 2.3) *a) If T is r-pointed and $T^* \subseteq T$, T^* perfect, and $T^* \leq_r T$, then $T^* =_r T$, and T^* is r-pointed. b) If T is r-pointed, and $T \leq_r f$, then there is a perfect $T^* \subseteq T$, r-pointed, and such that $T^* =_r f$.*

Let M and λ be as in definition 2.10. Part a) of the next lemma is just an application of the last fact with $\leq_r$ as $\leq_{\lambda,M}$. Part b) is just a variation on the Sacks minimal degree construction performed over $L[M]$.

Lemma 2.14 *a)* (cf. *[Sacks [10] 2.3]) Let T be λ-M-pointed. Let $f \in {}^\omega\omega$. Then there is an λ, M-pointed $T^* \subseteq T$, such that $T^* =_{\lambda,M} (T, f)$ b)(*cf. *[Sacks [9], 1.4 & 3.1]) Let T be perfect, M a mouse with $On \cap M < \lambda$, and suppose $g : \lambda \overset{(1-1)}{\longrightarrow} \omega, \lambda$ p.r. closed. Then there is a perfect subtree T^* of T, such that $T^* \in L_{\lambda+\omega}[M, g]$ and $\forall f \in T^*((f, T)$ is λ-M-minimal over $T)$.*

Definition 2.15 *An active properly small premouse N is* reasonable *if there is an N-cardinal, $\lambda = \lambda_N > \omega$, with H^N_λ closed under the sharp operation.*

Let $S \subseteq \omega \times^\omega \omega$ be a universal Σ^1_3 set; and further require that for any $h \in {}^\omega\omega$, $S^h = \{e|S(e,h)\}$ is a complete $\Sigma^1_3(h)$ subset of ω.

Suppose $S(e,h) \longleftrightarrow \exists g P(e,h,g)$ where P is Π^1_2. For $e \in \omega$ and for N reasonable, let $S^{N,e}_\lambda$ be the Martin-Solovay tree on $\omega^2 \times u^N_\omega$ defined in N whose projection is contained in P. (For definiteness, let us take the definition of this "MS"-tree as that of S_2 in [6] §2.2.) Here we have set $u^N_1 = \lambda_N$, and $\langle u^N_i | i \leq \omega\rangle$ enumerates the first ω members of

$$C^N = \bigcap_{\substack{a \subseteq \gamma < \lambda_N \\ a \in N}} C^a \backslash \lambda_N$$

where C^a is the class of Silver indiscernibles for $L[a]$. This sequence is definable in N, and we use these indiscernibles to construct the tree $S_\lambda^{N,e}$.

Remark: (1) A straightforward Löwenheim-Skolem argument using the full Martin-Solovay tree in K for P constructed on the first ω-uniform indiscernible above the lower measurable $\kappa = u_1^K$, shows that if $S(e,h)$ then there is a countable, reasonable N with some $g \in {}^\omega\omega$ so that $(g,h) \in p[S_{\lambda_N}^{N,e}]$.
(2) If $N \in M$ or $N = M$ are reasonable, and $\exists g(h,g) \in p[S_{\lambda_N}^{N,e}]$, and $\lambda_N \leq \lambda_M$ then $\exists g(h,g) \in p[S^{M,e}]$.
Proof: Clearly $H_{\lambda_N}^N \subseteq H_{\lambda_M}^M$; hence by "stretching" the functions $f \in N$, $f : [u_\omega^N]^k - u_\omega^N$ (each defined by $L[a]$ terms for $a \in H_{\lambda_N}^N$ with $a^\#$ existing in N) to functions $\overline{f} \in M$ $\overline{f} : [u_\omega^M]^k \to u_\omega^M$ we get the desired conclusion.

Definition 2.16 *(i)* $F(e,h) =_{df}$
$\{N \mid N$ *is a* $\leq_*$ *-least reasonable mouse so that* $\exists g S_{\lambda_N}^{N,e}(h,g)$ *is illfounded* $\}$.
(ii) $\widetilde{\Delta}_3^1(h) = \bigcup\{F(e,\bar{h}) \cap \Delta_3^1(h) \mid e \in \omega \wedge \bar{h} \leq_3 h\}$.

(i) above fulfills the role of $F(e,h)$ of [12]. Here $\leq_*$ is the natural mouse ordering: $M <_* N$ if in the comparison process, *via* ω-maximal iteration trees $\mathcal{T},\mathcal{U}$ to a common model $M_\theta^{\mathcal{T}} = M_\theta^{\mathcal{U}}$, then there has been a 'drop' in degree along the main branch $[0,\theta]_U$ or in model, *i.e.* $D^{\mathcal{U}} \cap [0,\theta]_U \neq \emptyset$. The next lemma shows that the mice of $\widetilde{\Delta}_3^1(h)$ are thus the "witnessing mice" for the $\Sigma_3^1(h)$ complete set. For the Dodd-Jensen core model $\bigcup \widetilde{\Delta}_3^1(h) = \bigcup \Delta_3^1(h)$ although there seems no *a priori* reason for this to be true here.

Lemma 2.17 $N \in F(e,h) \cap HC \longleftrightarrow$
$\exists g S_{\lambda_N}^{N,e}(h,g)$ *is illfounded* $\wedge \forall t(t$ *codes an iteration tree* $\mathcal{T}$ *on* N *with reasonable last model* $M = M_\theta^{\mathcal{T}}$ *and with a drop in degree or model along* $[0,\theta]_T \longrightarrow \forall g S_{\lambda_M}^{M,e}(h,g)$ *is wellfounded* . "$N \in F(e,h)$" *is thus a* $\Pi_2^1(N,h)$ *relation.*

Proof: $(\leftarrow)$ If $F(e,h) \neq \emptyset$ but $N \notin F(e,h)$, this is because there is M (which we may assume countable) with $M <_* N$ and g with $S_{\lambda_M}^{M,e}(h,g)$ illfounded. Let (M,N) be compared with resulting trees $\mathcal{U},\mathcal{T}$. Then we see, if $P = M_\theta^{\mathcal{U}}$, the last model on $\mathcal{U}$ with $D^{\mathcal{U}} \cap [0,\theta]_U = \emptyset$, that if $\gamma = i_{0,\theta}^{\mathcal{U}}(\lambda_M)$, $\exists g$"$S_\gamma^{P,e}(h,g)$ is illfounded"; as we may map up the branch, sequence by sequence, to one in $S_\gamma^{P,e}$ using the iteration map $i_{0,\theta}^{\mathcal{U}}$. As $M <_* N$ but $M_\theta^{\mathcal{T}} = M_\theta^{\mathcal{U}}$, there has been a drop in degree or in model, along the main branch $[0,\theta]_T$. This contradicts our assumptions. $(\rightarrow)$ is straightforward.
Q.E.D.

Lemma 2.18 *(i) If* $e \in S^h$ *then* $F(e,h) \cap \Delta_3^1(h) \neq \emptyset$. *(ii)* $\widetilde{\Delta}_3^1(h)$ *is* $\leq_*$ *cofinal in* $\Delta_3^1(h)$.
(iii) $\forall M \in \Delta_3^1(h) \exists N \in \widetilde{\Delta}_3^1(h)(M \in L_{\delta_3^1(h)}[N,h])$. *Hence* $|K_{\delta_3^1(h)}^h| = \bigcup_{N \in \widetilde{\Delta}_3^1(h)} |L_{\delta_3^1(h)}[N,h]|$.
(iv) $\forall M \in \Delta_3^1(h) \exists N \in \widetilde{\Delta}_3^1(h) \forall N' \in \Delta_3^1(h)(N' \geq_* N \to M \in L_{\delta_3^1(h)}[N,h])$.

Proof: (i) follows from the last sentence of Lemma 2.17 since $e \in S^h \longleftrightarrow \exists \alpha K_\alpha^h \models \exists N \in$

$F(e,h) \longleftrightarrow \exists\alpha < \delta^1_3(h) K^h_\alpha \models \exists N \in F(e,h)$.

Suppose for (ii) the given set is not $\leq_*$-cofinal but there is $M_0 \in \Delta^1_3(h)$ such that $\forall e(e \in S^h \to [\forall N \in F(e,h) N \leq_* M_0]$.

Claim "$e \in S^h$" is a $\Delta^1_3(h)$ relation (contradicting the assumption on S^h that it is a complete $\Sigma^1_3(h)$ set).
Proof:
(1) $e \in S^h \longleftrightarrow \exists \mathcal{T} \in K^h_{\delta^1_3(h)}$, a countable iteration tree $\mathcal{T}$ on M_0 with last model $M^{\mathcal{T}}_\infty$ so that "$\exists g S^{M^{\mathcal{T}}_\infty, e}_\lambda(h,g)$ is illfounded ".
Proof: Suppose $e \in S^h$ and let $N \in K^h_\delta$ where $\delta = \delta^1_3(h)$, witness this using (i) above. Let $\mathcal{U}, \mathcal{T}$ be the trees of length θ resulting from the comparison of N, M_0. As the latter are both countable, $\mathcal{U}, \mathcal{T} \in K^h_{\omega_1}$, and hence in K^h_α for some $\alpha < \delta$. As $N \leq_* M_0$ $D^{\mathcal{U}} \cap [0,\theta] = \emptyset$, and so, if $S^{N,e}_\lambda$ is the tree defined in N, then $i^{\mathcal{U}}_{0,\theta}(S^{N,e}_\lambda) = S^{P,e}_\gamma$ where $P = N^{\mathcal{U}}_\theta$ is the last common model on $\mathcal{T}$. If $\gamma = i^{\mathcal{U}}_{0,\theta}(\lambda)$, and if $(g,h,f) \in [S^{N,e}_\lambda]$, we may again map up the branch, sequence by sequence, to one in $S^{P,e}_\gamma$ using the iteration map $i^{\mathcal{U}}_{0,\theta}$. Hence $(g,h) \in p[S^{P,e}_\gamma]$, and as $M^{\mathcal{T}}_\theta = M^{\mathcal{U}}_\theta = P$ we have shown the right hand side. The converse direction is immediate. **Q.E.D.(2)**

By Shoenfield absoluteness, we have that the following is true:
(2) $e \in S^h \longleftrightarrow$
$\exists \alpha\, (V^{Coll(\omega,\alpha)})^{L[M_0,h]} \models$ "$\exists$ a countable iteration tree $\mathcal{T}$ on M_0 so that $L[M^{\mathcal{T}}_\infty, h] \models$ "$e \in S^h$" "

Hence "$e \in S^h$" is computable in $L[M_0,h]$, and we may write $e \in S^h \longleftrightarrow n_0(e) \in (M_0,h)^\#$ for some recursive $n_0 \in {}^\omega\omega$. But the latter sharp is in K^h_δ - as M_0, h are. **Q.E.D.(ii)**

To see that (iii) holds, let $M \in \Delta^1_3(h)$. Let $X \in K^h_\delta$ be a $\Delta^1_3(h)$ code for M with $X \subseteq \omega$. Let $e_0, e_1 \in \omega$ be such that $n \in X \longleftrightarrow R(e_0,n) \longleftrightarrow \neg R(e_1,n)$ where R is a universal $\Sigma^1_3(h)$ subset of $\omega \times \omega$. Let $\alpha < \delta$ be such that $X \in K^h$ and that $\exists \bar{N} \in K^h_\alpha \exists \lambda$ an $\bar{N}$-cardinal, with $K^h_\alpha \models$ "$\langle e_0, n\rangle \in R \longleftrightarrow L[\bar{N},h] \models$ "$\langle e_0,n\rangle \in p[S^R_\lambda]$" ", where S^R_λ is a version of the MS tree for the relation R defined in $\bar{N}$. Let $N \geq_* \bar{N}$ with $N \in F(e,h)$ for some $e \in \omega$. But now argue as at (2) in part (ii) above that
(3): $\langle e_0, n\rangle \in R \longleftrightarrow$
$\exists \beta (V^{Coll(\omega,\beta)})^{L[N,h]} \models$ "$\exists$ a countable iteration tree $\mathcal{T}$ on N so that $L[M^{\mathcal{T}}_\infty, h] \models$ "$\langle e_0,n\rangle \in R$" "
hence X and so M is computable in $L_{\delta^1_3(h)}[N,h]$. For the last sentence of (iii) just note that what we did for M we could have done for any $Y \in |K^h_{\delta^1_3(h)}|$.

For (iv) note that we made no intrinsic use of "$N \in F(e,h)$" in (3): any $N' \geq_* \bar{N}$ would do. Since $N' \in \Delta^1_3(h)$ we can bound the "$\exists\beta$" computation, as there, by $\delta^1_3(h)$. **Q.E.D.**

We define a reducibility ordering to replace that of "$\Delta^1_3(T)$" from [12]:

Definition 2.19 *Set* $f \leq_{\widetilde{\Delta}^1_3(T)} g \Longleftrightarrow \quad \exists \vec{M} = M_0, \ldots, M_k \in \widetilde{\Delta}^1_3(T) \quad f \in L_{\delta^1_3(T)}[\vec{M}, g]$

Again there are corresponding notions of $\widetilde{\Delta}^1_3(T)$-degree, minimality, and perfect $\widetilde{\Delta}^1_3(T)$-pointed trees. Note that the latter notion of pointedness is absolute between V and K^T.

Lemma 2.20 (cf. *[12] Lemma 9) Let T be a perfect tree, $f \in T$ and let $F(e,f) = N$ for some $e \in \omega$, but $f \notin K^T$. Then there are a perfect $T^* \subseteq T$, $T^* \leq_3 T$, and $M \in \Delta^1_3(T)$ such that $\forall f \in T^*(F(e,f) \neq \emptyset \wedge \forall P \in F(e,f)\, P \leq_* M)$. If additionally T is $\widetilde{\Delta}^1_3(T)$-pointed and Δ^1_3-pointed then also $T^* =_3 T$ and T^* is $\widetilde{\Delta}^1_3(T^*)$-pointed and Δ^1_3-pointed.*

Proof: Let $N \in F(e,f)$. Let $\bar{N} \in K^T$ be $\leq_*$-least with $\exists g S^{\bar{N},e}_\lambda(f,g)$ illfounded for some $\bar{N}$-cardinal λ. Then $\bar{N} \geq_* N$. Let $\alpha = On \cap \bar{N}$. Let G be a $Coll(\omega,\alpha)$-generic collapse over $L[\langle T,\bar{N}\rangle]$, with $f \notin L[G,\langle T,\bar{N}\rangle]$. From G define $h \subseteq \omega \times \omega$ coding $\bar{N}$, with $f \notin L[\langle h,T\rangle]$. Thus

$$(1) \qquad \exists h (h \text{ codes a mouse} N_h \wedge \exists f \in T(\exists g S^{N_h,e}_\lambda(f,g) \text{ illfounded } \wedge f \notin L[\langle h,T\rangle]))$$

This is $\Sigma^1_3(T)$. Hence there is such an $h, N_h \in K^T_{\delta^1_3(T)}$, and such an f as in (1) with $f \notin L[\langle h,T\rangle]$. For this h, N_h then set

$$E = \left\{ f \mid f \in T \wedge \exists g S^{N_h,e}_\lambda(f,g) \text{ is illfounded} \right\}$$

Then E is $\Sigma^1_2(\langle h,T\rangle$, and contains elements not in $L[\langle h,T\rangle]$. Hence there is a perfect set (given by a $T^* \subseteq T$) of such f, recursive in $\langle h,T\rangle^\#$. Hence $T^* \leq_3 (h,T) \leq_3 T$. For the last part, if T is Δ^1_3-pointed, $T \leq_3 T^*$. $\widetilde{\Delta}^1_3(T^*)$-pointedness of T^* is then immediate (as is Δ^1_3-pointedness) **Q.E.D.**

Lemma 2.21 *(cf. [F] Lemma 8.) Let T be Δ^1_3-pointed, and $\widetilde{\Delta}^1_3(T)$-pointed, and suppose $f \in \omega^\omega$. Then there is a $\widetilde{\Delta}^1_3(T)$-pointed $T^* \subseteq T$ such that $T^* =_3 (T,f)$. Any T^* satisfying these conditions is Δ^1_3-pointed and $\widetilde{\Delta}^1_3(T^*)$-pointed.*

Proof: By general pointedness arguments, since T is itself $\widetilde{\Delta}^1_3(T)$-pointed, there exists a $\widetilde{\Delta}^1_3(T)$-pointed $T^* \subseteq T$ such that
(1) $T^* =_{\widetilde{\Delta}^1_3(T)} (T,f)$.
Consequently, setting $\bar{\delta} = \delta^1_3(T)$, for some finite sequence of mice $\vec{M} \in \widetilde{\Delta}^1_3(T)$, $T^* \in L_{\bar{\delta}}[\vec{M},(T,f)] \subseteq K^{(T,f)}_{\delta^1_3((T,f))}$, so $T^* \leq_3 (T,f)$. Conversely note that $T \leq_3 T^*$ (since $T \leq_3 g$ where g is the leftmost branch of T^* which is recursive in T^*). As $\widetilde{\Delta}^1_3(T) \subseteq \Delta^1_3(T) \subseteq \Delta^1_3(T^*)$ (1) yields $(T,f) \leq_3 T^*$. As $\widetilde{\Delta}^1_3(T) \subseteq \widetilde{\Delta}^1_3(T^*)$ and $\bar{\delta} \leq \delta^1_3(T^*)$ we have T^* is $\widetilde{\Delta}^1_3(T^*)$-pointed.

As $g \in T^* \to g \in T$ and T is Δ^1_3-pointed, $T \leq_3 g$, and then $T \in K^g_{\delta^1_3(g)}$. Thus $\widetilde{\Delta}^1_3(T) \subseteq \Delta^1_3(g)$. As T^* is $\widetilde{\Delta}^1_3(T)$-pointed, for some mice $\vec{M} \in \widetilde{\Delta}^1_3(T)$ $T^* \in L_\delta[\vec{M},g]$. But the latter is contained in $|K^g_{\delta^1_3(g)}|$, by the above. Hence $T^* \leq_3 g$ and so is Δ^1_3-pointed. **Q.E.D.**

Lemma 2.22 *Suppose additionally in Lemma 2.14 b), that T is $\widetilde{\Delta}^1_3(T)$-pointed and Δ^1_3-pointed, and that $\omega < \lambda < \delta^1_3(T)$ is p.r. closed. Then, if $M \in \Delta^1_3(T)$ with $On \cap M < \lambda$, the T^* of the conclusion of b) can be taken to be $\widetilde{\Delta}^1_3(T^*)$ pointed and with $T^* =_3 T$ (and so Δ^1_3-pointed).*

Proof: Set $\gamma = \delta^1_3(T)$. $K^T_\gamma \models \forall \alpha\, \overline{\overline{\alpha}} = \omega$. As $\lambda < \gamma$ pick $g \in K^T_\gamma, g : \lambda \overset{(1-1)}{\longrightarrow} \omega$. Find $T^* \in L_{\lambda+\omega}[M,g]$ as in 2.14 b) with the required λ-M-minimality. Then $T^* \in K^T_{\delta^1_3(T)}$, so $T^* \leq_3 T$.

As $T^* \subseteq T$, T^* is Δ^1_3-pointed. As f, the leftmost branch of T^* is recursive in T^*, we have $T \leq_3 T^*$, so $T =_3 T^*$. As T is $\widetilde{\Delta}^1_3(T)$-pointed, if $f \in T^*$, then $T \in L_\gamma[\vec{M}, f]$ for some $\vec{M} \in \widetilde{\Delta}^1_3(T)$. As $M, g \in K^T_\gamma$, by 2.18 (iii), there is $\vec{N} \in \widetilde{\Delta}^1_3(T)$ with $(M, g) \in L_\gamma[\vec{N}, T]$. As $T^* \in L_\gamma[M, g]$ we have $T^* \in L_\gamma[\vec{M}, \vec{N}, f]$ and so T^* is $\widetilde{\Delta}^1_3(T) = \widetilde{\Delta}^1_3(T^*)$- pointed. **Q.E.D.**

The following lemmas are used to control the growth of $\Delta^1_3(f)$ in our construction. They are used just as [1] Lemmas 2 & 3 are.

Lemma 2.23 *Suppose a) f is $\Delta^1_3(D)$-minimal over D, b) $\Delta^1_3(D) \subseteq \Delta^1_3(f)$, and c) $\forall M \in \widetilde{\Delta}^1_3(f) \exists N \in \Delta^1_3(D)$ $(M \leq_* N)$. Then f is Δ^1_3-minimal over D.*

Proof: By b) clearly $\forall n(d_n \leq_3 f)$. And by a) $\forall n(f \not\leq_3 d_n)$ (for suppose $f \in M = K^{d_n}_\alpha$ for some $\alpha < \delta^1_3(d_n)$; then f is not $\Delta^1_3(D)$-minimal over D!) Now suppose $\forall n(d_n <_3 h) \wedge h \leq_3 f$. Then $\Delta^1_3(D) \subseteq \Delta^1_3(h) \subseteq \Delta^1_3(f)$. Then, setting $\Delta = \Delta^1_3(D)$:
(1) $\forall n(d_n \leq_\Delta h)$.
But $h \leq_\Delta d_n \rightarrow \exists k \exists M \in \Delta^1_3(d_k) \exists \alpha < \delta^1_3(d_k)$ $h \in L_\alpha[M, d_n]$. Letting $m = \max\{n, k\}$ $h \in L_{\delta^1_3(d_m)}[M, d_m]$. Thus $h \leq_3 d_m$, contradicting our supposition. Hence:
(2) $\forall n(d_n <_\Delta h)$.
(3) $h \leq_\Delta f$.
$h \in K^f_{\delta^1_3(f)}$ implies by Lemma 2.18(iii) $\exists M \in \widetilde{\Delta^1_3}(f) \exists \alpha < \delta^1_3(f)$ $h \in L_\alpha[M, f]$. By b), c) and 2.18(iv) we can pick $N \geq_* M$, $N \in \Delta \subseteq \Delta^1_3(f) \wedge h \in L_{\delta^1_3(f)}[n, f]$. Hence (3) holds. By a) then, $f \leq_\Delta h$. So $f \in L_\alpha[M, h]$ some $M \in \Delta \subseteq \Delta^1_3(h)$ and $\alpha < \delta^1_3(h)$. Hence $f \in K^h_{\delta^1_3(h)}$. Thus $f \leq_3 h$ as required. **Q.E.D.**

For the next lemma as we have $\Delta^1_3(d_n) \subseteq \Delta^1_3(d_{n+1}) \subseteq \Delta^1_3(D)$, we assume we have $\langle M_n \mid n < \omega \rangle$ an enumeration of $\Delta^1_3(D)$ in such a way that $\forall i, j < \omega \exists m > i, j$ $M_i, M_j \in L_\lambda[M_m]$ where λ is a p.r. closed ordinal $> On \cap M_m$. So let $\langle \lambda_m \mid m < \omega \rangle$ be an ascending sequence of p.r. closed ordinals with $\lambda_m > On \cap M_m$, so that λ_m witnesses this, and with $\sup_m \lambda_m = \delta$. By thinning out the $\langle d_n \mid n < \omega \rangle$ sequence, we may assume that for any n $\lambda_n < \delta^1_3(d_n)$.

Lemma 2.24 *Suppose a) $\forall k$ (f, d_k) is λ_k-M_k-minimal over d_k; b) $\Delta^1_3(D) \subseteq \Delta^1_3(f)$; c) of 2.23 holds; and d) $\forall n$ $(d_n <_3 f)$. Then f is Δ^1_3-minimal over D.*

Proof: Set $\Delta = \Delta^1_3(D)$. First note that $\forall n(d_n <_\Delta f)$ (by b) & d), and using the argument of (2) of Lemma 2.23). We show that f is Δ-minimal, and then the result follows by 2.23. So suppose
e) $\forall n(d_n <_\Delta h)$ and
f) $h \leq_\Delta f$.
By f), for some k
(1) $h \in L_{\lambda_k}[M_k, f]$ (using our presumed properties on M_m & λ_m).
Similarly e) shows: (2) $d_k \in L_{\lambda_n}[M_n, h]$ for some $k \leq n < \omega$.
Now consider (h, d_k). $d_k \in L_{\lambda_k}[M_k, (h, d_k)]$ whilst $(h, d_k) \notin L_{\lambda_k}[M_k, d_k]$ by e). By (1) $(h, d_k) \in L_{\lambda_k}[M_k, (f, d_k)]$. By λ_k-M_k-minimality of (f, d_k) over d_k, we conclude $(f, d_k) \in L_{\lambda_k}[M_k, (h, d_k)]$.

Using (2) we have $f \in L_{\lambda_n+\lambda_k}[M_n, M_k, h]$. For some sufficiently large $m > n, k$, $f \in L_{\lambda_m}[M_m, h]$. That is, $f \leq_\Delta h$. Hence f is Δ-minimal as required. **Q.E.D.**

We now have set up all the machinery, to run the main argument of [12] Lemma 13, keeping roughly to the same notations.

We state this as follows:

Lemma 2.25 *Let D, $\{d_n\}$, $\delta, z, \lambda_n, M_n$ be as above. There is a perfect set $T_0 \in K^z$ so that a) $\forall f \in T_0 \forall n(d_n \leq_3 f)$; b) $\forall f \in T_0 (f \notin \bigcup_n K^{d_n} \to f$ is Δ^1_3-minimal over $D \wedge \forall M \in \widetilde{\Delta}^1_3(f) \exists N \in \Delta^1_3(D)\, M \leq_* N)$.*

Proof: We only sketch the construction which takes place in K^z, defining a binary system of Δ^1_3-pointed trees $\langle T_s | s \in 2^{<\omega}\rangle$. Let e_i be an enumeration of ω in which every integer ocurs infinitely often. Let $T_\emptyset = 2^{<\omega}$. Assume T_s has been defined $\forall s \in 2^{<\omega}$ $(lh(s) \leq i)$ so that
a) $lh(s) = j \leq i \to T_s =_3 d_j$
b) T_s is $\widetilde{\Delta}^1_3(T_s)$-pointed and Δ^1_3-pointed.
For $lh(s) = i$, define disjoint T_{sa0}, T_{sa1} disjoint subtrees of T_s according to the following recipe:
1) Split T_s into two disjoint subtrees $T^*_s =_3 T^{**}_s =_3 d_{i+1}$, perforce both $\widetilde{\Delta}^1_3(T_s)$-pointed,
2) Then find $T^o_s \subseteq T^*_s$, T^o_s $\widetilde{\Delta}^1_3(T^o_s)$-pointed , with $T^o_s =_3 T^*_s =_3 d_{i+1}$ and so that

$$\forall f \in T^o_s((f, T^o_s) \text{ is } \lambda_{i+1}\text{-}M_{i+1}\text{-minimal over } T^*_s)$$

Define T^{oo}_s entirely similarly using T^{**}_s, replacing * with ** and ^{o} with ^{oo} throughout.
3) Find $T_{sa0} =_3 T^o_s =_3 d_{i+1} =_3 T^{oo}_s =_3 T_{sa1}$, T_{saj} $\widetilde{\Delta}^1_3(T_{saj})$-pointed, so that for some $N_{saj} \in \widetilde{\Delta}^1_3(T_s)$ $\forall f \in T_{sa0}(\forall M \in F(e_i, f), M \leq_* N)$ if such a T_{sa0} exists. Otherwise set $T_{sa0} = T^0_s$. Similarly define T_{sa1}. This induction can take place in any initial segment of K^z, which is a model of ZFC^-, containing $z, \{K^{d_n}_{\delta^1_3(d_n)}\}$ (and so the $\widetilde{\Delta}^1_3(T)(d_n)$ etc.)

The perfect tree T_0 of the lemma is that arising from the fusion of the $\langle T_s | s \in 2^{<\omega}\rangle$, and so $[T_0] = \{f \mid \bigcup_{g \in 2^\omega} \bigcap_{N \in \omega} f \in [T_{g^\frown n}\}$. The argument that T_0 is properly defined follows that of [12] with minor changes (replacing $K[d_n]$ with K^{d_n} for example, and replacing $\Delta^1_3(T)$ by $\widetilde{\Delta}^1_3(T)$ throughout). By asking for $\widetilde{\Delta}^1_3(T_s)$ pointed trees at each stage we have ensured the construction is absolute between K^z and V, and one may show by induction on $lh(s)$, using 2.20-2.22, that that the trees are in fact Δ^1_3-pointed in V. We have ensured $\forall f \in T_0 \forall i(lh(s) = i \to ((f, T^o_s)$ is λ_{i+1}-M_{i+1}-minimal over d_{i+1} (a notion absolute between K^z and V) to fulfill requirement a) of Lemma 2.24. As clearly $d_n \leq_3 f$ for any $f \in T_0$ (by Δ^1_3-pointedness of T_s where $f \in T_s \wedge lh(s) = n$), we have requirement b) of 2.24 (and a) of the current lemma). Now if $f \in T_0 \wedge f \notin \bigcup_n K^{d_n}$, by Lemma 2.20, if $F(e_i, f) \neq \emptyset$ then $\forall M \in F(e_i, f)\, M \leq_* N$ for some $N \in \Delta^1_3(D))$. We have that f, D satisfy 2.23 c) and the result follows by that lemma. **Q.E.D.**

The following theorem analogous to [12] Thm 14 is proven similarly *mutatis mutandis.*

Theorem 2.26 *(i) For every $f \in T_0$, f is an upper bound for $\{d_n\}_{n\in\omega}$; if additionally $f \in T_0 \backslash \bigcup_{i\in\omega} K^{d_i}$ then f is a minimal upper bound for $\{d_n\}_{n\in\omega}$.*

(ii) There exists a minimal upper bound of $\{d_n\}_{n\in\omega}$.
(iii) There is a least upper bound of $\{d_n\}_{n\in\omega}$ *iff* $\exists i_0 \in \omega(\mathrm{R} \subseteq K^{d_{i_0}})$.

We remark now on how to remove the assumption that there are two measurable cardinals $\kappa < \Omega$. The essential ingredient of Lemma 2.25 is to have a ZFC^- model M, containing z which is Σ^1_3-correct (and so contains all $\Delta^1_3(d_n)$ - thinking of this as the mice coded by $\Delta^1_3(d_n)$ subsets of ω), and so has some ordinal height $\theta > \sup_n\{\delta^1_3(d_n)\} = \delta)$. Given then an enumeration of $\Delta^1_3(D) = \bigcup_n \Delta^1_3(d_n)$, and a sequence $\lambda_n < \delta^1_3(d_n)$ with $\sup \lambda_n = \delta$, satisfying the requirement before Lemma 2.24 (which we can define over M), we define a sequence of trees $\langle T_s \rangle$ as above by induction in M. We establish the existence of such a suitable M by the following form of argument (due to Woodin? - this is the kind of argument that Woodin &Hauser uses to lift the theorem of [13], on all Π^1_3 sets of reals containing Π^1_3-singletons being equivalent with all reals residing in the Dodd-Jensen K_{DJ} to the context of the Steel Core Model. They prove the following theorem: (we should like to thank Kai Hauser for explaining this argument to us.)

Theorem 2.27 *(Hauser-Woodin) [2] ($ZFC + \forall r \subseteq \omega\ r^\#$ exists.) Suppose every non-empty $\Pi^1_3(x)$ set contains a $\Pi^1_3(x)$ singleton, for every real x. Then either (1) $\boldsymbol{\Delta^1_2}$-Determinacy holds, or (2) there exists a real z with $\mathrm{R} \subseteq K^z$.*

We may assume $\forall y\, y^\dagger$ exists (otherwise, setting $d_0 = y$ satisfying $\neg y^\dagger$ we conclude Thm 1.3 by relativizing the argument of 1.1 using the model $K_{DJ}[y]$ as is done in [12].) But then, we can consider a (or any) canonical inner model arising from $y^\dagger$ with a measure μ on some $\kappa > \omega_1$ containing y, $L[\mu, y]$ say. Using as Ω any of the upper indiscernibles from $y^\dagger$ and, as in [11] 7.7, construct a Σ^1_3-correct model $P = K^z$ in $L[\mu, y]$. Let $\tau = \omega_1^P$, and let $M(y) = P|\tau$. Note that for any $y' \geq_T y$ a trivial comparison argument shows $M(y)$ an initial segment of $M(y')$. Let M be the union over all such $y \geq_T z$. The following claim shows that M has sufficient properties for the induction to go through.
Claim (i) $z \in M \models ZFC^-$; (ii) M is Σ^1_3-correct.
For (i), clearly M is of the form $\langle J^{E^z}_\theta, \in, E^z \rangle$ for some $\theta \leq \omega_1$. So assume $\theta < \omega_1$ and that ZFC^- fails. Let w code M and place ourselves in a $L[\mu, w]$. Again a trivial comparison shows us that M is a proper initial segment of $K^z_{\omega_1^{K^z}}$, but M is the union of such! For (ii), if $t \in J^{E^z}_\alpha$ (the latter coded by some real y say) and if B is a non-empty set of $\Pi^1_2(t)$ reals, let $s \in B$, and place ourselves in $L[\mu, s \oplus y]$: $t \in J^{E^z}_\alpha$ is an initial segment of K^z and the latter is Σ^1_3-correct. Hence there must be an $s' \in B$ with $s' \in K^z$. Hence $s' \in M$ as required. **Q.E.D.**(Theorem 1.3)

Finally we remark that by reordering the induction of 2.25 we may use the analogue of the predicate $\varphi(x)$ of [13], and the methods here, also to prove 2.27.

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