**Archive for Mathematical Logic manuscript No.**
(will be inserted by the editor)

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# On unfoldable cardinals, $\omega$-closed cardinals, and the beginning of the inner model hierarchy

**Abstract.** Let $\kappa$ be a cardinal, and let $H_\kappa$ be the class of sets of hereditary cardinality less than $\kappa$; let $\tau(\kappa) > \kappa$ be the height of the smallest transitive admissible set containing every element of $\{\kappa\} \cup H_\kappa$. We show that a $ZFC$-definable notion of long unfoldability, a generalisation of weak compactness, implies in the core model $K$, that the mouse order restricted to $H_\kappa$ is as long as $\tau$. (It is known that some weak large cardinal property is necessary for the latter to hold.) In other terms we delimit its strength as follows:

**Theorem** $Con(ZFC + \omega^2\text{-}\Pi^1_1\text{-Determinacy}) \Rightarrow$
$\Rightarrow Con(ZFC + V = K + \exists$ a long unfoldable cardinal $\Rightarrow$
$\Rightarrow Con(ZFC + \forall C(X^\#$ exists) + "$\forall D \subseteq \omega_1$ ($D$ is universally Baire $\Leftrightarrow \exists r \in \mathbb{R}(D \in L(r)))$", and this is set-generically absolute).

We isolate a notion of $\omega$-closed cardinal which is weaker than an $\omega_1$-Erdős cardinal, and show that this bounds the first long unfoldable:

**Theorem** Let $\kappa$ be $\omega$-closed. Then there is a long unfoldable $\lambda < \kappa$.

## 1. Introduction

The study of *inner models* of $ZFC$ and the natural hierarchy of the canonical inner models $L, L[O^\#], L[O^{\#\#}], \ldots, L[\mu], L[O^\dagger], \ldots, L[\mathbf{E}], \ldots$ generated by embeddings is a central theme of contemporary set theory. The definitions of the first few models in this series are indeed now venerable. The phenomena of the effect of large cardinals on, *e.g.*, the constructible power set of $\omega_1$ was demonstrated initially through Silver's and Rowbottom's results on *partition properties* of cardinals. Some authors (*cf.* Baumgartner and Galvin [**?**]) tried reduced even further

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The author wishes to gratefully acknowledge support from Nato Grant PST.CLG 975324.

*Key words or phrases:* large cardinals–constructibility–inner models–descriptive set theory

*Mathematics Subject Classification (2000):* 03E45,03E15,03E55,03E60

the homogeneity properties required of a partition cardinal to obtain, for example sharps for reals.

Although the modern perspective is that the generating principle for such models is usually the existence of some small iterable structure, such partition properties are nevertheless useful as benchmarks for obtaining consistency results of other principles. (We are thinking here of examples such as those concerning weak Chang's Conjecture, due to Donder and Koepke [3], and further results of Donder and Lewinsky [2] on flat filters and the like.) There are also phenomena of a descriptive set theoretic nature that seem to bracket, and in turn be bracketed by, the generating iterable structures (the so-called "mice") of these inner models. These phenomena, (*e.g.* such as stated as Fact 3.4 below using results from [10], and the consistency strength of a certain level in the difference hierarchy on $\Pi^1_1$, [12]) although of consistency strength greater than that of "For every set $X$, $X$ has a sharp", are nevertheless rather weak and seem to fall below all known "(weak) large cardinal" properties. A useful vantage point in this study appears to be the stage where the following ordinals coincide: (i) the first uniform indiscernible for bounded subsets of some uncountable cardinal $\kappa$; (ii) the height of the least admissible set containing $\kappa$ and all sets of herditary cardinality smaller than $\kappa$; (iii) the length of the mouse order for mice of size less than $\kappa$; (in fact the equality of the first two implies their joint equality with the third). It seems apparent that there is a wealth of detail below say the first measurable cardinal as these canonical inner or "core" models develop, and it is interesting to watch the interplay between these phenomena as the hierarchy unfolds.

It is not the purpose of this article to survey these results in any detail but to contribute further to them, by ascertaining how a small large cardinal definition due to Villaveces [9] fits into this scheme. One can give a description of the strength of the notion in terms of small large cardinals (that is Theorem 1.3 below). Once one has the cardinal notion inside a canonical core model, one can use the hierarchical structure of the model (that is the mouse ordering) to show that in this situation the phenomenon is indeed weak: it seems to require descriptive set theoretic principles alone to find bounds for the consistency strength (this is Theorems 1.4 and 1.6, combined as the first Theorem of the abstract.)

The notion of an *$\omega$-closed cardinal* is introduced as follows.

**Definition 1.1.** *(I) A structure $\mathcal{A} = \langle A, \in, F\rangle$ is called an $\omega$*-closed structure *if it satisfies the following clauses for some $\kappa$:*
*(i) $\langle A, \in\rangle$ is a transitive $ZFC^-$-model, with $H^{\mathcal{A}}_\kappa \in A$;*
*(ii) $\mathcal{A}$ is amenable and $\mathcal{A} \models$ "$F$ is a normal measure on $\kappa$".*
*(iii) $F \subseteq A$ is an $\omega$-closed filter on $\mathcal{P}(\kappa)_A$, that is if $\langle X_n \mid n < \omega\rangle$ is a sequence of sets in $F \cap A$ then $\cap_n X_n \neq \emptyset$.*
*(II) Let $\kappa$ be a cardinal. Then $\kappa$ is $\omega$*-closed *if there is a $\omega$-closed structure $\mathcal{A}$ with $H_\kappa = H^{\mathcal{A}}_\kappa$.*

This is admittedly a rather technical definition, but inner model theorists will recognise this as deriving from the inductive definition of the core model: an $\omega$-closed cardinal occurs, when we add a filter $F$ onto the extender sequence $E$ we have built, in the inner model $L[E, F]$ the critical point of $F$ is not collapsed and $F$ is an $\omega$-closed filter.

We shall show:

**Theorem 1.1.** *Let $\kappa$ be $\omega_1$-Erdős. Then there is a stationary set of $\omega$-closed cardinals $\gamma < \kappa$.*

(We shall recall the definition of $\omega_1$-Erdős at 2.2 below.)

Villaveces [9] defines the following notion of *unfoldability* of a cardinal. We shall investigate this by means of $\omega$-closure.

**Definition 1.2.** *(i) Let $\kappa$ be inaccessible, and $\lambda > \kappa$.*
*$\kappa$ is $\lambda$*-unfoldable $\Leftrightarrow \forall M(M \textit{ transitive}, |M| > \kappa,\ \kappa \in M \vDash ZF^- \to$
$\to \exists j, N(N \textit{ transitive}, \lambda \in N,\ j : M \to N, \textit{crit}\ (j) = \kappa\ \wedge\ j(\kappa) \geq \lambda))$.

*(ii) $\kappa$ is* unfoldable $\Leftrightarrow$ *it is $\lambda$-unfoldable for all $\lambda$.*

Actually [9] 's definition is stated in terms of there being elementary chains of extensions of $(V_\kappa, \in S)$ for any $S \subseteq \kappa$, but this is easily seen to be equivalent to the above.

Clearly this is a generalisation of weak compactness. Villaveces (see [9]) shows the following: $\kappa$'s unfoldability relativises to $L$; if $\lambda$ is subtle then $V_\lambda \vDash$ "$\exists$ unfoldable cardinals"; and in $L$ unfoldables are $\Pi^n_m$ indescribable for any $m, n < \omega$. More particularly for us here:

**Theorem 1.2.** *(Villaveces [9]) If $\kappa$ is weakly compact and $\kappa \to (\omega_1)^{<\omega}_2$, then the first unfoldable is less than $\kappa$.*

We recall that the first cardinal that satisfies $\kappa \to (\omega_1)^{<\omega}_2$ is $\omega_1$-Erdős , but is not weakly compact.

[9] also gives a definition of "$On$-unfoldable", which is formalisable in Gödel-Bernays (but not ZFC), and essentially is the following: $\kappa$ is *$On$-unfoldable*, if for any $S \subseteq \kappa$, $\langle V_\kappa, \in, S\rangle$ has an $On$-length chain of elementary extensions. He shows that if $\lambda$ is ineffable then $L_\lambda$ satisifes the second order formula that implies there are "$On$-unfoldables". $L$ itself may contain cardinals that are $On$-unfoldable (for example if $0^{\#}$ exists, every Silver indescernible is $On$-unfoldable) but, as already mentioned, this can only be expressed in Gödel-Bernays.

As 1.2 involves cardinals less than the first weakly compact uncountable partition cardinal satisfying $\kappa \to (\omega_1)^{<\omega}_2$, and the latter relativises to the core model $K$, the definition of $On$-unfoldability in the core model invites comparison with the mouse order and this was the initial motivation for this paper. The definition appeared not without independent utility however: for example D.Asperó had shown that $ZFC+$ Bounded Martin's Maximum $+\exists$ an $\omega_1$-Erdős cardinal $\vdash 2^{\aleph_0} = \aleph_2$. The use of $\omega_1$-Erdős can be replaced by $\omega$-closed, as this notion also produces sufficiently long sequences of good indiscernibles. (Subsequently Todorčević ([8]) has removed this hypothesis altogether.)

Villaveces calls "$On$-unfoldable" long unfoldable, but we shall use that term for a definition that is formalisable in ZFC, and hence somewhat stricter. We ask that there is a class $t = \{\langle M, T, \lambda\rangle \mid \varphi(M, T, \lambda)\}$ for some formula $\Sigma_n$

$\varphi(v_0, v_1, v_2)$ (possibly with other set parameters) in the language of set theory $\mathcal{L}_{\{\dot\in,\dot=\}}$ that gives the elements of a chain of models under elementary extension.

**Definition 1.3.** *$\kappa$ is* long unfoldable *if $\forall S \subseteq \kappa$ there is a class given by a fixed $\Sigma_n$ formula $\varphi(v_0, v_1, v_2)$ (possibly containing further set parameters) so that*

*(i) $\varphi(M, \overline{S}, \lambda) \Rightarrow \overline{S} \subseteq \lambda$, $On \cap M = \lambda$, $M$ is a transitive set and $\langle V_\kappa, \in S\rangle \prec \langle M, \in, \overline{S}\rangle$.*
*(ii) $\forall \lambda, \lambda'\ \varphi(M_0, S_0, \lambda) \wedge \varphi(M_1, S_1, \lambda')\ \wedge\ \lambda < \lambda' \rightarrow \langle M_0, \in, S_0\rangle \prec \langle M_1, \in, S_1\rangle$.*
*(iii) There are arbitrary large $\lambda$ so that $\exists M, \bar{S}\ \varphi(M, \bar{S}, \lambda)$.*

$L$ cannot contain any long unfoldables according to this definition, as otherwise satisfaction for $L$ would be $L$-definable. More generally:

**Fact 1.1.** *Suppose $\kappa$ is long unfoldable, taking $S = \emptyset$, and we take a formula $\varphi$ defining a class as above. Suppose that $V_\kappa \prec M_\lambda \prec M_{\lambda'} \prec \ldots$ is an elementary chain through $On$. Then for all sufficiently large $\lambda$ $M_\lambda \neq V_\lambda$.*

*(Otherwise we should have $V_\kappa \prec V_\lambda \prec \cdots \prec V$, and thus be able to define a satisfaction relation for $V$ which is impossible.)*

*Example* If $L[\mu]$ is the canonical inner model for a single measurable cardinal and if $\kappa$ is the measurable cardinal, then $\kappa$ is long unfoldable using iterated ultrapowers of the structure $\langle H_{\kappa^+}, \in, \mu\rangle$: For any $S \subseteq \kappa$, let $\varphi(R_i, S_i, \kappa_i)$ express the following (in the parameters $S, \mu, H_{\kappa^+}$): "Set $H_0 = H_{\kappa^+}$, $\pi_{0,i} : H_0 \longrightarrow H_i$ with $\langle H_i, \in, \mu_i\rangle$ the $i$'th iterated ultrapower of $H_0$ by $\mu$, with ultrapower maps $\pi_{i,j}$, and let $\kappa_i$ be the critical point of $\mu_i$ and $R_i = (V_{\kappa_i})_{H_{\kappa_i}}$; finally set $S_i = \pi_{0,i}(S)$".

As the example shows, in $L[\mu]$ there will clearly be (stationary many) unfoldable cardinals below the measurable cardinal. This is the canonical example to keep in mind in the sequel.

We improve Villaveces' theorem 1.2 on the least unfoldable to:

**Theorem 1.3.** *If $\kappa$ is $\omega$-closed then the first long unfoldable is less than $\kappa$.*

We thus gain doubly: by obtaining the result for long unfoldables, as well as weakening the large cardinal assumption.

We investigate the consistency strength of long unfoldables a little more closely. By Theorem 1.3 we see that it is less than that of an $\omega_1$-Erdős cardinal. However such cardinals fail to exist in $L$. It is thus instructive to identify some inner core model between $L$ and $L[\mu]$ in which such a cardinal can exist. We calibrate this strength, accordingly between two descriptive set theoretic principles, as traditional weak large cardinals are a little thin on the ground in this region. To wit:

**Theorem 1.4.** $ZFC + \omega^2\Pi^1_1$ *- Determinacy* $\vdash$
*"There is an inner model of: $ZFC + \exists$ a Mahlo class of long unfoldables"*

**Remark:** The existence of an $\omega$-closed cardinal suffices to prove (boldface) $\omega^2\boldsymbol{\Pi^1_1}$-Determinacy. Here the $\omega^2\Pi^1_1$ denotes the level $\omega^2$ of the Hausdorff difference hierarchy on $\Pi^1_1$. In fact the weaker hypothesis of an "almost $< \omega_1$-Erdős " suffices (Theorem 1.15 of [12].) By "Mahlo" in the conclusion, we mean "definably Mahlo", that is, stationary with respect to definable closed and unbounded classes.

**Theorem 1.5.** $ZFC + V = K$ + *"$\exists$ a long unfoldable"* $\vdash$
$\forall D \subseteq \omega_1$ *($D$ is universally Baire* $\Leftrightarrow \exists r \subseteq \omega\ D \in L[r]$).

The hypothesis of the last theorem proves that it is consistent that the stated characterization of universally Baire subsets of $\omega_1$, is set-generically absolute:

**Theorem 1.6.** $ZFC + V = K+$ *"$\exists$ a long unfoldable"* $\vdash$
*"There is an inner model of:* $ZFC + \forall X(X \subseteq On \longrightarrow X^{\#}$ *exists)* +
*"$\forall D \subseteq \omega_1$ ($D$ is universally Baire $\Leftrightarrow \exists r \in \mathbb{R}(D \in L(r))$)",*
*and this is set-generically absolute."*

Universally Baire subsets of $\omega_1$ are defined and discussed in [5] where it is shown that this characterisation follows from the existence of a measurable cardinal. In [10] this is reduced to an "almost $< \omega_1$-Erdös" cardinal (we shall not use this definition here and so refer the reader to this paper for a definition). Thus, by virtue of 1.4 and the remark following it, 1.5 represents an improved upper bound over [10] for this characterisation to be provable. In fact in $K$ the characterisation holds if there is a cardinal $\kappa$ so that the mouse order, $<^*$, restricted to $H_\kappa$ stretches out as far as the height of the least admissible set conatining $\{\kappa\} \cup H_\kappa$. It is known that $\forall X(X \subseteq On \longrightarrow X^{\#}$ exists) alone is insufficient.

Theorems 1.5 and 1.6 are actually consequences of what we shall actually prove:

**Theorem 1.7.** $ZFC + V = K + \kappa$ *long unfoldable* $\vdash$

*(i) $\kappa$ is admissibly measurable*
*(ii) There is an inner model of: $ZFC+$ "$\exists$ a proper class of admissibly measurable cardinals".*

(Admissibly measurable is defined at Definition 3.5 below) In [10] it is shown that the conclusions of Theorems 1.5 and 1.6 follow from those of Theorem 1.7 (i) and (ii).

We close this introduction by making a few observations, and mentioning some theorems about the general notion of unfoldability before proceeding to the proofs of the theorems above.

Like weak compacts no large cardinal strength is needed to get the failure of the GCH at an unfoldable:

**Theorem 1.8.** *(Hamkins)[6]* $Con(ZFC + \exists$*an unfoldable* $) \Rightarrow$
$\Rightarrow Con(ZFC + \exists\kappa$ *unfoldable* $+ 2^\kappa > \kappa^+)$

Villaveces asked in [9] whether it is possible, over a $GCH$ model say, to destroy the unfoldability of a cardinal but preserve its weak compactness. We were able to do this by some Jensen coding methods. Thus:

**Theorem 1.9.** *Let $M \models GCH + \kappa$ unfoldable. Then there is a cardinal preserving class forcing extension $M[G] \models V = L[r], r \subseteq \omega \wedge \kappa$ weak compact but not unfoldable.*

By brute force one can obtain this effect on a non-stationary class of unfoldables. However Sy Friedman, in response to this query, proved the following theorem that allows one to get this effect everywhere.

**Theorem 1.10.** *(S. Friedman) Let $M \models GCH$. Then in a coding generic extension $M[G] \models V = L[r], r \subseteq \omega$, cofinalities are preserved, as is $\Pi^1_1$-indescribability. However all instances of $\Pi^1_2$-indescribability are lost.*

Since the extension is a model of $V = L[r]$ and Villaveces shows that in $L$, or $L[r]$, all unfoldables are $\Pi^n_m$ indescribable, a full positive answer to Villaveces' question is a corollary of Friedman's theorem.

It should be pointed out that the *consistency strength* of long unfoldability on its own, is rather weak: it does not imply strong indescribability properties in $V$, only in canonical inner models. Similarly only in inner models of "$V = K$" does it imply that the mouse order is long. Outside of such models it does not imply even that $O^{\#}$ exists. Set models of long unfoldability can be easily constructed. For example:

**Theorem 1.11.** $Con(ZFC + V = L + \exists$ *an ineffable cardinal)* $\vdash$ *Con*($ZFC + \exists A \subseteq \kappa^+(V = L[A]) + \kappa$ *is long unfoldable but* $\Sigma^2_2$*-describable*).

**Proof:** This is essentially a result of Villaveces stated in our terms plus some coding: if $\lambda$ is ineffable, then he remarks ([9] p.1121 following Prop.1.4) that over $V_\lambda$ there is $\kappa < \lambda$ which is $On$-unfoldable with respect to $V_\lambda$; that is for each $S \subseteq \kappa$ there is a cub class $C_S \subseteq \lambda$ of ordinals, and a class $T^S = \langle T^S_\gamma \mid \gamma \in C_S \rangle$ of sets with $\gamma \in C_S \longrightarrow \langle L_\kappa, S \rangle \prec \langle L_\gamma, T_\gamma \rangle$. Let $\widetilde{C} = \bigcap_{S \in \mathcal{P}(\kappa)} C_S$, and let $D = \{ \langle S, \langle \gamma, \xi \rangle \rangle \mid S \in \mathcal{P}(\kappa) \wedge \gamma \in C_S \wedge \xi \in T^S_\gamma \rangle \}$. Then $D$ is amenable to $L_\lambda$. Let $\mathbb{P}$ be the Jensen coding conditions for coding over $\langle L_\lambda, \in, D \rangle$ the predicate $D$ by a subset $A$ of $\kappa^+$. Then in $\langle L_\lambda[A], \in, A \rangle$ we can decode from $A$ the classes $\widetilde{C}, T$. $\mathcal{P}(\kappa)$ is unchanged. Now using $A$ we can define the requisite definable classes $t_S$ for $S \in \mathcal{P}(\kappa)$. As $\kappa$ is least with a non-constructible subset of $\kappa^+$ we obtain the above describability of $\kappa$. **Q.E.D.**

(We note that there are versions of the above theorem preserving sharps for all sets using a $\#$-preserving Coding Theorem, providing this holds in the ground model.) On the other hand to prove the existence of such a cardinal such as in Theorem 1.3, seems to require in $V$ more than one requires in $K$, as shown by Theorem 1.4, where there are lots of iterable structures around to help.

We finally remark here that actually there is rather a lot of "slack" between hypotheses and conclusions in both Theorems 1.4 and 1.6. In particular there is a whole hierarchy of "$\alpha$-hyper-unfoldable" that one could define and investigate implied by the determinacy hypothesis of 1.4.

Our notations are standard, and any unexplained terms we hope can be found in [7]. $H_\kappa$ will denote the set of sets of hereditary cardinality $< \kappa$.

## 2. $\omega$-closed cardinals and a generalised indiscernibility lemma

Suppose that $\langle \mathcal{A}, F\rangle$ is an $\omega$-closed structure. By (ii) and (iii) in Definition 1.1 we may form the iterated ultrapowers of $\mathcal{A}$ by $F$. The following are then routinely verified:

(1) $H_\kappa^{\mathcal{A}} \models ZFC$; hence if $\pi_{i,j} : \mathcal{A}_i \longrightarrow \mathcal{A}_j$ is the iteration map, then $\pi_{i,j}(H_{\kappa_i}^{\mathcal{A}_i}) = H_{\kappa_j}^{\mathcal{A}_j}$ and $\pi_{i,j} \restriction H_{\kappa_i}^{\mathcal{A}_i} : H_{\kappa_i}^{\mathcal{A}_i} \longrightarrow_{\Sigma_\omega} H_{\kappa_j}^{\mathcal{A}_j}$ where $\mathcal{A}_j \models$ "$F_j$ is a normal measure on $\kappa_j$."

(2) If $H = \bigcup_i H_{\kappa_i}^{\mathcal{A}_i}$ then $H \models ZFC+$"Every set has a sharp".

A direct limit argument taken from the proof of the Jensen Indiscernibles Lemma ([1], Ch.16) will be used below to obtain $\omega$-closed structures from good sequences of indiscernibles of uncountable order type.

**Definition 2.1.** *(cf [1] Ch.16) Suppose $\mathcal{A} = \langle L_\delta[A], \in, A, B\rangle$ is an amenable structure. $I \subseteq \delta$ is a* good sequence of indiscernibles *for $\mathcal{A}$ if for all $\gamma \in I$:*

*(i) $\langle L_\gamma[A \cap \gamma], \in, A\cap\gamma, B\cap\gamma\rangle \prec \mathcal{A}$; (ii) $I\backslash\gamma$ is a set of indiscernibles for the structure $\langle L_\delta[A], \in, A, B, \langle \xi\rangle_{\xi\leq\gamma}\rangle$.*

*$I$ is said to be* remarkable *if it satisfies (ii).*

**Remark:** It seems not to have been explicitly stated in the literature known to us, but if the good sequence $I$ is infinite then each $\gamma \in I$ is ineffable in $\mathcal{A}$. This is similar to the argument that Silver indiscernibles are ineffable in $L$; or see the argument in [7] at p.395 where it is shown that the least $\omega$-Erdős cardinal has ineffables below it.

The following gives the version of $\alpha$-Erdős we shall use in this paper.

**Definition 2.2.** *Let $\kappa$ be a regular cardinal and $\alpha$ a limit ordinal. Then*
*(i) $\kappa$ is $\alpha$-*Erdős *if every structure of the form $\mathcal{A} = \langle L_\kappa[A], \in, A, B\rangle$ has a good sequence of indiscernibles of order type $\alpha$.*
*(ii) $\kappa \longrightarrow (\alpha)_\mu^{<\omega}$ if and only if for every function $f : [\kappa]^{<\omega} \longrightarrow \mu$ there is $X \subseteq \kappa$ with $X$ homogeneous for $f$,* i.e., *$\forall n|f``[X]^n| = 1$, and $ot(X) = \alpha$.*

The least cardinal satisfying (ii) is additionally $\alpha$-Erdős .

**Proof of Theorem 1.1** Let $C \subseteq \kappa$ be any cub set. As any $\alpha$-Erdős cardinal is strongly inaccessible, we may choose $E \subseteq \kappa$ be such that $H_\kappa = L_\kappa[E]$. By the $\omega_1$-Erdős property, let $I \subseteq C$ be a sequence of good indiscernibles of order type $\omega_1$. We shall claim that $\gamma^* =_{df} \sup(I)$ is $\omega$-closed. Standard arguments show that each $\gamma \in I$ is strongly inaccessible, and hence $\mathcal{A} \restriction \gamma = \langle L_\gamma[E], E \restriction \gamma\rangle \models ZFC$. Closely following the original proof of [4], we set $\widetilde{\mathcal{A}}_\gamma^n = \mathrm{Hull}_{\mathcal{A}}(\gamma + 1 \cup \boldsymbol{\gamma})$ where $\boldsymbol{\gamma}$ is any member of $[I\backslash\gamma+1]^n$; $\mathcal{A}_\gamma^n = \widetilde{\mathcal{A}}_\gamma^n \cap H_{\gamma^+}$ is a transitive $ZFC^-$ model. Set $\mathcal{A}_\gamma = \cup_n \mathcal{A}_\gamma^n$ and $\delta^n = On \cap \mathcal{A}_\gamma^n$. Note that $\delta^n$ is definable in $\mathcal{A}_\gamma^{n+1}$. Hence:

(1) $\delta^n < \delta^{n+1}$ and $\mathcal{A}_\gamma^n \prec \mathcal{A}_\gamma^{n+1} \prec \mathcal{A}_\gamma \models ZFC^-$.

For $\gamma < \gamma' \in I$ we define $\pi_{\gamma\gamma'}^n : \mathcal{A}_\gamma^n \longrightarrow_e \mathcal{A}_{\gamma'}^n$ by applying shift maps on indiscernibles, to terms in the language appropriate to $\mathcal{A}$:

$$\pi_{\gamma\gamma'}^n(t_{\mathcal{A}}(\boldsymbol{\nu}, \gamma, \boldsymbol{\gamma})) = t_{\mathcal{A}}(\boldsymbol{\nu}, \gamma', \boldsymbol{\gamma}) \text{ for } \boldsymbol{\gamma} \in [I\backslash\gamma'+1]^{<\omega} \text{ and } \boldsymbol{\nu} \in [\gamma]^{<\omega}.$$

The choice of $\gamma$ is not important due to their indiscernibility. Define

$$\mathcal{U}^n_\gamma = \left\{ X \in \mathcal{P}(\gamma) \cap \mathcal{A}^n_\gamma \mid \gamma \in \pi^n_{\gamma\gamma'}(X), \text{ for some } \gamma' > \gamma,\ \gamma' \in I \right\}.$$

Then $\mathcal{U}^n_\gamma \in \mathcal{A}^{n+1}_\gamma$ and, if we set $\mathcal{U}_\gamma = \cup_n \mathcal{U}^n_\gamma$, then we conclude:

(2) $\langle \mathcal{A}_\gamma, \mathcal{U}_\gamma \rangle \models$ "$\mathcal{U}_\gamma$ is a normal measure on $\gamma$", and is an amenable structure.

Set

$$\langle \langle A_{\gamma^*}, \mathcal{U}_{\gamma^*} \rangle, \langle \pi_{\gamma\gamma^*} \rangle_{\gamma \in I} \rangle = \varinjlim \langle \langle A_\gamma, \mathcal{U}_\gamma \rangle, \langle \pi_{\gamma\gamma'} \rangle_{\gamma < \gamma' \in I} \rangle$$

where $\pi_{\gamma\gamma'} = \cup_n \pi^n_{\gamma\gamma'}$. As $\mathcal{U}_{\gamma^*}$ is generated by $I$ as a final segment filter on $I$, and $\mathrm{cf}(otp(I)) = \omega_1$:

(3) $\mathcal{U}_{\gamma^*}$ is $\omega$-closed.

Let $\mathcal{A}_{\gamma^*} = \langle L_{\beta^*}[E^*],\ E^* \rangle$. Clearly $E^* \restriction \gamma^* = E \restriction \gamma^*$, and, using (2), $\langle L_{\beta^*}[E^*],\ \mathcal{U}_{\gamma^*} \rangle$ is amenable, and, using (3), iterable. Further $E \restriction \gamma^* \in \mathcal{A}_{\gamma^*} \models ZFC^-$; hence, using (i) of the definition good indiscernibles as $L_\gamma[E \restriction \gamma] \prec L_\kappa[E]$ for any $\gamma \in I \cup \{\gamma^*\}$:

(4) $H_{\gamma^*} = |L_{\gamma^*}[E^*]| \in \mathcal{A}_{\gamma^*}$.

But then this together with (3) and the comments after it show that $\gamma^*$ is $\omega$-closed. **Q.E.D.**

Note then that an $\omega$-closed cardinal need not be regular (in $V$), but only have uncountable cofinality. Note also in the above construction we do not claim that $\langle \mathcal{A}_{\gamma^*}, \mathcal{U}_{\gamma^*} \rangle \models KP$ as this is a much stronger property. But the concept of $\omega$-closure is what is needed to get back an uncountable sequence of good indiscernibles:

**Theorem 2.1.** *Let $H_\kappa = L_\kappa[B]$ where $B \subseteq \kappa$ and $L_\kappa[B] \in A$ where $\mathcal{A} = \langle A, \in, F \rangle$ witnesses the $\omega$-closure of $\kappa$. Then there is $I \subseteq \kappa$, $ot(I) \geq \omega_1$, with $I$ a good set of indiscernibles for $\langle L_\kappa[B], \in, B \rangle$.*

**Proof:** Let $C \subseteq \kappa$ be closed and unbounded such that $\gamma \in C \longrightarrow L_\gamma[B \cap \gamma] \prec L_\kappa[B]$ with $C \in A$. The argument is standard, but has to be conducted without using $\Sigma_1$-Collection (which in general will not hold over $\mathcal{A}$), and we simply check below that these details can be done. Let $\langle \varphi_m \mid m \in \omega \rangle$ be a recursive enumeration of the formulae of $\mathcal{L}_{\{\dot\in, \dot=, \dot B\}}$. Let $f : [C]^{<\omega} \longrightarrow \kappa$ be defined by:

$$\begin{aligned} f(\{\alpha_1, \ldots, \alpha_{2n}\}) = {} & \mu\lambda < \alpha_1 (\exists \delta_1 < \cdots < \delta_k, \exists m < \omega\ \lambda = \prec m, \delta_1, \ldots \delta_k \succ \wedge \\ & L_\kappa[B] \not\models \varphi_m(\boldsymbol{\delta}, \alpha_1, \ldots, \alpha_n) \leftrightarrow \varphi_m(\boldsymbol{\delta}, \alpha_{n+1}, \ldots, \alpha_{2n})) \\ & \text{if such exists,} \\ = {} & 0 \text{ otherwise.} \end{aligned}$$

Then $f \in A$ and is regressive. A routine argument shows that if $I \subseteq C$ is infinite and is homogeneous for $f$ then $f$ must take value 0 everywhere, and hence $I$ is a good sequence of indiscernibles of the required type. The construction of $I$ is by induction on $n$ defining homogeneous sets $I_n$ for $f^n =_{df} f \restriction [C]^n$. We assume as inductive hypothesis that, for $n = m$ and for any sequence $\langle g_\alpha \mid \alpha < \kappa \rangle \in A$ where each $g_\alpha : [C]^m \longrightarrow \kappa$ is regressive, there is a sequence $\langle X_\alpha \mid \alpha < \kappa \rangle \in A$ with $X_\alpha \in F$ homogeneous for $g_\alpha$. (Thus $X_\alpha$ has type $\kappa$ as it has $F$-measure 1. For the base case of $n = 0$ *i.e.* for functions of zero variables, we of course set $X_\alpha = \kappa$.)

Thus let $\langle f_\beta \mid \beta < \kappa \rangle \in A$ be a sequence of regressive functions on $[C]^n$ with $n = m + 1$. Define

$$\begin{aligned} f^\alpha_\beta(x) &= f_\beta(\{\alpha\} \cup x) \text{ if } \alpha < \min(x) \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then $\langle f^\alpha_\beta \mid \alpha, \beta < \kappa \rangle \in A$ and by the inductive hypothesis there is $\langle X^\alpha_\beta \mid \alpha, \beta < \kappa \rangle \in A$ with each $X^\alpha_\beta \in F$ and homogeneous for $f^\alpha_\beta$. As $\mathcal{A} \models ZFC^-$ if we set $X_\beta =_{df} \triangle_{\alpha<\kappa} X^\alpha_\beta$ then $\langle X_\beta \mid \beta < \kappa \rangle \in A$, and by normality of $F$, each $X_\beta \in F$.

Suppose $\beta < \kappa$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}, \{\alpha_1, \alpha'_2, \ldots, \alpha'_n\} \in [X_\beta]^n$. Then define

$$\gamma^{\alpha_1}_\beta =_{df} f^{\alpha_1}_\beta(\{\alpha_2, \ldots, \alpha_n\}) = f^{\alpha_1}_\beta(\{\alpha'_2, \ldots, \alpha'_n\}) < \alpha_1.$$

Then for each fixed $\beta$, $\alpha \to \gamma^\alpha_\beta$ is a function, $h_\beta$, in $A$ with domain $X_\beta$ and together $\langle h_\beta \mid \beta < \kappa \rangle \in A$. If we set $Y^\gamma_\beta =_{df} \{\alpha \in X_\beta \mid h_\beta(\alpha) = \gamma\}$ then $U =_{df} \{Y^\gamma_\beta \mid \gamma, \beta < \kappa\}$ is in $A$. By amenability of $\mathcal{A}$ $U \cap F \in A$ and so by normality for each $\beta < \kappa$ $\exists Y \in U \cap F, \exists \gamma < \kappa\, Y = Y^\gamma_\beta$. Thus, using a wellorder of $U \cap F$ in $\mathcal{A}$, we may define a function $G : \kappa \to U$, $G(\beta) =$ some $Y \in U \cap F$ on which $h_\beta$ is constant. Let $Z_\beta = G(\beta) \cap X_\beta$. Then $\langle Z_\beta \mid \beta < \kappa \rangle \in A$, each $Z_\beta \in F$ and is homogeneous for $f_\beta$. This completes the inductive step.

Now returning to our original regressive $f : [C]^{<\omega} \longrightarrow \kappa$ we may inductively define for each $n$ a sequence of sets $\langle I_m \mid m \leq n \rangle \in A$ homogeneous for $\langle f^m \mid m \leq n \rangle$. In general a requisite sequence $n \longrightarrow I_n$ may not exist in $A$. However we do have that each $I_n \in F$ and thus by $\omega$-closure $I =_{df} \cap_n I_n \neq \emptyset$. By considering tail segments it is easy to see that $I$ is unbounded in $\kappa$ - as cf$(\kappa) \geq \omega_1$ we get our desired conclusion. **Q.E.D.**

Putting these last two together gives us an equivalence:

**Corollary 2.1.** *Let $\kappa$ be a cardinal. Then the following are equivalent:*

*(i) $\kappa$ is the least cardinal which is $\omega$-closed;*

*(ii) $\kappa$ is the least cardinal so that $\exists B \subseteq \kappa \wedge H_\kappa = L_\kappa[B] \wedge \exists I \subseteq \kappa$, ot$(I) = \omega_1$, $I$ cofinal in $\kappa$, with $I$ a good set of indiscernibles for $\langle H_\kappa, \in, B \rangle$.*

We remark that the proof of 1.1 does not actually require the height of $\mathcal{A}$ to be a cardinal, or for the domain of $\mathcal{A}$ to be an initial rank of the $V$-hierarchy. It in fact shows the following (using the notation of that proof):

**Theorem 2.2.** *Suppose $\mathcal{A} = \langle L_\delta[A], \in, A, B\rangle$ is a structure with $A, B \subseteq \delta$, and let $I \subseteq \delta$ be a good sequence of indiscernibles for $\mathcal{A}$ of limit order type with supremum $\gamma^*$ where $cf(\gamma^*) > \omega$, then there is an iterable, amenable structure $\langle \mathcal{A}_{\gamma^*}, \mathcal{U}\rangle$ extending $\mathcal{A} \restriction \gamma^* = \langle L_\gamma[A], A \cap \gamma^*, B \cap \gamma^*\rangle$.*

## 3. The consistency strength of long unfoldables

### *3.1. Preliminaries*

We recall first some basic facts about $Q$-structures in inner core models. The reader may refer to [12] where this is done for models with many measurables, or in a more abbreviated form for the Dodd-Jensen $K$ in [10]. By "$K$" we shall mean the core model built under some anti-large cardinal assumption, but in practice as we are proving some consistency results which are weaker than that of the existence of a measurable cardinal we shall really be working with the internal structure of a core model above the largest measurable of its extender sequence $E$, and below its dagger $E^\dagger$. As this level of generality would yield essentially nothing new we simply work below $O^\dagger$, and use the Dodd-Jensen $K$ (however using modern fine structure, not that of [1].).

We shall adopt the mouse notation and concepts of [12]. A *premouse* is a fully amenable, acceptable $J$-model of the form $M = \langle J_\gamma^{E^M}, E^M, F^M\rangle$ with $E^M = \{\langle X, \nu\rangle | \nu < \gamma \wedge X \in E_\nu^M\}$ such that any proper initial segment $M|\beta =_{df} \langle J_\beta^{E^M}, E^M, E_\beta^M\rangle$ is *sound*. If $E_\beta^M \neq \emptyset$ then there is $\kappa < \beta$ with $E_\beta^M$ an $M|\beta$-normal measure on $\mathcal{P}(\kappa)^{M|\beta}$. If $\pi : M|\beta \longrightarrow_{E_\beta} Ult(M|\beta, E_\beta) = N$, then $E^N \restriction \beta = E^M \restriction \beta$, and $(\kappa^+)^N = \beta$. If $E_\gamma^M = F^M \neq \emptyset$ then we call $M$ *active*, otherwise it is *inactive*. A *mouse* is an iterable premouse, that is one in which iterated ultrapowers of $M$ (by its topmost measure $F$, or any initial extenders on the $E$-sequences arising, truncating the model when required) are wellfounded. We let $<^*$ denote the canonical wellordering of active mice under coiteration.

**Definition 3.1.** *Let $M = \langle J_\gamma^E, E, F\rangle$ be an active mouse. $K_M$, the inner model generated by $M$ is*

$$K_M =_{df} \bigcup_{i<On} H_{\kappa_i}^{M_i}$$

*where*

$$M_i = \langle J_{\gamma_i}^{E_i}, E_i, F_i\rangle \models \text{“}crit\ (F_i) = \kappa_i\text{”}$$

*is the $i$'th iterate of $M$ by the top extender $F_i$. We set $C_M = \{\kappa_i \mid i \in On\}$.*

$K_M$ is then an inner $L[E]$ model, with $C_M$ a cub class of indiscernibles for it.

**Definition 3.2.** *(i) For $\kappa$ a cardinal, let $F_\kappa$ be the cub filter on $\kappa$.*
*(ii) Let $W$ be an inner model, $\kappa$ a cardinal of $W$, $F_\kappa = (F_\kappa)_W$, then:*
*(a) set $\theta_\kappa^W$ = maximal $\theta$ so that $\langle J_\theta^{F_\kappa}, \in F_\kappa\rangle \models$ “$F_\kappa$ is a normal measure on $\kappa$”.*
*(b) Then also set:*

$$Q_\kappa^W =_{dt} \langle J_{\theta_\kappa^W}^{F_\kappa}, \epsilon, F_\kappa\rangle.$$

*If $W = K_M$, then we replace $W$ and write $\theta_\kappa^M, Q_\kappa^M \ldots$ etc.*

**Fact 3.1.** *Suppose there is no inner model with a measurable cardinal. If* $\mu_\kappa = ot\{\{M \in H_\kappa\}, <^*\}$*, the length of the mouse ordering restricted to* $H_\kappa$*, then* $\theta_\kappa = \kappa(1 + \mu_\kappa)$.

For $b$ a set of ordinals, we set

$$\begin{aligned} I^b &= \text{class of Silver indiscernibles for } L[b] \text{ if } b^{\#} \text{ exists;} \\ &= On \text{ otherwise.} \end{aligned}$$

**Definition 3.3.** *For* $\kappa$ *a cardinal*

*(i)* $C(\kappa) =_{df} \bigcap_{a \subseteq \gamma < \kappa} I^a$;

*(ii)* $u_2(\kappa) =_{df}$ *least element of* $C(\kappa)$ *which is greater than* $\kappa$.

If all bounded subsets of $\kappa$ have sharps, $u_2(\kappa) = \sup\{\kappa^{+L[a]} | a \in H_\kappa\}$.

**Definition 3.4.** *(i) Let* $\mathcal{A} = \mathcal{A}_\kappa$ *be the smallest transitive admissible set with* $H_\kappa \cup \{\kappa\} \subseteq |\mathcal{A}_\kappa|$.

*(ii)* $\tau(\kappa) =_{df} On \cap \mathcal{A}_\kappa$ *- the height of the smallest transitive admissible set containing* $\kappa$ *and every element of* $H_\kappa$.

**Definition 3.5.** $\kappa$ *is* admissibly measurable *if there is a transitive, iterable, and amenable structure* $\langle \mathcal{A}, F \rangle \models$ *"*$KP_{\{\dot\in, \dot F\}} + F$ *is a normal measure on* $\kappa$*" with* $H_\kappa \subseteq |\mathcal{A}|$.

Notwithstanding the appearance of $KP$ in the language including $\dot F$, the notion of admissibly measurable is considerably weaker than $\omega$-closure. This is due to the fact that $H_\kappa$ is only required to be a class, and not a set of $|\mathcal{A}|$.

**Fact 3.2.** *In the core model below (or including) a single measurable cardinal, if* $\kappa$ *is a cardinal, then* $\theta_\kappa \le u_2(\kappa) \le \tau(\kappa)$.

**Remark:** In $K$,$\theta_\kappa < u_2(\kappa) < \tau(\kappa)$, $\theta_\kappa = u_2(\kappa) < \tau(\kappa)$, as well as $\theta_\kappa = u_2(\kappa) = \tau(\kappa)$ are all possible (even assuming that $H_\kappa$ is closed under #'s). (Example 1 of [10], using $L^{\#}$, yields an example of the first possibility here; the third possibility occurs in $L^\mu$, - see Example 3 of [10]. Lastly we sketch how the second possibility can arise: consider the least inner model $K_0$, of $V = K$ where for some $\kappa$ we have $u_2(\kappa) = \tau(\kappa)$; then by Fact 3.3 (ii) to come, we have that the structure $\mathcal{A} = \langle Q_\kappa, \in, F_\kappa \rangle$ is admissible; then the mouse order $<^* \restriction H_\kappa$ has order type $\theta_\kappa = \tau(\kappa)$. Let $\langle M^\beta | \beta < \tau(\kappa) \rangle$ be the enumeration of these mice in ascening $<^*$ order. Let $F : \tau(\kappa) \longrightarrow \tau(\kappa)$ be the function defined by $F(\beta) =$ the critical point of the $\kappa + 1$ iterate of $M^\beta$. Then $ran(F) \subseteq (\kappa, \tau(\kappa))$; further $F$ is $\Delta_1^{Q_\kappa}$. Hence there is some some proper initial level $J_\nu^{F_\kappa}$ of $Q_\kappa$ which is closed under the function $F$. This $\nu$ becomes "$\theta_\kappa = u_2(\kappa)$" of the inner model $W = L[J_\nu^{F_\kappa}]$, which is then a proper inner model of $K_0$. By the leastness in the choice of $K_0$, we have that "$\nu < \tau(\kappa)$" holds in $W$.)

(Note: Assuming there is no inner model with a measurable cardinal, by $\Sigma^1_3$-correctness arguments for $\kappa \ge \omega_2$, $u_2(\kappa)_V = u_2(\kappa)_K$, and, for $\kappa > \omega_2$, by the Covering Lemma for $K$, $\theta_\kappa = \theta_\kappa^K$ .)

**Fact 3.3.** *[10] (Lemma 4)* Suppose $V = K$, $H_\kappa$ is closed under #'s, and $u_2(\kappa) = \tau(\kappa)$. Then
(i) $\tau(\kappa) = \theta_\kappa = u_2(\kappa)$;
(ii) In $K$, $\mathcal{A} = \langle Q_\kappa, \in, F_\kappa\rangle \models KP$ and so $\kappa$ is admissibly measurable.

Let $\Psi$ be the sentence
"$\forall D \subseteq \omega_1$ ($D$ is universally Baire $\iff \exists r \in \mathbb{R}(D \in L[r])$)".

**Fact 3.4.** *([10] Theorem 1, Lemma 4) (i) If there exists an admissibly measurable, then $\Psi_K$.*
*(ii) In $K$ the following are equivalent:*
*(a) "There exist arbitrarily large admissibly measurable cardinals",*
*(b) "$\Psi$ is set generically absolute."*

To obtain the conclusions of Theorems 1.5 and 1.6 , it thus suffices to show that in $K$ there is an admissibly measurable cardinal, or an inner model with a proper class of them.

To prove Theorem 1.7 we shall use the equivalence of $\omega^2$-$\Pi^1_1$-Determinacy from [12]. It is shown there (Main Theorem 3) that this amount of determinacy is equivalent to the existence of a "clever" mouse. This is essentially a mouse with the critical point of its top measure $\kappa$ that reflects a $\Sigma_1$-Rowbottom property of its measure into the $Q$-structure of the generated model at $\kappa$: $\langle Q^M_\kappa, \in F^M_\kappa\rangle$. However we shall take as our definition here another equivalent notion, that of "inwardly reflecting." ([12] Lemma 1.10 show they are equivalent.)

**Definition 3.6.** *Let $M = \langle J^{E^M}_\alpha, E^M, F^M\rangle$ be an active mouse with $crit(F^M) = \kappa = \kappa_M$. $M$ is* inwardly reflecting *if for any $p \in H^M_\kappa$, for any $\varphi \in \Sigma_1 \cap \mathcal{L}_{\{\dot\in, \dot=, \dot F\}}$, there is $N \in H^M_\kappa$ so that $\forall \boldsymbol{\kappa} \in [C_M]^r$ (where $FV(\varphi) \subseteq \{v_0, \ldots, v_{r-1}\}$), and* $\forall\lambda \in C_M$, $\max \boldsymbol{\kappa} < \lambda$

$$Q^M_\lambda \models \varphi(\boldsymbol{\kappa}, p) \iff Q^N_\lambda \models \varphi(\boldsymbol{\kappa}, p)$$

An interpretation of this is that below an inwardly reflecting mouse, given any inner model $K^{\widetilde{M}}$ generated by a mouse $\widetilde{M}$, there is a parameter $p \in H^{\widetilde{M}}_\kappa$ and a $\Sigma_1$-formula $\varphi$ so that $\varphi$ is true of $p$ on the indiscernibles for $K^{\widetilde{M}}$ in some (or any) sufficiently large $Q$-structure, but this fails to be true in any smaller inner model (meaning precisely, that if $<^*$ is the mouse ordering and $N <^* M$, and $K_N$ is generated by $N$, then $\varphi$ is not true of $p$ on indiscernibles given by the class of iteration points of $N$, $C_N$). Once one has an inwardly reflecting mouse, such inner models cannot be characterised by sets of Gödel numbers of $\Sigma_1$-formulae, with parameters, true on indiscernibles in $Q$-structures. We have

**Fact 3.5.** *([12, Theorem 3, Lemma 1.10])*
$ZFC \vdash \omega^2$-$\Pi^1_1$-Determinacy $\longleftrightarrow$ *there exists an inwardly reflecting mouse.*

### *3.2. The conclusions*

**Proof of Theorem 1.5** We assume $V = K$ and that $\kappa$ is long unfoldable. Note that $H_\kappa$ is closed under #'s: otherwise $V = L[B]$ for $B$ a bounded subset of $\kappa$. However this is impossible by the argument of Fact 1.1, that there would then be a definable satisfaction relation for $L[B]$. By Fact 3.4 it suffices to show that $\kappa$ is admissibly measurable, that is, to show that Theorem 1.7 (i) holds. Suppose not, then by Fact 3.2 & 3.3 $\theta_\kappa \leq u_2(\kappa) < \tau(\kappa)$ and $Q_\kappa = \langle J^{F_\kappa}_{\theta_\kappa}, \in, F_\kappa\rangle$ is inadmissible. Hence there is a parameter $p \in H_\kappa = H^{Q_\kappa}_\kappa$ and a $\Sigma_1$ formula $\varphi(v_0, v_1, p) \in \mathcal{L}_{\{\dot\in, \dot= \dot F\}}$ defining a map $\Sigma_1$ over $Q_\kappa$, $G : \kappa \longrightarrow \theta_k$, which is cofinal and monotone. The latter can be coded by a subset of $\kappa$, $S \subseteq \kappa$ say, and then by definition of long unfoldability, there are $\langle M_\lambda, \in S_\lambda\rangle$ with $\lambda \subseteq M_\lambda$ and $\langle V_\kappa, \in, S\rangle \prec \langle M_\lambda, \in S_\lambda\rangle$. By elementarity $S_\lambda$ codes $Q^{M_\lambda}_\lambda$ - a level of the $J^{F_\lambda}$-hierarchy which is accordingly an initial segment of the full $Q_\lambda$. Further $\varphi$ defines a $\Sigma_1$ cofinal map of $\lambda$ into $On \cap Q^{M_\lambda}_\lambda$, in the parameter $p$.

If $W$ is the tower obtained by taking the union of all the models $M_\lambda \vDash V = K$ in a long chain, we have a proper inner model $W$ of $V = K$, in which $\varphi$ with $p$ witnesses the inadmissibility of $Q^W_\lambda$ on cofinally many $\lambda$. Let $N$ be the $<_*$ -least mouse not in $W$. Then $N$ generates $W$: $W = K_N$.

*Claim* $C = \{\lambda \mid \varphi$ defines a cofinal map from $p$ $\Sigma_1$ over $Q^W_\lambda$ into $\theta^W_\lambda\}$ is closed and unbounded.

**Proof:** (of claim) As we already have unboundedness, suppose $\lambda$ is a limit point of $C$. Then $\lambda$ is a limit cardinal of $W$, and $Q^W_\lambda$ is the union of the $\lambda$'th iterates of the structures $Q^W_\mu$ for $\mu < \lambda$ (taking iterated ultrapowers of $Q^W_\mu$ by the measures $F^W_\mu$ and their images). But then, as the ultrapower maps are $\Sigma_1$-preserving

$$\forall\gamma < \theta^W_\lambda\, \exists\delta > \gamma\, \exists\mu < \lambda\, \exists\bar\delta < \theta^W_\mu\, \exists\xi < \mu \quad \left(\varphi_{Q_\mu}(\xi, \bar\delta, p) \wedge \delta = i_{Q_\mu,(Q_\mu)_\lambda}(\bar\delta)\right)$$

shows that $\varphi_{Q_\lambda}(v_0, v_1, p)$ defines a function that cannot be bounded in its range below $\theta^W_\lambda$. **Q.E.D.(Claim)**

Consequently this property of $\varphi$, $p$ characterises $Q^N_\kappa$ at each of its iteration points, since it will have an iteration point in the cub class $C$.

As $p \in H_\kappa$ with $\kappa$ inaccessible, a Löwenheim-Skolem argument shows that such an $N$ can be found with $N \in H_\kappa$.

Suppose $N = \langle J^{E^N}_\gamma, E^N, F^N\rangle$ with $crit(F^N) = \delta < \kappa$. If $i : N \longrightarrow N_\kappa$ is the iteration map obtained by taking $\kappa$-many iterated ultrapowers by the top measure $F^N$, $i(Q^N_\delta) = Q^{N_\kappa}_\kappa$, and $i \restriction Q^N_\delta : Q^N_\delta \longrightarrow Q^{N_\kappa}_\kappa$ is elementary. Hence $\varphi(v_0, v_1, p)$ defines a map $G : \kappa \longrightarrow On \cap Q^{N_\kappa}_\kappa$ which is cofinal and $\Sigma_1(Q^{N_\kappa}_\kappa)$. But $N \in H_\kappa$, and so $Q^{N_\kappa}_\kappa$ is a proper initial segment of $Q_\kappa$. But $\varphi$ is supposed to define an unbounded map $G : \kappa \longrightarrow \theta_\kappa$! This is a contradiction.

**Q.E.D.(Theorem 1.5)**

**Proof of Theorem 1.6**. Using the notation of the proof of the last theorem, as we have established that $Q_\kappa$ is admissible, then $S_\lambda$ codes an admissible $Q^{M_\lambda}_\lambda$. Thus in the model $W$ we have arbitrarily large $\lambda$ with $Q^W_\lambda$ admissible. Then $W$ is a model satisfying the conclusion of Theorem 1.7 (ii). **Q.E.D.(Theorem 1.6)**.

**Proof of Theorem 1.4** As mentioned in §2.1, $\omega^2$-$\Pi^1_1$-Determinacy implies there is an inwardly reflecting mouse $M$. Let $W = K_M$, the model generated by $M$. let $C_M$ be the class of critical points of the top measure $F^{M_i}$ of the iterates of $M$. We shall show that each $\kappa_i \in C_M$ is long unfoldable in $W$. Let $\kappa = \kappa_0 = \min C_M$. Let $S \subseteq \kappa, S \in W$. Consider the $\omega$'th iterate of $M$ beyond $\kappa_0$ with critical point $\kappa_\omega$. Let $\pi^M_{i,j} : M_i \longrightarrow_{\Sigma_1} M_j$ be the iteration maps. Then $\pi^M_{i,j}$ extends to a map $\widetilde{\pi}^M_{i,j} : W \longrightarrow_{\Sigma_\omega} W$. Applying $\pi^M_{0,2}(\langle W_{\kappa_0}, S\rangle) = \langle W_{\kappa_2}, S_{\kappa_2}\rangle$. We have then $\langle W_{\kappa_0}, S\rangle \prec \langle W_{\kappa_2}, S_{\kappa_2}\rangle$. Let $p = \langle W_{\kappa_0}, S\rangle \in H^{M_1}_{\kappa_1}$ be our parameter in the definition of the inwardly reflecting property as applied to the active mouse $M_1$. We see that

$$Q^M_{\kappa_\omega} \models \exists \bar{M}, \bar{S}\ (On \cap \bar{M} = \kappa_2 \supseteq \bar{S} \wedge \langle W_{\kappa_0}, S\rangle \prec \langle \bar{M}, \bar{S}\rangle).$$

This is a $\Sigma_1$ statement about $p$ and $\kappa_2 \in C_M$, $\varphi(p, \kappa_2)$ say. By the definition of inwardly reflecting, there is $N \in H^{M_1}_{\kappa_1}$ so that $Q^N_{\kappa_\omega} \models \varphi(p, \kappa_2)$. If $\langle M', S'\rangle \in K_N$ witnesses this, and if $\pi^N_{i,j} : N_i \longrightarrow N_j$ are the iteration maps associated to $N$, we may use these to construct a formula defining elementary extensions of $\langle M', S'\rangle$. Specifically, as $N \in H^{M_1}_{\kappa_1}$, $N <^* M$, and hence $C_M \backslash \{\kappa_1\} \subseteq C_N$. Let $\lambda$ be a cardinal of $W > \kappa_2$; then set $\langle M_\lambda, S_\lambda\rangle = \pi^N_{\kappa_2,\lambda}(\langle M', S'\rangle)$. Putting this together, we have inside $W$, a formula defining an $On$-length chain of elementary extensions of $\langle W_{\kappa_0}, S\rangle$ of the form $\langle M_\lambda, S_\lambda\rangle$. As $S$ was arbitrary, we have that $\kappa_0$ is long unfoldable in $W$. As $\kappa_0 = \min C_M$, and the latter is a cub class of indiscernibles for $W$, we have in $W$ that the class of long unfoldables is definably Mahlo. **Q.E.D.(Theorem 1.4)**

As shown in [12] Lemma 1.11, an inwardly reflecting mouse is rather weak, even in the $\Sigma_1$-mouse hierarchy: if $M$ is an active mouse, $\kappa = crit(F^M)$, and $\rho^1_M$ (the first projectum of $M$) $= \kappa$, then $M$ is inwardly reflecting. Consequently in any inner model of $V = K$, in which there is an active sound mouse with $crit(M) = \kappa$, and $\kappa$ an actual cardinal of the model, we shall have that $\rho^\omega_M \geq \kappa$ for all $n < \omega$ and $\omega^2\boldsymbol{\Pi}^1_1$-Determinacy.

The argument of the Indiscernibles Lemma provides just such mice given good indiscernibles over $K$ of sufficient length. In particular, if in $K$ $\kappa$ is $\omega$-closed, thence taking a good sequence $I$ of indiscernibles for $\langle K_\kappa, \in, E^K\rangle$ of order type $\omega_1$, with $\sup I = \gamma^* \leq \kappa$ say, then each $\gamma \in I$ is inaccessible in $K$, and the lemma's proof provides an *$\omega$-closed mouse*, a sound mouse $M = \langle J^{E^M}_\beta, E^M, F^M\rangle$ with $crit(F^M) = \gamma^*$, $E^M \upharpoonright \gamma^* = E^K \upharpoonright \gamma^*$ and therefore $\rho^n_M \geq \gamma^*$ for all $n < \omega$. The core model "below an $\omega$-closed mouse" has been partially investigated in [11]. For Theorem 1.3, if we additionally assume $V = K$, all that is needed is some such $\omega$-closed $F$ from the $K$-sequence on some $K$-cardinal $\kappa$, in order for there to be long unfoldables below $\kappa$.

**Proof of Theorem 1.3** We assume that $\kappa$ is $\omega$-closed. By Theorem 2.1 we have a sequence $I$ whose order type is of uncountable cofinality, of good indiscernibles for $\langle V_\kappa, \in, B\rangle$ where $B \subseteq \kappa$ codes $H_\kappa$: that is, for any strongly inaccessible $\lambda \leq \kappa$ we have that $V_\lambda = L_\lambda[B \cap \lambda]$. Each $\gamma \in I$ is ineffable (see the remark above after Definition 2.1). Fix now $\gamma$ as $\min(I)$. By Villaveces' argument the ineffability of $\gamma$ implies that there are $\lambda < \gamma$ which are $On$-unfoldable over $V_\gamma$. that is for any $S \subseteq \lambda$ there is a chain $E^\gamma$, of transitive $ZFC$ models $M_\delta \in V_\gamma$, and subsets $S_\delta \subseteq \delta$ with $\langle V_\lambda, \in, S\rangle \prec \langle M_\delta, \in, S_\delta\rangle \prec \langle M_{\delta'}, \in, S_{\delta'}\rangle$ for $\lambda \leq \delta \leq \delta' < \gamma$ (We may set $\langle M_\lambda, S_\lambda\rangle = \langle V_\lambda, S \oplus B\rangle$.) Note that we may define a final model for this chain $\langle M_\gamma, \in, S_\gamma\rangle$ with $On \cap M_\gamma = \gamma$ and $\langle M_\gamma, S_\gamma\rangle = \bigcup_{\delta<\gamma}\langle M_\delta, S_\delta\rangle$.

Let $\gamma^* = \sup I$. By the goodness of the indiscernibles we have that $\gamma < \gamma'$ in $I$ implies that $V_\gamma \prec V_{\gamma'} \prec V_{\gamma^*} \prec V_\kappa$. We thus have extensions of the chain $E^\gamma$ to $E^{\gamma^*}$.

Let $\mathcal{A} = \langle A_{\gamma^*}, \in, \mathcal{U}_{\gamma^*}\rangle$ be the iterable structure of the proof of Theorem 1.1. Let $H = L[\widetilde{B}]$ be the inner model that it generates (in the sense of (2) at the beginning of §2), we see that there is an embedding $\pi : H \longrightarrow H$, with $crit(\pi) = \gamma^*$. Hence $\mathcal{A}$ is a model of "There is a chain $E^\gamma$ with last model $\langle M_\gamma, \in, S_\gamma\rangle$" for $\gamma < \gamma^*$". But in $H$ we have $V_\gamma \prec V^H_{\gamma^*} \prec V^H_{\kappa_i}$, for any $i$ where the $i$'th model $\mathcal{A}_i$ has critical point $\kappa_i$.

Hence using iterations of $\mathcal{A}$ we can stretch this chain to have length $On$, and have $\kappa_i$'th models $\langle M_{\kappa_i}, S_{\kappa_i}\rangle$ for $i < \infty$. Thus, using $\mathcal{A}$ as a parameter, we then can construct a formula, just as in the canonical example of $L[\mu]$, defining a proper class, which gives us a tower of extensions of $\langle V_\lambda, \in, S\rangle$ as demanded by the definition of long unfoldability.

**Q.E.D.(Theorem 1.3)**

In the above proof if we could have told in advance which $\lambda$ it was that would be $On$-unfoldable over $V_\gamma$ we could use a weaker partition property - for example if $\kappa$ satisfied $\kappa \longrightarrow (< \omega_1)^{<\omega}_{2^\lambda}$, (we refer the reader to [2] for the precise definition) this should be more than enough. However Villaveces' argument is nonconstructive. So we have to leave the matter open.

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