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I. Σ^* Fine Structure

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1. Σ^* Fine Structure

Why is fine structure necessary? One answer is that, although the constructible hierarchy is an extremely uniform creation, it is not *that* uniform. We have for any level of our J_α hierarchy uniform Σ_1-Skolem functions. In attempting to generalise this and prove the existence of Σ_n-uniformising functions one runs immediately into the difficulty that these cannot be given a completely uniform definition. The following example taken from [17] shows why (see also the related [8] pp.106-107).

For any $\tau < \omega_1$, $\{\zeta \mid J_{\omega_1+\tau} \models$ "$\omega_1 + \zeta$ does not exist"$\}$ is a Π_1-relation over $J_{\omega_1+\tau}$ whose least member, i.e. τ itself, has a Σ_2 definition via applications of a posited uniformly definable Σ_2-Skolem function. For such τ we should have a resulting Σ_2 definition, which, by Fodor's Lemma, on a stationary set $D \subseteq \omega_1$ of τ, would be given by the same Σ_2 formula over J_τ, $\exists u \forall v \Phi(u, v, t)$ say. By Fodor again we may shrink D further to a stationary E so that for some constant δ, and some fixed $u_0 \in J_{\omega_1+\delta}$, and taking any $\eta < \omega_1$, if η is in $E \setminus (\delta + 1)$ then $(\forall v \Phi(u_0, v, \eta))$ holds in $J_{\omega_1+\eta}$. In particular for any other $\gamma \in E \setminus (\delta + 1)$ below η this Π_1 fact about η also holds in $J_{\omega_1+\gamma}$, where γ is supposed to be the unique solution. This is absurd.

We deduce from this that parameters must be involved in defining Σ_2-Skolem functions. Jensen's solution in [24] is to reduce the problem of Σ_2-uniformisation over J_τ to that of Σ_1-uniformisation over a *projectum structure* $\langle J_{\rho_\tau}, A_\tau \rangle$ with A_τ a Σ_1-*mastercode* essentially coding up Σ_1 truth over J_τ. $\Sigma_2(J_\tau) \cap J_{\rho_\tau}$ relations are transformed to $\Sigma_1(\langle J_{\rho_\tau}, A_\tau \rangle)$ relations, which can be Σ_1-uniformised, and such a uniformising function can be translated back up to a Σ_2 function over J_τ. This is because ρ_τ has been chosen so that there is a uniform (in τ) Σ_1-Skolem function mapping (a subset of) J_{ρ_τ} onto J_τ (although uniform, a *parameter* is inserted here dependent on τ). In this, admittedly very scantily sketched, manner we can effectively Σ_2-uniformise all of J_τ. This machinery can be generalised for all $n \geq 2$, and thus prove Jensen's Σ_n-*Uniformisation Theorem*: *Any* $\Sigma_n(J_\tau)$ *relation* *can be uniformised by a* $\Sigma_n(J_\tau)$ *function.* However as we have observed, there can be no uniform way of performing Σ_2-uniformisations: there was the notion of the projectum, and the parameter in the above, which will vary between J_τ's. All the objects concerned (the projecta, the parameters etc.) nevertheless have definitions over J_τ; however these definitions are not at the same level of complexity in the Levy hierarchy of formulae as the level we are working at, and at which we would wish. The Σ^* hierarchy of formulae replaces the Levy hierarchy and seeks to encapsulate the idea that somehow Σ_{n+1} is "Σ_1-in-Σ_n" much as the above talked about Σ_2 (with $2 = n + 1$) as "Σ_1-in-Σ_n" over the *first reduct* or projectum structure $\langle J_{\rho_\tau}, A_\tau \rangle$. Then we think of $\Sigma_1^{(n+1)}$ as Σ_1 in $\Sigma_1^{(n)}$ – and that is why $\Sigma_1^{(n)}$ plays a part in the "atomic" clause definition of $\Sigma_1^{(n+1)}$ below. $\Sigma_1^{(n)}$ formulae then refer to nth reducts only indirectly through the type of the variables.

The Σ^* hierarchy then, is not at all necessary for an exposition of a fine structure theory for L: we have chosen on illustrative grounds to rework the proofs of, say, global $\Box$ for L below, using it. Although for a structure M the $\Sigma_1^{(n)}(M)$ hierarchy of relations is different from the usual $\Sigma_n(M)$ hierarchy *qua* hierarchy, the totality of relations $\boldsymbol{\Sigma^*}(M) = \cup_{n<\omega} \boldsymbol{\Sigma_1^{(n)}}(M)$ are the same: $\boldsymbol{\Sigma_\omega}(M)$. The gain in the Σ^* hierarchy comes chiefly in the development of the theory of fine structure suitable for mice, for fine-structural ultrapowers, and $L[E]$ models. One should further make the remark that much of the work in defining the notions of projecta, good and very good parameters (the sets P_M and R_M), n'th codes, and reducts etc., is prior to the notion of $\Sigma_1^{(n)}$ relation: we shall exploit this here by referring back to the chapter [46] of Schindler and Zeman in this Handbook for these concepts.

Each section consists of one or more fine structural constructions followed by some discussion on variants, extensions, and a necessarily brief list of some sample applications. Thus this first section will continue after these introductory remarks to introduce the $\Sigma_1^{(n)}$ hierarchy of formulae, and the accompanying relations on acceptable structures. Although these pages look notationally complex the proofs are essentially elementary. The notion of $\Sigma_k^{(n)}$*-preserving embeddings* as those embeddings that preserve formulae of the correct type, is of course natural and by 1.15 we can restate Downwards Extension of Embeddings Lemma and Jensen's uniformisation result as a $\Sigma_1^{(n)}$-Uniformisation Theorem. Notions of $\Sigma_1^{(n)}$-Skolem function and of the Condensation Lemma for L in a strong form follow.

In Section 1.1 variant fine structures are discussed: the original form of fine structure for the Dodd-Jensen core model ([10]); the use of Skolem hulls to potentially replace parts of the fine structure apparatus, and the hyperfine structure theory. In Section 1.2 the important theory of Σ^* ultrapowers is developed. This is a natural notion of ultrapower based on the $\Sigma_1^{(n)}$ hierarchy. Section 1.3 reworks this material for so-called pseudo-ultrapowers, otherwise known as long extender ultrapowers. Section 2 is devoted to using this apparatus to a proof of Global $\Box$ in L. As remarked later, we do not actually need the $\Sigma_1^{(n)}$ hierarchy and the corresponding analysis of the constructible hierarchy to effect this. However it is illustrative of the kind of arguments one uses in this arena. Section 2.1 gives some variants of $\Box$ principles. We only very briefly mention some applications, but do discuss consequences of the failure of $\Box_\kappa$. The fascinating question of whether it can be proven that $\Box$ holds in fine structural inner models is discussed in section 2.2. Recent work identifies for which κ in certain classes of $L[E]$ models we can have $\Box_\kappa$. Section 3 discusses morasses and gives constructions of both a gap-1 morass and the simplifying variant of a coarse gap-1 morass in L.

We shall not discuss in any great depth the two major applications of contemporary fine structure: the Covering Lemma and $L[E]$-Inner Models. Although we shall want to refer to these later, these two major topics are properly covered elsewhere in the Handbook, in the chapters by Mitchell and by Schimmerling.

Acknowledgements Although this chapter is an expansion of a brief outline in [56] that was based on Jensen's original [28], the authoritative published reference to the $\Sigma_1^{(n)}$ theory is Zeman's [62]. We are very much indebted to his volume for much of the early part of the exposition here, as well as to the authors of [18] and [32], and would like to thank them. We should like to thank those who have commented helpfully on previous versions of this chapter, and again we are particularly grateful to Martin Zeman in this regard.

By "Lim" ("Sing") we mean the class of limit ordinals (singular limit ordinals, respectively), by "Card" we mean the class of cardinals, and by "Reg", we shall mean the class of regular cardinals. For h_M a Σ_1-Skolem function (see [46] Theorem 1.15) for a structure M, we shall denote by $h_M(X)$ the Skolem hull inside M generated from the set $X \subseteq M$; more properly we should have written $h_M``(\omega \times [X]^{<\omega})$, where in turn $[X]^{<\omega}$ denotes the class of finite subsets of X. We use $\vec{y}$ to denote the list $y_0, \ldots, y_h$, and $\langle \vec{y} \rangle =_{\mathrm{df}} \langle y_0, \ldots, y_h \rangle$ is the ordered sequence, i.e. a finite function. Order type is abbreviated as ot. If $X \subseteq \mathrm{On}$ then X^* denotes the class of limit points of X. We use the following rudimentarily defined wellorder on $[\mathrm{On}]^{<\omega}$: $a <^* b \leftrightarrow \max(a \triangle b) \in b$.

In what follows we assume that $M = \langle J_\alpha^A, \in, A, B \rangle$ is an acceptable J-structure in the sense of [46] Definition 1.20 with J_α^A a level of a relativised J-hierarchy. That is, it is a structure that satisfies the *axiom of acceptability*:

$\forall\tau < \alpha\forall\xi < \tau$:

$$\mathcal{P}(\xi) \cap J^A_{\tau+1} \backslash J^A_\tau \neq \varnothing \longrightarrow (\exists f \in J^A_{\tau+1})\,(f : \xi \stackrel{onto}{\to} \tau\,.)$$

We further recall from there that being acceptable can be expressed as a *Q-property* (see [46] 1.18). We shall write $\rho_M = \rho(M)$ as usual for the Σ_1-projectum of M. ([46] prefers the notation $\rho(M)$ here; we have also adhered to Jensen's notation, and that of [62], in writing here "J_α" for what [46] would name "$J_{\omega\alpha}$"). Similarly we shall write for the $(n+1)$*st projectum* $\rho^{n+1}_M =_{\rm df} \min\{\rho_{M^{n,p}} \mid p \in \Gamma^n_M\}$. ([46] would write $\rho_{n+1}(M)$ for ρ^{n+1}_M here, and similarly $\rho(M^{n,p})$ for $\rho_{M^{n,p}}$; as here, we keep in general to the notation of [62].) We can, and do, assume that parameters are finite sets of ordinals (cf. the comment of [46] before 6.3). This applies as well to the *nth-standard parameter* and the *standard parameter* ([46] 6.3 & 6.6) denoted here p^n_M, p_M respectively for a structure M as above. For the notions of *soundness* and *n-soundness* the reader can refer to [46] Definition 5.2 – and see Lemma 6.8.

The Σ^* Hierarchy of Formulae

Notions of "fine-structural" preserving maps (and ultimately ultrapowers) can be smoothly presented in terms of Jensen's Σ^* hierarchy of formulae. As already intimated, this is a hierarchy of definability over a J-structure, with a different order of stratification than the usual Levy hierarchy of formulae. Although the relations Σ^*-definable over M are also the usual Σ_ω-definable relations, the intermediate levels of Σ_n-definability do not in general correspond to those within Σ^*. More complex initially, the Σ^* hierarchy possesses nice properties the Levy hierarchy lacks: again for example, over L, Jensen's Σ_n-Uniformisation Theorem states that such relations may be Σ_n-uniformised albeit *in a parameter*. When given the Σ^* analysis, $\Sigma^{(n)}_1(J_\alpha)$ relations enjoy *uniform* $\Sigma^{(n)}_1(J_\alpha)$-uniformising functions. Moreover the hierarchy encapsulates the notion of elementarity that the reader of [46] Sections 3 & 4 on Downwards and Upwards Extensions of Embeddings will have already seen expressed by those results. As mentioned above, the hierarchy and all the results here of this first section were first exposited in [28] by Jensen.

We start by defining the language $\mathcal{L}^* = \mathcal{L}^*_{\{\dot\in, \dot=, \dot A, \dot B\}}$. This has *variables* v^i_j $(i, j < \omega)$ of *type i*. The *atomic formulae* are those of the form $v^i_j \in \dot A, v^i_j \in \dot B,\ v^i_j \in v^k_l,\ v^i_j = v^k_l (i, j, k, l < \omega)$. The *formulae* are those obtained from the atomic formulae by closing under $\neg, \wedge$ and *typed quantification* $\exists v^i_j, \forall v^i_j$. An $\mathcal{L}^*$-*structure* is then a structure of the form $\mathfrak{H} = \langle H, A, B, \in, H^0, H^1, \ldots\rangle$ with $H = H^0$ and $H^i \supseteq H^j \neq \varnothing$ for $i \leq j < \omega$, all H^j being transitive. The variables v^n_m are intended to range over H^n. An acceptable J-model $M = \langle J^A_\alpha, \in, A, B\rangle$ can be viewed as the $\mathcal{L}^*$-structure $M = \langle J^A_\alpha, \in, A, B, H^0_M, H^1_M, \ldots\rangle$, which amounts to the *standard interpretation*. Here the variables v^n_m are intended to range over $H^n_M =_{\rm df} H^M_{\omega\rho^n_M}$. Note that

$n \leq m$ implies $\rho_M^n \geq \rho_M^m$, so this makes sense. In what follows we shall use extra variable symbols such x^n, u^m ad lib.

The stratification we referred to earlier is as follows:

1.1 Definition. (a) The $\Sigma_0^{(n)}$ *formulae* are the formulae in the smallest class $\Sigma \subseteq \mathcal{L}^*$ such that:

(i) Σ contains the atomic formulae, and the formulae $\Sigma_1^{(m)}$ for any $m < n$;

(ii) Σ is closed under $\neg, \wedge$, and quantification which binds variables of type n by a higher type; that is if $\varphi \in \Sigma$, so are $\forall x^n \in y^m \varphi$, $\exists x^n \in y^m \varphi$, where $m \geq n$;

(b) $\Sigma_k^{(n)}$ *formulae* are obtained by alternating k blocks of quantifiers of type n: $\exists \vec{x}_1^n \forall \vec{x}_2^n \dots \mathfrak{Q} \vec{x}_k^n \varphi$ where $\varphi \in \Sigma_0^{(n)}$ (and $\mathfrak{Q}$ is $\forall$ or $\exists$ depending on whether k is even or odd).

(c) $\Pi_0^{(n)} = \Sigma_0^{(n)}$, and $\Pi_k^{(n)}$ *formulae* as those of the form: $\forall \vec{x}_1^n \exists \vec{x}_2^n \dots \mathfrak{Q} \vec{x}_k^n \varphi$ for the appropriate $\mathfrak{Q}$; and we define:

$$\Sigma^* = \bigcup_n \Sigma_0^{(n)} \ (= \bigcup_n \Sigma_1^{(n)}) .$$

The notion of $\Sigma_k^{(n)}$ relation and function are defined fairly straightforwardly, using these formulae. We need to analyse such relations and functions, and we need a notion of *good* $\Sigma_1^{(n)}$ *functions* as those functions that may be substituted into a $\Sigma_1^{(n)}$ relation and still yield a $\Sigma_1^{(n)}$ relation. These will be defined below.

Over a model M $\Sigma_k^{(n)}$ relations involve a defining $\Sigma_k^{(n)}$ formula whose variables are of a certain *type*.

1.2 Definition.

(i) Let M be a J-structure; $R(x_1^{i_1}, \dots, x_k^{i_k})$ is a $\Sigma_k^{(n)}(M)$ *relation of type* $\langle i_1, \dots i_k \rangle$ *in parameters* $\vec{q} \in M^{<\omega}$, if and only if R is defined over M by a $\Sigma_k^{(n)}$ formula $\varphi(v_1^{i_1}, \dots, v_k^{i_k}, \vec{q})$. (If we wish to mention the parameters we shall say that R is "$\Sigma_k^{(n)}(M)$ in the parameters $\vec{q}$.")

(ii) R is $\boldsymbol{\Sigma}_k^{(n)}(M)$ if and only if it is $\Sigma_k^{(n)}(M)$ in some parameters $\vec{q}$.

(iii) *The* $\Sigma^*(M)$ *relations* (also written $\Sigma_1^{(\omega)}(M)$), are those $\Sigma_0^{(n)}(M)$ for some n, and $\boldsymbol{\Sigma^*}(M)$ relations are defined analogously.

A relation more formally speaking is thus a pair: $\langle R, \langle i_1, \dots, i_k \rangle \rangle$, consisting of the actual graph of the relation as a subset of $H^{i_1} \times \cdots \times H^{i_k}$ together with its type: $\langle i_1, \dots, i_k \rangle$.

1.3 Definition. A function F is a $\Sigma_k^{(n)}(M)$ *function to* H_M^i *of argument type* $\langle i_1, \dots, i_k\rangle$ if and only "$y^i = F(x_1^{i_1}, \dots, x_k^{i_k})$" is a $\Sigma_k^{(n)}(M)$ relation of type $\langle i, i_1, \dots, i_k\rangle$.

Part of our analysis will show that although, for example, $\Sigma_k^{(n)}(M)$ relations are not just graphs but come with the additional baggage of a type, for most intents and purposes, we shall not have to worry about the type. As the domains $H_M^i \supseteq H_M^j$ for $j \geq i$ are decreasing, replacing a variable in a formula defining a relation, by one of higher type, just reduces the domain specified, and so the resulting relation may be regarded as a *specialisation* of the original: if $\overline{R}(x_1^{j_1}, \dots, x_k^{j_k}), R(x_1^{i_1}, \dots, x_k^{i_k})$ are relations on H, then $\overline{R}$ is a *specialisation* of R, if $j_l \geq i_l (0 < l \leq k)$ and $\overline{R} = R \cap H^{j_1} \times \cdots \times H^{j_k}$.

1.4 Lemma. *If* $R(x_1^i, \vec{x}\,)$ *is* $\Sigma_k^{(n)}(M)$*, and* $j \geq i$*, then so is the specialisation* $R(x_1^j, \vec{x}\,)$.

1.5 Lemma. *If* $R(x_1^j, \vec{x}\,)$ *is* $\Sigma_k^{(n)}(M)$ *(in* $\vec{q}$*) and* $j \geq i \geq n$*, then* R *is a specialisation of a* $\Sigma_k^{(n)}(M)$ *(in* $\vec{q}$*) relation* $R(x_1^i, \vec{x}\,)$.

The proofs of the above are both straightforward inductions on the structure of the defining formula for R. Notice that this implies that any $\Sigma_k^{(n)}(M)$ relation can be considered a specialisation of a $\Sigma_k^{(n)}(M)$ relation whose arguments are all of type less than or equal to n. Note also that trivial operations like permutation of variables, or insertion of dummy ones, in a $\Sigma_k^{(n)}(M)$ relation leave it in the same class of definability.

The next lemma tells how $\Sigma_k^{(n+1)}(M)$ relations R can be expressed as Σ_k relations on a suitably extended $(n+1)$st reduct domain. It is important for the reader to note the uniformities implicit in the statement of the lemma. Firstly note that although the lemma talks about $\Sigma_k^{(n+1)}(M)$ relations, the structure M really does not enter into consideration and the result is independent of it. (The reader should be aware of this kind of ostensible dependence on a structure M, of which it is in reality independent, as this reoccurs in several places in the development below.) The result is really a purely *syntactic* one about the defining formula for R. This is the second uniformity: the relations S^i that appear in the conclusion of the lemma are derived entirely from the matrix of this defining formula, and depend only on it.

1.6 Lemma. $R(\vec{x}^{\,n+1}, \dots, \vec{x}^{\,0})$ *is* $\Sigma_k^{(n+1)}(M)$ *if and only if there are* $\Sigma_1^{(n)}(M)$ *relations* $S^i (i \leq m)$ *such that for all* $\overrightarrow{x} = \vec{x}^{\,n}, \dots, \vec{x}^{\,0} \in M$:

$R_{\overrightarrow{x}} =_{\mathrm{df}} \{\langle \vec{x}^{\,n+1}\rangle \mid R(\vec{x}^{\,n+1}, \overrightarrow{x})\}$ is uniformly $\Sigma_k(\langle H_M^{n+1}, \in, Q^0_{\overrightarrow{x}}, \dots, Q^m_{\overrightarrow{x}}\rangle)$,

where each $Q^i_{\overrightarrow{x}}$ *(for* $i \leq m$*) has the form*

$$Q^i_{\overrightarrow{x}} = \{\langle \vec{y}^{\,n+1}\rangle \mid S^i(\vec{y}^{\,n+1}, \overrightarrow{x})\}\,.$$

Proof. We should first note that the "uniformly" here (a further uniformity) refers to the fact that the same Σ_k formula ψ of the conclusion works for every choice of $\vec{x}$. Although notationally complicated the lemma is actually rather simple: the relations S^i correspond to the fact that $\Sigma_1^{(n)}$ formulae play a role of atomic formulae in the definition of $\Sigma_0^{(n+1)}$ and are in fact the relations defined by those components. To continue in the proof of ($\Longrightarrow$), we may write the formula $\varphi(\vec{x}^{\,n+1},\ldots,\vec{x}^{\,0})$ defining R in prenex form:

$$(+)\quad \exists v_1^{n+1}\forall v_2^{n+1}\cdots(\exists w_1^{n+1}\in u_1^{n+1})(\forall w_2^{n+1}\in u_2^{n+1})\cdots\psi(\vec{v}^{\,n+1},\vec{w}^{n+1},\vec{x}^{\,n+1},\vec{x}\,)$$

(the bounding variables u_j^{n+1} being amongst the $\vec{v}^{\,n+1},\vec{x}^{\,n+1}$). As intimated ψ is a propositional combination of $\Sigma_1^{(n)}$ formulae $\varphi_0,\ldots,\varphi_m$ which define for us (relabelling the type $n+1$ variables as $\vec{y}^{\,n+1}$) the $S^i(\vec{y}^{\,n+1},\vec{x}\,)(i\leq m)$. Let $Q(\vec{y}^{\,n+1},\vec{x}\,)$ be the relation corresponding to that propositional combination of the S^i mirroring the structure ψ. Then $R_{\overrightarrow{x}}(\vec{t}\,)$ can be seen to be expressible as:

$$(*)\qquad \exists v_1\forall v_2\cdots(\exists w_1\in u_1)(\forall w_2\in u_2)\cdots Q_{\overrightarrow{x}}(\vec{v},\vec{w},\vec{t}\,)$$

where now the bounding variables u_j are amongst the $\vec{v},\vec{t}$.

For the converse, if $R_{\overrightarrow{x}}$ is $\Sigma_k(\langle H_M^{n+1},\in,Q^0_{\overrightarrow{x}},\ldots,Q^m_{\overrightarrow{x}}\rangle)$, then it has the form of $(*)$. We now just unwind the previous process, thinking of each $Q^j_{\overrightarrow{x}}(\vec{v},\vec{w},\vec{t}\,)$ as a propositional combination of $Q^i_{\overrightarrow{x}}$; we now replace $Q^i_{\overrightarrow{x}}$,$\vec{v}$,$\vec{w}$, $\vec{t}$, by φ_i, $\vec{v}^{\,n+1}$, $\vec{w}^{n+1}$, $\vec{x}^{\,n+1}$ and obtain the $\Sigma_k^{(n+1)}$ form of $(+)$. $\dashv$

The next lemma relates the last to the reduct structures $M^{n,p(\vec{x}\,)}$.

1.7 Lemma.

(i) *Let* $0<k$. $R(\vec{x}^{\,n},\ldots,\vec{x}^{\,0})$ *is* $\Sigma_k^{(n)}(M)$ *if and only if the relation* $R_{\overrightarrow{x}}=\{\langle\vec{x}^{\,n}\rangle\mid R(\vec{x}^{\,n},\overrightarrow{x})\}$ *of the last lemma is uniformly* $\Sigma_k(M^{n,p(\vec{x}\,)})$*, where* $p(\overrightarrow{x})=_{\rm df}\langle\langle\vec{x}^{\,0}\rangle,\ldots,\langle\vec{x}^{\,n-1}\rangle\rangle$.

(ii) *If* $R(\vec{x}^{\,n},\ldots,\vec{x}^{\,0})$ *is* $\Sigma_0^{(n)}(M)$ *then* $R_{\overrightarrow{x}}$ *is uniformly rudimentary in* $M^{n,p(\vec{x}\,)}$*; conversely if* $R_{\overrightarrow{x}}$ *is* $\Sigma_0(M^{n,p(\vec{x}\,)})$ *then* $R(\vec{x}^{\,n},\ldots,\vec{x}^{\,0})$ *is* $\Sigma_0^{(n)}(M)$.

Proof. By an induction on n. For $n=0$ it is trivial. For (i), we suppose this is true for m, and we shall prove it for $n=m+1$.

For the forward direction, let $R(\vec{x}^{\,n},\ldots,\vec{x}^{\,0})$ be $\Sigma_k^{(n)}(M)$. By appealing to, and using the notation of, the last lemma, there are $\Sigma_1^{(m)}(M)$ relations $S^0,\ldots,S^t$ so that for $\overrightarrow{x}=\vec{x}^{\,m},\ldots,\vec{x}^{\,0}\in M$, $R_{\overrightarrow{x}}$ is uniformly $\Sigma_k(\langle H_M^n,\in,Q^0_{\overrightarrow{x}},\ldots,Q^t_{\overrightarrow{x}}\rangle)$. However each $S^i(\vec{z}^{m+1},\overrightarrow{x})$ can be considered a specialisation of a $\Sigma_1^{(m)}(M)$ relation $\widetilde{S}^i(\vec{z}^{m},\overrightarrow{x})$. We use the notation

$\overrightarrow{y} = \vec{x}^{m-1}, \ldots, \vec{x}^0$ and $p(\vec{y}) =_{\mathrm{df}} \langle\langle\vec{x}^0\rangle, \ldots, \langle\vec{x}^{m-1}\rangle\rangle$. Then we can rewrite: $\widetilde{S}^i(\vec{z}^m, \overrightarrow{x})$ as $\widetilde{S}^i(\vec{z}^m, \vec{x}^m, \overrightarrow{y})$. The inductive hypothesis implies that

$$Q^i_{\overrightarrow{y}} =_{\mathrm{df}} \{\langle\vec{z}^m, \vec{x}^m\rangle \mid \widetilde{S}^i(\vec{z}^m, \vec{x}^m, \overrightarrow{y})\}$$

is uniformly $\Sigma_1(M^{m,p(\vec{y})})$. There is thus a fixed Σ_1-formula $\varphi_{j(i)}$ such that

$$\begin{aligned}\langle\vec{z}^m, \vec{x}^m\rangle \in Q^i_{\overrightarrow{y}} &\iff M^{m,p(\vec{y})} \models \varphi_{j(i)}(\vec{z}^m, \vec{x}^m)\\ &\iff \langle j(i), \vec{z}^m\rangle \in A^{\vec{x}^m}_{M^{m,p(\vec{y})}}\end{aligned}$$

where the latter is of course the standard code predicate occurring in the nth reduct determined by $p(\vec{x})$: $M^{n,p(\vec{x})}$ (cf. [46] Definition 5.1). However $\langle\vec{z}^m\rangle \in Q^i_{\overrightarrow{x}} \iff \langle\vec{z}^m, \vec{x}^m\rangle \in Q^i_{\overrightarrow{y}}$, so if we replace every occurrence of "$\langle\vec{z}^m\rangle \in Q^i_{\overrightarrow{x}}$" by "$\langle j(i), \vec{z}^m\rangle \in A^{\vec{x}^m}_{M^{m,p(\vec{y})}}$" in the $\Sigma_k(\langle H^n_M, \in, Q^0_{\overrightarrow{x}}, \ldots, Q^t_{\overrightarrow{x}}\rangle)$ definition of $R_{\overrightarrow{x}}$, we obtain a $\Sigma_k(M^{n,p(\vec{x})})$ definition for it. These substitutions and translations did not depend on $\overrightarrow{x}$ or M, but only on the definition of R, and so are themselves uniform and effective.

For the converse, suppose that $R_{\vec{x}}$ is uniformly $\Sigma_k(M^{n,p(\overrightarrow{x})})$ definable by some formula Φ say. This will contain atomic components of the form "$\langle j, \langle\vec{x}^n, \vec{z}^n\rangle\rangle \in A^{\vec{x}^m}_{M^{m,p(\vec{y})}}$" (continuing with the same notation as from the first part). If we can show that these atomic formulae are expressible as $\Sigma_1^{(m)}$ formulae (in variables $j, \vec{z}^n, \vec{x}^n, \ldots, \vec{x}^0$) then we may effectively transform Φ into a $\Sigma_k^{(m)}$ formula by substituting these $\Sigma_1^{(m)}$ formulae for the atomic components. We have however:

$$\langle j, \langle\vec{x}^n, \vec{z}^n\rangle\rangle \in A^{\vec{x}^m}_{M^{m,p(\vec{y})}} \iff M^{m,p(\vec{y})} \models \varphi_j(\langle\vec{x}^n, \vec{z}^n\rangle, \vec{x}^m)\,.$$

We consider the right side here a specialisation of the relation: $M^{m,p(\vec{y})} \models \varphi_j(\langle\vec{w}^m, \vec{z}^m\rangle, \vec{x}^m)$ and the latter is (since Σ_1 satisfaction over J-models is uniformly Σ_1-definable) $\Sigma_1(M^{m,p(\vec{y})})$ in the variables $j, \vec{w}^m, \vec{z}^m, \vec{x}^m$ uniformly for all $\vec{y}$, and by the inductive hypothesis it is $\Sigma_1^{(m)}(M)$. Hence by Lemma 1.4 the atomic component is $\Sigma_1^{(m)}(M)$ also, This suffices for (i).

For (ii), the converse direction works as above, but in the forwards direction, when we perform the replacement "$\langle\vec{z}^m\rangle \in Q^i_{\overrightarrow{x}}$" by "$\langle j(i), \vec{z}^m\rangle \in A^{\vec{x}^m}_{M^{m,p(\vec{y})}}$" as the latter is in general not Σ_0 in the predicate $A^{\vec{x}^m}_{M^{m,p(\vec{y})}}$ we obtain only that it is rudimentary in $M^{n,p(\vec{x})}$ – but that is the only difference. $\dashv$

We can argue similarly to the above that $(n+1)$st standard codes are themselves uniformly $\Sigma_1^{(n)}$-definable:

1.8 Lemma. *There is an* $A^*(\vec{x}^{n+1}, \ldots, \vec{x}^0)$ *which is* $\Sigma_1^{(n)}(M)$ *uniformly for all* J*-structures* M *such that (again with* $p(\vec{x})$ *as in the last lemma, and*

$\overrightarrow{x} = \vec{x}^{\,n}, \ldots, \vec{x}^{\,0})$

$$\langle \vec{x}^{\,n+1} \rangle \in A_M^{n+1,p(\vec{x}\,)} \Longleftrightarrow A^*(\vec{x}^{\,n+1}, \ldots, \vec{x}^{\,0})\,;$$

i.e. $A_M^{n+1,p(\vec{x}\,)} = A^*_{\vec{x}}$.

In general $\Sigma_1^{(n)}(M)$ relations are not closed under substitution of $\Sigma_1^{(n)}(M)$ functions. We shall need to define a class of functions, the *good functions* which do permit this kind of substitutability. As a preliminary:

1.9 Lemma. *Let $m \leq n, 0 < k$. Let $R(\vec{x}^{\,n}, \ldots, \vec{x}^{\,0})$ be $\Sigma_k^{(n)}(M)$. Let $\vec{F}^n, \ldots, \vec{F}^0$ be such that each $F_j^m(\vec{z}^0, \ldots, \vec{z}^m)$ is a (possibly partial) $\Sigma_1^{(m)}(M)$ function to H_M^m. Then $R(\overrightarrow{F^i(\overrightarrow{z})})$ is uniformly $\Sigma_k^{(n)}(M)$.*

Proof. By induction on n. We assume it holds for $l < n$. For the sake of brevity, we consider just a single $F^m(\vec{z}\,)$ of value type $m \leq n$, this illustrates the idea and the reader will see that the rest is merely complication of notation. We can consider $R(F^m(\vec{z}\,), \vec{x}\,)$ as defined by:

$$M \models \exists x^m (x^m = F^m(\vec{z}\,) \wedge R(x^m, \vec{x}\,))\,.$$

If $m = n$ this already shows that $R(F^m(\vec{z}\,), \vec{x}\,)$ is $\Sigma_k^{(n)}(M)$. So suppose $m < n$ and hence $0 < n$. Write out a prenex form of the definition of $R(x^m, \vec{x}\,)$ with a $\Sigma_0^{(n)}$ matrix φ as:

$$M \models \exists \vec{v}_1^n \forall \vec{v}_2^n \cdots Q \vec{v}_k^n \varphi(\vec{v}, x^m, \vec{x}\,)\,.$$

The free variable x^m only occurs in the $\Sigma_1^{(n-1)}$ "atomic constituents" of the $\Sigma_0^{(n)}$ formula φ. We shall apply the induction hypotheses to these constituents: they can be effectively listed, and if $\psi(x^m, \vec{y}\,)$ is a typical member of this list, then apply the inductive hypothesis to $\psi(F^m(\vec{z}\,), \vec{y}\,)$ to yield some $\Sigma_0^{(n)}$ formula. If this is done throughout φ the resultant formula $\widetilde{\varphi}(\vec{v}, \vec{z}, \vec{x}\,)$ is itself now $\Sigma_0^{(n)}$. Hence $R(F^m(\vec{z}\,), \vec{x}\,)$ is definable over M by

$$\exists \vec{v}_1^n \forall \vec{v}_2^n \cdots Q \vec{v}_k^n (\exists x^m = F^m(\vec{z}\,) \wedge \widetilde{\varphi}(\vec{v}, \vec{z}, \vec{x}\,))\,.$$

The process is clearly effective and independent of M. $\dashv$

1.10 Definition. The *good* $\Sigma_1^{(n)}(M)$ *functions* form the smallest class $\mathfrak{F}$, such that, taking $i, j_1, \ldots, j_k \leq n$,

(i) Each partial $\Sigma_1^{(i)}(M)$ function to H_M^i of the form $F(x_k^{j_k}, \ldots, x_1^{j_1}) = x^i \in \mathfrak{F}$;

(ii) If $F(x_k^{j_k}, \ldots, x_1^{j_1}) = x^i \in \mathfrak{F}$ and $G_i(\vec{y}\,) = z^{j_i} \in \mathfrak{F}$ (and the $\vec{y}$ all have type $\leq n$), then $F(G_k(\vec{y}\,), \ldots, G_1(\vec{y}\,)) \in \mathfrak{F}$.

The previous lemma together with an induction on the scheme generating the good $\Sigma_1^{(n)}$ functions proves:

1.11 Lemma. *Let $R(x^i, \vec{x}\,)$ be $\Sigma_k^{(n)}(M), k \geq 1, n \geq i$. Let $F^i(\vec{y})$ be a good $\Sigma_1^{(n)}(M)$ function of value type i. Then $R(F^i(\vec{y}\,), \vec{x}\,)$ is uniformly $\Sigma_k^{(n)}(M)$.*

Again we should remark that the $\Sigma_k^{(n)}$ definition of the resultant relation is uniformly obtained from the scheme generating $\Sigma_1^{(n)}$ good functions, and the definition of $R(x^i, \vec{x}\,)$. It has nothing to do with M (or $\vec{x}\,$). The following corollary then shows that $\Sigma_1^{(n)}$ relations are after all characterisable by their graphs alone.

1.12 Corollary.

(i) *Let $R(x_1^{i_1}, \ldots, x_k^{i_k})$ and $R(y_1^{j_1}, \ldots, y_k^{j_k})$ have the same graph. Then*

$$R(x_1^{i_1}, \ldots, x_k^{i_k}) \in \Sigma_1^{(n)}(M) \Longleftrightarrow R(y_1^{j_1}, \ldots, y_k^{j_k}) \in \Sigma_1^{(n)}(M)\,.$$

(ii) *In particular, if $R(x_1^{i_1}, \ldots, x_k^{i_k}\,) \in \Sigma_1^{(n)}(M)$, then it is a specialisation of $R'(x_1^0, \ldots, x_k^0) \in \Sigma_1^{(n)}(M)$ with the same graph, and all of whose arguments are of value type 0.*

Proof. To see (i) we may substitute into $R(x_1^{i_1}, \ldots, x_k^{i_k})$ the good $\Sigma_1^{(n)}(M)$-projection functions $x_1^{i_1} = y_1^{j_1}, \ldots, x_k^{i_k} = y_1^{j_k}$. (ii) is then a special case of this. $\dashv$

$\Sigma_k^{(n)}$-Preserving Embeddings.

We may define $\Sigma_k^{(n)}$-preserving embeddings in a natural way: Let $\bar{M}, M$ be J-models, $n, l < \omega$.

(i) $\pi : \bar{M} \longrightarrow_{\Sigma_l^{(n)}} M$ iff $\pi : \bar{M} \longrightarrow M$ and whenever $\varphi(v_1^{j_1}, \ldots v_m^{j_m}) \in \Sigma_l^{(n)}, x_i \in H_{\bar{M}}^{j_i}$ $(1 \leq i \leq m)$ then $\pi(x_i) \in H_M^{j_i}$, and

$$\bar{M} \models \varphi(\vec{x}\,) \Longleftrightarrow M \models \varphi(\overrightarrow{\pi(x)})\,.$$

(ii) $\pi : \bar{M} \longrightarrow_{\Sigma^*} M$ iff $\pi : \bar{M} \longrightarrow_{\Sigma_1^{(n)}} M$ for all $n < \omega$.

(iii) $\pi : \bar{M} \longrightarrow_{\Sigma_0^{(n)}} M$ *cofinally* iff $\pi : \bar{M} \longrightarrow_{\Sigma_0^{(n)}} M$ and $H_M^n = \bigcup \pi``H_{\bar{M}}^n$.

If F is a good $\Sigma_1^{(n)}(\bar{M})$-definable function, then one may show that F has such a definition that is "functionally absolute"; i.e. a definition that defines a function over any acceptable structure N, and thus is robust under $\Sigma_1^{(n)}$-preserving maps. (See [62] 1.8.10.) The key to this is that the canonical

Σ_1-Skolem function has such a definition over any acceptable J-structure, in particular for those of the form $M^{n,p}$. Thence the same holds for any other $\Sigma_1^{(n)}(M)$ function to H^n_M also: briefly if $f(\overrightarrow{x})$ where $\overrightarrow{x} = \vec{x}^{\,n}, \ldots, \vec{x}^{\,0}$ is a $\Sigma_1^{(n)}(M)$ function to H^n_M defined by some formula φ, we may define $g_{\overrightarrow{x}}(\vec{x}^{\,n}) \simeq f(\overrightarrow{x})$ and $p(\vec{x}) =_{\mathrm{df}} \langle\langle \vec{x}^{\,0}\rangle, \ldots, \langle \vec{x}^{\,n-1}\rangle\rangle$. This makes the definition of $g_{\overrightarrow{x}}(\vec{x}^{\,n})$ $\Sigma_1(M^{n,p(\vec{x})})$. However this depends only on φ and not $\overrightarrow{x}$. Hence there is a single fixed i so that $g_{\overrightarrow{x}}(\vec{x}^{\,n}) \simeq h_{M^{n,p(\vec{x})}}(i, \vec{x}^{\,n})$ and the latter is a uniform functionally absolute definition. In short, concerning embeddings, any $\Sigma_1^{(n)}(\bar{M})$ functionally absolute definition applied over $\bar{M}$ where $\pi : \bar{M} \longrightarrow_{\Sigma_1^{(n)}} M$ also yields a $\Sigma_1^{(n)}$ function over M.

1.13 Lemma. *Let $\bar{M}, M$ be acceptable J-structures. Then: $\pi : \bar{M} \longrightarrow_{\Sigma_k^{(n)}} M$ iff $\pi : \bar{M} \longrightarrow M$ and whenever $\bar{p} \in \Gamma^n_{\bar{M}}$ then $p = \pi(\bar{p}) \in \Gamma^n_M$, and $\pi{\restriction}H^n_{\bar{M}} : \bar{M}^{n,\bar{p}} \longrightarrow_{\Sigma_k} M^{n,p}$.*

Proof. This is a direct consequence of Lemma 1.7. ⊣

The following two lemmata correspond to [46] 5.8 and 5.9 expressed in our language.

1.14 Lemma. *Let $\bar{M}, M$ be acceptable J-structures. Let $\pi : \bar{M} \longrightarrow M$ be the n-completion of $\pi{\restriction}H^n_{\bar{M}} : \bar{M}^{n,\bar{p}} \longrightarrow_{\Sigma_k} M^{n,p}$ where $\bar{p} \in R^n_{\bar{M}}$ and $p = \pi(\bar{p}) \in \Gamma^n_M$. Then $\pi : \bar{M} \longrightarrow_{\Sigma_k^{(n)}} M$.*

Proof. Let $\bar{q} \in \Gamma^n_{\bar{M}}$ be arbitrary. As $\bar{p} \in R^n_{\bar{M}}$ we have that for any $q = \pi(\bar{q}) \in \Gamma^n_M$ that $A^{n,\bar{q}}_{\bar{M}}$ is rudimentary in $A^{n,\bar{p}}_{\bar{M}}$ in some parameter r say. (See the proof of [46] 5.3(b).) Then $A^{n,q}_M$ is rudimentary in $A^{n,p}_M$ in $\pi(r) \in H^n_M$ by the same rudimentary definition. So $\pi{\restriction}H^n_{\bar{M}} : \bar{M}^{n,\bar{q}} \longrightarrow_{\Sigma_k} M^{n,q}$. As $\bar{q}$ was arbitrary, the previous lemma shows $\pi : \bar{M} \longrightarrow_{\Sigma_k^{(n)}} M$. ⊣

1.15 Lemma (Downwards Extension of Embeddings Lemma). *Let $\bar{M}, M$ be acceptable J-structures. Let $\pi{\restriction}H^n_{\bar{M}} : \bar{M}^{n,\bar{p}} \longrightarrow_{\Sigma_k} M^{n,p}$ where $\bar{p} \in R^n_{\bar{M}}$. Then there is a unique $\widetilde{\pi} \supseteq \pi$ such that $\widetilde{\pi}(\bar{p})$=p and $\widetilde{\pi} : \bar{M} \longrightarrow_{\Sigma_k^{(n)}} M$.*

Some of the fruits of this analysis are seen in the following two lemmas, the first of which is the analogue of Jensen's classical Σ_n-Uniformisation Lemma.

1.16 Lemma ($\boldsymbol{\Sigma_1^{(n)}}$-Uniformisation Theorem). *Let $R(y^n, x_1^{j_1}, \ldots x_m^{j_m}) \in \Sigma_1^{(n)}(M)$ with $j_i \leq n$. Then there is a $\Sigma_1^{(n)}(M)$ function F (uniformly definable with respect to any such M) into H^n_M (which is thus good) uniformising R. Namely:*

(a) $\mathrm{dom}(F) = \{\langle x_1^{j_1}, \ldots x_m^{j_m}\rangle \in M \mid \exists y^n \in M R(y^n, \vec{x})\}$ *(where $\vec{x} = x_1^{j_1}, \ldots, x_m^{j_m}$).*

(b) $\forall \vec{x} = x_1^{j_1}, \dots x_m^{j_m} \exists y^n R(y^n, \vec{x}\,) \leftrightarrow R(F(\vec{x}\,), \vec{x}\,)$.

Proof. Writing R in the form $R(y^n, \vec{x}^{\,n}, \dots, \vec{x}^{\,0})$ and setting again $\overrightarrow{x} = \vec{x}^{\,n-1}, \dots, \vec{x}^{\,0}$, we have that if $R_{\overrightarrow{x}} =_{\rm df} \{\langle y^n, \vec{x}^{\,n}\rangle \mid R(y^n, \vec{x}^{\,n}, \overrightarrow{x})\}$, then Lemma 1.7(i) shows that $R_{\overrightarrow{x}}$ is uniformly $\Sigma_1(M^{n,p(\vec{x}\,)})$, where, as before, $p(\vec{x}\,) =_{\rm df} \langle\langle\vec{x}^{\,0}\rangle, \dots, \vec{x}^{\,n-1}\rangle\rangle$. If φ_i is a Σ_1 formula yielding this definition, then we may define the partial function

$$F(\vec{x}^{\,n}, \overrightarrow{x}) = F_{\overrightarrow{x}}(\vec{x}^{\,n}) \simeq h_{M^{n,p(\vec{x})}}(i, \langle\vec{x}^{\,n}\rangle)$$

Then $F_{\overrightarrow{x}}$ is a uniformly $\Sigma_1(M^{n,p(\vec{x}\,)})$-definable Skolem function for $R_{\overrightarrow{x}}$, and by Lemma 1.7(i) F is $\Sigma_1^{(n)}$ with value type n, and will do the job. ⊣

1.17 Lemma. *Let $n < \omega$. There is a $\Sigma_1^{(n)}$ formula defining a good $\Sigma_1^{(n)}(M)$ function $F_n(u, v)$ into M (definable uniformly with respect to any J-structure M), so that*

$$\forall p\, (p \in R_M^{n+1} \longrightarrow |M| = F_n \text{“} H_M^{n+1} \times \{p\})\,.$$

Proof. By induction on n. For $n = 0$, we have that $p(0) \in R_M$ and thus $|M| = h_M\text{“}(\omega \times (H_M^1 \times \{p(0)\}))$. Thus we can take $F_0(\langle i, v\rangle, w) = h_M(i, \langle v, w\rangle)$. Suppose the statement of the lemma holds for $n-1$ as witnessed by $F_{n-1}(u^{n-1}, w^0)$ and we prove it for n. Let $p \in R_M^{n+1}$. Then $p{\restriction}n \in R_M^n$. Let $x \in |M|$ be arbitrary; then $x = F_{n-1}(z, p{\restriction}n)$ for some $z \in H_M^n$; z in turn equals $h_{M^{n,p\restriction n}}(j, \langle y, p(n)\rangle$ for some $y \in H_M^{n+1}$ since $p(n) \in R_{M^{n,p\restriction n}}$ as well. $F_{n-1}(u^n, w^0)$ is also good $\Sigma_1^{(n-1)}(M)$ (as it is obtained from the good $\Sigma_1^{(n-1)}(M)$ function $F_{n-1}(u^{n-1}, w^0)$ by specialising the first variable u^{n-1} to u^n; it is a general fact goodness is preserved by specialisation). Hence we may define

$$F_n(u^n, w^0) \simeq F_{n-1}(h_{M^{n,w^0\restriction n}}((u^n)_0, \langle (u^n)_1, w^0(n)\rangle), w^0{\restriction}n)\,.$$

As we are substituting good $\Sigma_1^{(n)}(M)$ functions (with value type $\leq n$) into $F_{n-1}(u^n, w^0)$ we end up with a good $\Sigma_1^{(n)}(M)$ function F_n. ⊣

To bring out the nature of the above argument, we may define

$$g_n(\langle j, y^{n+1}\rangle, p) = h_{M^{n,p\restriction n}}(j, \langle y^{n+1}, p(n)\rangle)$$

and more generally

$$g_i(\langle j, y^{i+1}\rangle, p) = h_{M^{i,p\restriction i}}(j, \langle y^{i+1}, p(i)\rangle)$$

and this is uniformly lightface $\Sigma_1^{(i)}(M)$ in the two variables $w^{i+1} = \langle j, y^{i+1}\rangle$ and $x^0 = p$ for $i \leq n$. We may compose these functions and if $p \in R_M^{n+1}$ then any $x \in |M|$ is the value of such an iterated composition. This is expressed below at part (ii) of the next Definition and the Fact that follows it.

1.18 Definition ($\Sigma_1^{(n)}$-Skolem Functions). Let M be an acceptable J-structure, and let $p \in \Gamma_M^n$.

(i) $h_M^{n,p} = h_{M^{n,p}}$;

(ii) $\widetilde{h}_M^n(w^n, x^0) = g_0(g_1(\cdots g_{n-1}((w^n)_0, \langle (w^n)_1, x^0(n-1)\rangle) \cdots x^0(0)\rangle)$.

Thus $\widetilde{h}_M^n$ is $\Sigma_1^{(n-1)}$ uniformly over all M. The Σ_1 hull of a set $X \subseteq M^{n,p}$ we shall denote by $h_M^{n,p}(X)$ (and is thus the set $\{h_M^{n,p}(i,x)) \mid i \in \omega, x \in X\}$). Note that $\widetilde{h}_M^1(\langle j, y^0\rangle, p(0)) = g_0(j, \langle y, p(0)\rangle) = h_M(j, \langle y, p(0)\rangle)$. If $p \in R_M^n$ then every $x \in M$ is of the form $\widetilde{h}_M^n(z,p)$ for some $z \in H_M^n$. We may similarly form hulls using $\widetilde{h}_M^n$: again if $X \subseteq M^{n,p}$ say, and $q \in M$ then the $\Sigma_1^{(n-1)}$ hull of $X \cup \{q\}$ is the set $\{\widetilde{h}_M^n(x,q)) \mid x \in X\}$). The following states some of these facts and are now easy to establish (see [62] p.29):

Fact: *Let M be acceptable, and $p \in R_M^n$.*

(i) *If $\omega\rho_M^n \in M$ and $p \in R_M^n$ then $\widetilde{h}_M^n$ is a good, uniformly defined, $\Sigma_1^{(n-1)}(M)$ function mapping H_M^n onto M: $M = \{\widetilde{h}_M^n(u^n,p) \mid u^n \in H_M^n\}$.*

(ii) (a) *every $A \subseteq H_M^n$ which is $\boldsymbol{\Sigma}_1^{(n)}(M)$ is $\boldsymbol{\Sigma_1}(M^{n,p})$;*

(b) *$\rho_M^{n+1} = \rho_{M^{n,p}}$.*

1.19 Lemma. *Let M be an acceptable J-structure.*

(i) $\boldsymbol{\Sigma}^*(M) \subseteq \boldsymbol{\Sigma}_\omega(M)$.

(ii) *If M is sound then $\boldsymbol{\Sigma^*(M) = \Sigma_\omega(M)}$.*

Proof. For (i) we just have to see that any typed variable v^n can be replaced by $v \in H_M^n$ and the latter is definable (not necessarily at the nth level of the Levy-hierarchy of complexity!). For (ii) we prove ($\supseteq$). Suppose by induction we have shown that for every $\Sigma_m(M)$ formula $\varphi(\vec{y})$ there is a $\boldsymbol{\Sigma_1^{(m)}}(M)$ formula $\widetilde{\varphi}(\vec{y^0})$ such that $(\forall \vec{x} \in M)((\varphi(\vec{x}))_M \Leftrightarrow (\widetilde{\varphi}(\vec{x}))_M$. Let $n = m+1$ and suppose $p \in R_M^n$ (as M is sound). Let $\widetilde{h}_M^n$ be as at (i) of the last Fact. This is a good $\Sigma_1^{(m)}$ function of value type 0 (and hence good $\Sigma_1^{(n)}$ also) such that

$$\begin{aligned}(\exists x \psi(x,\vec{y}))_M &\Longleftrightarrow (\exists u \in H_M^n)(\psi(\widetilde{h}_M^n(u,p),\vec{y}))_M \\ &\Longleftrightarrow (\exists u \in H_M^n)(\widetilde{\psi}(\widetilde{h}_M^n(u,p),\vec{y}))_M\end{aligned}$$

where $\widetilde{\psi}$ is the $\boldsymbol{\Pi_1^{(m)}}(M)$-formula given by the inductive hypothesis. Thence $(\widetilde{\psi}(\widetilde{h}_M^n(u,p),\vec{y}))_M$ is a $\boldsymbol{\Sigma_1^{(n)}}(M)$ property and $(\exists u^n \widetilde{\psi}(\widetilde{h}_M^n(u,p),\vec{y}))_M$ is a $\boldsymbol{\Sigma_1^{(n)}}(M)$ relation. $\dashv$

The following is a standard result. For its proof see [62] 1.11.2.

1.20 Lemma. *Suppose* $\pi : \bar{M} \longrightarrow_{\Sigma_1^{(n)}} M$ *and is such that (i)* $\pi \restriction \omega\rho_{\bar{M}}^{n+1} = \mathrm{id} \restriction \omega\rho_{\bar{M}}^{n+1}$ *and (ii)* $\mathrm{ran}(\pi) \cap P_M^* \neq \varnothing$ *. Then* π *is* Σ^**-preserving.*

Solidity Witnesses

Naturally the notion of *solidity witness* (see [46] Section 7) can be defined in this context. We shall not re-enter a discussion of these notions, but simply give the definition.

1.21 Definition. Let M be an acceptable J-structure, $p \subseteq M$ a finite set of ordinals, and $\nu \in M$. The *standard witness for* ν *with respect to* M, p is the J-structure $W = W_M^{\nu,p}$ where, if n is such that $\omega\rho_M^{n+1} \leq \nu < \omega\rho_M^n$:
$\sigma : W \cong X$ and $X = \widetilde{h}_M^{n+1}(\nu \cup p\backslash(\nu+1))$ and σ is the inverse of the transitive collapse.

The notion of generalised witness has an analogous definition mutatis mutandis. The properties of witnesses, the definitions of n-solidity etc, then all go through. It is easy to check for the L-hierarchy that if $M = J_\beta$, $p = p_M$, the standard parameter, and $\omega\rho_M^{n+1} \leq \nu < \omega\rho_M^n$, then $\nu \in p \iff W_M^{\nu,p} \in M$. We have for the L-hierarchy a strong form of condensation. We shall use the fact that the L-hierarchy is sound (cf. [46] Lemma 9.2.) Note as always for the pure J-hierarchy, the usual condensation property that hulls of a J_β always transitivise to some J_δ for a $\delta \leq \beta$.

1.22 Lemma (Condensation Lemma). *Let* $M = J_\beta$, $\pi : J_\delta \longrightarrow_{\Sigma_1^{(n)}} J_\beta$, $\omega\rho_M^{n+1} \leq \alpha < \omega\rho_M^n$, $\pi(\bar{\alpha}) = \alpha$, *and* $\pi(\bar{p}) = p_{J_\beta}\backslash\alpha$. *Then* $\bar{p} = p_{J_\delta}\backslash\bar{\alpha}$.

Proof. As π is $\Sigma_1^{(n)}$-preserving (and $\widetilde{h}_M^{n+1}(u^{n+1} \cup \{p_M\})$ is good $\Sigma_1^{(n)}(M)$) $J_\delta = \widetilde{h}_{J_\delta}^{n+1}(\bar{\alpha} \cup \{\bar{p}\})$. Thence follows that $\omega\rho_{J_\delta}^{n+1} \leq \bar{\alpha}$: for if this failed we should have a $\mathbf{\Sigma_1^{(n)}}(J_\delta)$ definable map of $\bar{\alpha} < \omega\rho_{J_\delta}^{n+1}$ onto the whole of J_δ. However this is impossible as the usual simple diagonalisation argument would yield a $\mathbf{\Sigma_1^{(n)}}(J_\delta)$-definable subset of $\bar{\alpha}$ which is not in J_δ. This would contradict the definition of $\omega\rho_{J_\delta}^{n+1}$. That $\bar{\alpha} < \omega\rho_{J_\delta}^n$ follows from the $\Sigma_1^{(n)}$ statement "$\exists u^n(u^n = \pi(\bar{\alpha}))$" and the $\Sigma_1^{(n)}$-preservation property of π.

As $J_\delta = \widetilde{h}_{J_\delta}^{n+1}(\bar{\alpha} \cup \{\bar{p}\})$, we have that $\bar{p} \in R_{J_\delta}^n$; this means that it can be lengthened to a parameter $p' \in P_{J_\delta}^{n+1}$ (cf. [46] Lemma 6.4). By the minimality properties of the standard parameter ([46] Definition 6.3) we have then that $p_{J_\delta}\backslash\bar{\alpha} \leq^* \bar{p}$. If we had $<^*$ instead of $\leq^*$ then we should have that $\bar{p} \in \widetilde{h}_{J_\delta}^{n+1}(\bar{\alpha} \cup \{p_{J_\delta}\backslash\bar{\alpha}\})$. Applying the $\Sigma_1^{(n)}$-preserving π we'd conclude that $p_M\backslash\alpha \in \widetilde{h}_M^{n+1}(\alpha \cup \{\pi(p_{J_\delta}\backslash\bar{\alpha})\})$ with $\pi(p_{J_\delta}\backslash\bar{\alpha}) = \pi(p_{J_\delta})\backslash\alpha <^* p_M\backslash\alpha$. A contradiction. $\dashv$

1.1. Variant Fine Structures

Although we may consider the $\Sigma_1^{(n)}$ hierarchy as emerging finally out of the original fine structure of [24], the historical line of development is not so direct. From the time of Jensen's proof of the Covering Lemma for L some attempts at simplifications were made by several people. Silver's development of *Silver Machines* built the L-hierachy in a quite different manner by using a very much slowed down version of construction based on creating hulls and using a different collection of Skolem functions. As opposed to steps in the L-hierachy, very little happens in the transition from M^δ to $M^{\delta+1}$ in this machine hierarchy. This is expressed by a *Finiteness property*: the hull in $M^{\delta+1}$ of a set A can be obtained by taking the hull in M^δ of $A \cap \delta$ together with just finitely many more ordinals less than δ (and lastly adding the point δ). This relatively simple apparatus enjoyed sufficient condensation properties that proofs of $\Box$ and Jensen's Covering Lemma for L could be attained (see [8] Ch.IX for an outline here, and [2] for a machine proof of $\Box$). Once the theory of core models came to the fore it was not apparent that this mechanism could be used as a substitute for the fine structure of mice with measures that was then emerging.

The fine structure for mice has also not had a direct development. We mention here some of the history of this fine structure, and the various uses of the word "acceptable" in the literature. The fine structure for the mice of the Dodd-Jensen core model [10] and [9], although used for later core models with sequences of measures [35], and even in manuscript form of Dodd's for models with extenders, would seem to be far from amenable to the type of approximation hierarchy of Silver. The Dodd-Jensen fine structure was based on previous unpublished work of Solovay who had built on Jensen's fine structure for L, to do the same for $L[U]$; the order of set construction was that of relative constructibility in the traditional "macro" sense, and two notions of "acceptability" and "strong acceptability" occurred here. The latter is now closer to the current, and by now standard, use of the word 'acceptable'. This detailed how sets appeared in such J^U_α hierarchies, and which expressed a strong and uniform version of the GCH. However they bore little resemblance to the current ordering of $L[E]$ hierarchies, and were becoming extremely difficult to work with, and, past a strong cardinal, particularly hair-raising. Various papers from the time were published using this older fine structure, and for very thin core models, that are inner models of K^{DJ}, the vestigial notions of "Q-structure" that were a part of the old theory still play a role (eg. [58]). However with the very successful reorganising of hierarchies due to Baldwin and Mitchell this form of fine structure became defunct.

Magidor replaced fine structure (in the sense of eliminating the use of projecta and mastercodes) by the use of more general Σ_n hulls and Skolem functions for Σ_n formulae that, albeit not Σ_n-definable, were preserved under condensations in reproving the Covering Lemma for L (this is the proof given in [8]) using the J-hierarchy, and even using the L-hierarchy alone [34].

One of the chief advantages of the Baldwin-Mitchell reorganisation of the hierarchy of construction from sequences of measures or extenders, was that every level of the models to be constructed was sound. Mitchell and Steel published an account [37] of a fine structure together with a model construction, of extenders whose comparison iteration required iteration trees (see the chapter by Steel in this Handbook). This was the first time that an account of fine structure for mice whose comparison required trees of iterations, allied with the Baldwin-Mitchell organisation of hierarchies, was published. The fine structure used there was inspired by the nature of the predicates that were being used. The notion of $r\Sigma_{n+1}$ formula is analogous to $\Sigma_1^{(n)}$ used here (and the notion of $r\Sigma_{n+1}$ embedding between structures with very good parameters is employed in [46] at 5.12). The success of this fine structure, together with the ability to build much larger models, has ensured its widespread use and represented a leap forward in the production of models that could contain many Woodin cardinals (cf. [50]). Jensen in a series of circulated manuscripts developed the theory for similarly large models using the Σ^* language [27],[25]. Another auxiliary variety of acceptability occurs in [14], where Feng and Jensen develop a theory of mice with some overlapping of extenders in the Σ^* language, in order to build a core model without assuming any 'technical hypothesis' in the form of a large cardinal Ω in the universe (see [41] Sect.1). In that paper a J-structure $M = \langle J_\alpha^A, \in, A\rangle$ is *strongly acceptable* if $\forall\tau < \alpha\forall\vec{\xi} < \tau\forall\varphi(v_0) \in \Sigma_1$:

$$J_{\tau+1}^A \models \varphi[\vec{\xi}] \wedge J_\tau^A \models \neg\varphi[\vec{\xi}] \Longrightarrow (J_{\tau+1}^A \models \mathrm{Card}(\tau) \leq \max\{\omega, \vec{\xi}\}))$$

holds in M. Despite its name the notion is supplemental to acceptability (which it does not in general imply).

We shall lastly mention the *hyperfine structure* of Friedman-Koepke [19]. This is an attractive elongation of the usual L_α-hierarchy by interspersing infinitely many stages between the usual L_α and $L_{\alpha+1}$. A crucial feature, as will be seen below, for many fine-structural arguments is the notion of *singularisation* of an ordinal, the place where an ordinal ν is first seen to definably singularised by some formula. If $\exists x < \alpha\varphi(\xi, x, p)$ defines over L_β a cofinal subset of ξ's forming $C \subseteq \nu$ (for some least $\beta = \beta(\nu)$ and according to some $<_{L_\beta}$-least finite parameter sequence of ordinals p) we can think of the triple $(\beta, \varphi, (p^\frown\alpha))$ as a minimal *location* where ν is singularised. If we define $S_\varphi(q, x) =$ the $<_L$-least ξ so that $\varphi(\xi, x, q)$, as being the term for the Skolem function for φ, we have that $C = S_\varphi^{L_\beta}$ "$\{(p, x) \mid x < \alpha\}$. We thus focus on structures of the form

$$L_{(\beta,\varphi_n,(p^\frown\alpha))} =_{\mathrm{df}} (L_\beta, \in, <_L, N, I, S, S^{L_\beta}_{\varphi_0}, S^{L_\beta}_{\varphi_1}, \ldots, S^{L_\beta}_{\varphi_n}\restriction\{w <_{\mathrm{lex}} p^\frown\alpha\}, \ldots)$$

(where $\varphi = \varphi_n$, the nth formula in some standard recursive enumeration, with all subformulae of φ_n being some φ_i for an $i \leq n$). We can think of this as the singularising structure for ν in this context, as ν is singularised by the last object in this structure: $S^{L_\beta}_{\varphi_n}\restriction\{w <_{\mathrm{lex}} p^\frown\alpha\}$. The functions N, I, S are included, where the first two are functions for naming and interpreting objects respectively, and S is a general Skolem function. Properly construed one may form hulls and, importantly, the Finiteness property of Silver's machines is valid here too; one may prove condensation - that hulls inside locations transitively collapse to other locations. Ultimately the theory allows for a particularly elegant and short proof of global $\Box$. This hyperfine structure can be extended to consider premice with a single measure - thus sufficient for forming the Dodd-Jensen K^{DJ}, but like other alternatives to "true" fine structure there seem to be real technical difficulties to going beyond that.

1.2. Σ^* Ultrapowers

We give an account of the formation of a fine-structure preserving ultrapower. [46] 8.4-5 gives the definition and construction of the Σ_0 ultrapower. We shall see here how to extend the usual notion of ultrapower that uses functions within the model M, to one using $\Sigma^*(M)$-definable functions f. Such a function f is not in general an element of the domain structure M, but with the correct assumptions, M has enough information about the measure or extender being used to form the ultrapower, that this can be sensibly done. The resulting target ultrapower stucture N may contain more objects than the ordinary Σ_0 ultrapower, in particular more ordinals (if we are starting with a set model M). We shall develop this theory for ultrapowers by short extenders that are weakly amenable with respect to the models M concerned. This is more than one needs for dealing with premice with measures of order 0. However, this greater generality will enable us to quickly dispose of "pseudo-ultrapowers" in the next subsection. In any case it is still basically the account of such ultrapowers from [28]. Suppose that $M = \langle J^A_\alpha, A, B\rangle$ is an acceptable J-model and E is a (κ, ν) extender over M with a single critical point $\mathrm{crit}(E) = \kappa < \mathrm{On}\cap M$ and $E = \langle E_a \mid \alpha \in [\nu]^{<\omega}\rangle$. We shall recall here for later the definition of *weakly amenability* of an extender E with respect to M. This is defined as for any $a \in [\nu]^{<\omega}$, $\langle X_\alpha \mid \alpha < \kappa\rangle \in M$, that $\{\alpha \mid X_\alpha \in E_a\} \in M$. This can be shown equivalent to saying that $\mathcal{P}(\kappa)\cap M = \mathcal{P}(\kappa)\cap \mathrm{Ult}_0(M, E)$ (the latter denoting the usual "ordinary" Σ_0 ultrapower of M by E). At some point we shall additionally assume that E is Σ_1-amenable: this in fact is just clause (1) of 8.9 of [46]. Thus such E will be "close to M" in that terminology.

Assuming only that E is an extender, we shall write $\pi_E : M \longrightarrow_E N = \mathrm{Ult}_0(M, E)$ for the ordinary ("coarse") ultrapower embedding. The map π_E is a Σ_0 and cofinal embedding and hence Σ_1-preserving. In general this is as much definable preservation that one could hope for. The notion of Σ^* ultrapower involves using functions that lie outside of M but that are definable classes using formulae of the language $\mathcal{L}^*$. Let M and E be as above.

1.23 Definition. Suppose that E is a (κ, ν) extender over M. The relation $\pi : M \longrightarrow^*_E N$ holds iff the following conditions are met:

(i) Whenever $\rho^k_M > \kappa$, $\pi : M \longrightarrow_{\Sigma^{(k)}_0} N$; where N is transitive.

(ii) Let $\rho^n_M = \min\{\rho^m_M \mid \rho^m_M > \kappa\}$, and $H = \bigcup_{x \in H^n_M} \pi(x)$. Then $\pi{\restriction}H^n_M : H^n_M \longrightarrow_E H$.

(iii) Whenever $\rho^{k+1}_M > \kappa$, N is the closure of $H \cup \mathrm{ran}(\pi)$ under good $\Sigma^{(k)}_1(N)$ functions.

1.24 Remark. (a) (ii) requires that $\pi{\restriction}H^n_M$ be the usual Σ_0 ultrapower map into H. Thus at this level we have the familiar ultrapower with functions taken from H^n_M. Hence $\mathrm{crit}(\pi) = \mathrm{crit}(E)$.

(b) If $\omega\rho^\omega_M > \kappa$ then $\pi : M \longrightarrow^*_E N$ implies $\pi : M \longrightarrow_{\Sigma^*} N$. If we define a good $\Sigma^{(-1)}_1$ function to be a function in M then clause (iii) makes sense even if $\rho^1.{}_M \leq \kappa$.

In most situations we shall want N to be wellfounded (hence the definition), but sometimes it is convenient (although we shall not be concerned with this here) to ask only that the wellfounded core of N (which will be assumed transitive) contains ν. The definition straightforwardly implies that if $\omega\rho^1_M < \kappa$ then $\pi : M \longrightarrow_E N$ if $\pi : M \longrightarrow^*_E N$. (We'll see that under this hypothesis the converse is also true.) We shall see that the existence of such a π, and an N with $\pi : M \longrightarrow^*_E N$ satisfying the above, implies that such π and N are unique. Why is this plausible? Suppose such a π and N exist. Sgap-1uppose $\kappa < \omega\rho^{m+1}_M$ for consideration as in (iii) above. Then for every $z \in N$ there is a good $\Sigma^{(m)}_1(N)$ function F with $z = F(u, \pi(x))$ for some $u \in H$ and $x \in M$ for some m. Let $y \in H^n_M$ be such that $u \in \pi(y)$. Let $\bar{F}$ have the same functionally absolute definition over M as F does over N. Then $\bar{F}$ is a good $\Sigma^{(m)}_1(M)$ function. Let $\varphi(v_0)$ be any Σ_0 formula. Then $M \models \varphi(\bar{F}(u, x))$ is a $\Sigma^{(m)}_1(M)$ property in u, x. Let $w = \{v \in y \mid M \models \varphi(\bar{F}(v, x))\}$. Then, as $w \in H^n_M$ (since $y \in H^n_M$ and we may assume $n > m$):

$$M \models (\forall v^n \in y)(\varphi(\bar{F}(v^n, x)) \longleftrightarrow v^{n+1} \in w)\,.$$

This is $\Pi_0^{(n)}$, and as π is Σ^*-preserving we have:

$$N \models (\forall v^n \in \pi(y))(\varphi(F(v^n, \pi(x))) \longleftrightarrow v^n \in \pi(w))\,;$$

in short, for u as above:

$$N \models \varphi(F(u, \pi(x)) \longleftrightarrow u \in \pi(w)\,.$$

This is tantamount to saying that Σ_0 facts about $(F(u, \pi(x)))_N$ are determined by checking whether $u \in \pi(w)$ for a $w \in H^n_M$ suitably formed as above. (And similarly for Σ_0 facts such as "$(F(u', \pi(x')) \in F(u, \pi(x)))_N$".) However $\pi{\restriction}H^n_M : H^n_M \longrightarrow_E H$ is the usual ultrapower of H^n_M by E, and this map is determined by M and by E. And thus so are the Σ_0 facts about N mentioned above. If the $\in$-diagram of N is determined then the whole map $\pi : M \longrightarrow N$ is determined – and thus unique – if it can be shown to exist. To put it another way, the above argument shows that there is an $\in$-isomorphism between any two such ultrapowers N, N'; however if the N, N' are taken as transitive, then this isomorphism is the identity.

We have implicitly in the above assumed that we can map across such good functions in a well-defined way. We justify this more formally in a moment.

We now proceed to describe ultrapowers formed by taking these extra class (over M) functions.

1.25 Definition. $\Gamma = \Gamma(\kappa, M)$ is the set of functions f with $\mathrm{dom}(f) = [\kappa]^k$ (some $k < \omega$) and either $f \in M$ or f is a good $\boldsymbol{\Sigma_1^{(n)}}(M)$ function for some n with $\omega\rho_M^{n+1} \geq \kappa$.

The extension of π to elements of Γ is effected as follows: if $f \in \Gamma$ and $\mathrm{dom}(f) = \kappa$, and f with $\mathrm{dom}(f) = \kappa$ has a functionally absolute $\Sigma_1^{(m)}(M)$ definition in a parameter r, where $\omega\rho_M^{m+1} > \kappa$, then we set $\pi(f)$ to be the function with $\mathrm{dom}(\pi(\kappa))$ defined over N by the same $\Sigma_1^{(m)}$ definition using $\pi(r)$. We need to argue that this is well-defined. Suppose $F^M(v^m, u^0)$ and $G^M(v^m, u^0)$ are two good $\Sigma_1^{(m)}(M)$ definitions in the parameters q, r respectively of the function f. Then $(\forall \xi < \kappa)(\ F^M(\xi, q) = G^M(\xi, r))$ holds over M. However this can be expressed as $\boldsymbol{\Pi_0^{(m+1)}}(M)$:

$$(\forall x^{m+1} < \kappa)(F^M(x^{m+1}, q) = G^M(x^{m+1}, r))\,.$$

As π is Π_0^{m+1} preserving, the same statement is true of $\pi(\kappa), \pi(q), \pi(r), F^N$, G^N over N. So $\pi(f)$ is indeed independent of the choice of its functionally absolute definition.

We shall form a suitable domain and a term model more or less exactly following the pattern for Σ_0 ultrapowers as is done in [46] Theorem 8.4, although we shall incorporate the functions from $\Gamma_M = \Gamma(\kappa, M)$, and here we only have the single relevant critical point κ (thus we set $\mu_a = \kappa$ for all $a \in \nu$ from their definition). Thus our domain will be:

$$D = \{\langle a, f\rangle \mid f \in \Gamma_M, a \in [\nu]^{<\omega}, f : [\kappa]^{\mathrm{card}(a)} \longrightarrow M\}.$$

The definitions of the $\sim$ and $\dot{\in}$ relations are then unaltered, and we use the same notation "$f^{a,b}$" where $\mathrm{dom}(f) = [\kappa]^n$, $a \subseteq b$, and $a = (u_{i_1}, \ldots, u_{i_n}) \subseteq b = (u_1, \ldots, u_m)$ to denote the function with domain $[\kappa]^m$ given by $f^{a,b}(v) = f(v_{i_1}, \ldots, v_{i_n})$.

$$\langle a, f\rangle \sim \langle b, g\rangle \iff \{u \in [\kappa]^{\mathrm{card}(c)} \mid f^{a,c}(u) = g^{b,c}(u)\} \in E_c \text{ for } c = a \cup b$$
$$\langle a, f\rangle \mathrel{\dot{\in}} \langle b, g\rangle \iff \{u \in [\kappa]^{\mathrm{card}(c)} \mid f^{a,c}(u) \in g^{b,c}(u)\} \in E_c \text{ for } c = a \cup b$$
$$\dot{C}(\langle a, f\rangle) \iff \{u \mid C(f(u)) \mid \in E_a\} \text{ (for } C = A, B).$$

1.26 Definition. The term model $\mathbb{D}$ is defined as: $\mathbb{D} = \langle D, \sim, \dot{\in}, \dot{A}, \dot{B}\rangle$.

We observe:

1.27 Lemma. *Let $\varphi(v_0^{i_0}, \ldots, v_k^{i_k})$ be either a $\Sigma_1^{(n)}$ formula where $\rho_M^{n+1} > \kappa$, or else a $\Sigma_0^{(n)}$ formula where only $\rho_M^n > \kappa$ is assumed. Let $\langle a_0, f_0\rangle, \ldots, \langle a_k, f_k\rangle \in D$, and let $b \in [\nu]^{<\omega}$ be such that $b \supseteq a_0 \cup \cdots \cup a_k$. Assume that for $j \le k$, f_j is a function to $H_M^{i_j}$.*

Then $\{u \mid M \models \varphi(f_0^{a_0 b}(u), \ldots, f_k^{a_k b}(u))\} \in M$.

Proof. Assume first $\rho_M^{n+1} > \kappa$ and that φ is $\Sigma_1^{(n)}$. As we have assumed the value types of the defining formulae of the $f_j \in \Gamma_M$ satisfy $i_j \le n$ for $j \le k$. Then $M \models \varphi(f_0^{a_0 b}(u), \ldots, f_k^{a_k b}(u))$ defines a $\boldsymbol{\Sigma}_1^{(n)}(M)$ relation. Now if φ is $\Sigma_0^{(n)}$ and if $\rho_M^{n+1} > \kappa$ still holds we have nothing to prove. However if $\rho_M^n > \kappa \ge \rho_M^{n+1}$ we have in this case that either the $f_i \in M$ or they are good $\Sigma_1^{(n-1)}$ and the set under consideration is a $\boldsymbol{\Sigma}_0^{(n)}$ subset of κ, and the result is immediate. ⊣

The following lemma is a kind of "uniformisation lemma" that we shall need in order to prove Łoś theorems.

1.28 Lemma. *Let $R(y^m, x^{i_0}, \ldots, x^{i_k})$ be a $\Sigma_1^{(m)}$ relation with $i_0, \ldots, i_k \le m$. Let $m \le n$ be such that $\rho_M^{n+1} > \kappa$, and let $f_0, \ldots, f_k \in \Gamma$ be good $\boldsymbol{\Sigma}_1^{(n)}(M)$ functions with $f_j : [\kappa]^l \longrightarrow H_M^{i_j}$. Then there is a good $\boldsymbol{\Sigma}_1^{(n)}(M)$ function $g : [\kappa]^l \longrightarrow H_M^m$, $g \in \Gamma$, such that for any $u \in [\kappa]^l$:*

$$M \models \exists y^m R(y^m, f_0(u), \ldots, f_k(u)) \iff M \models R(g(u), f_0(u), \ldots, f_k(u)).$$

Proof. By Lemma 1.16 there is a good $\boldsymbol{\Sigma}_1^{(m)}(M)$ function $F(x^{i_0}, \ldots, x^{i_k}) = y^m$ such that:

$$M \models \exists y^m R(y^m, x^{i_0}, \ldots, x^{i_k}) \iff M \models R(F(x^{i_0}, \ldots, x^{i_k}), x^{i_0}, \ldots, x^{i_k}).$$

So set $g'(u) \simeq F(f_0(u), \ldots, f_k(u))$. g' is then a good $\mathbf{\Sigma_1^{(n)}}(M)$ function. $\mathrm{dom}(g') \subseteq [\kappa]^l$ and by Lemma 1.27, we see that $a =_{\mathrm{df}} \mathrm{dom}(g') \in M$. Now define $G(v^0, w^0)$ by

$$y = G(v^0, w^0) \iff (y = v^0 \wedge w^0 \in a) \vee (y = 0 \wedge w^0 \notin a)\,.$$

Then G is good, $\mathbf{\Sigma}_0(M)$, and hence $g(u) =_{\mathrm{df}} G(g'(u), u)$ is good $\mathbf{\Sigma_1^{(n)}}(M)$. $\dashv$

Before proving a Łoś theorem later for $\Sigma_1^{(n)}$, we state one for Σ_0 :

1.29 Lemma. *Let $\varphi(v_0, \ldots, v_k)$ be a Σ_0 formula. Let $\langle a_0, f_0\rangle, \ldots, \langle a_k, f_k\rangle \in D$, and let $b \in [\nu]^{<\omega}$ be such that $b \supseteq a_0 \cup \cdots \cup a_k$. Then*

$$\mathbb{D} \models \varphi(\langle a_0, f_0\rangle, \ldots, \langle a_k, f_k\rangle) \iff \{u \mid M \models \varphi(f_0^{a_0 b}(u), \ldots, f_k^{a_k b}(u))\} \in E_b\,.$$

This is an induction on the structure of the Σ_0 formula φ using the last two lemmas. We shall give more detail in the $\Sigma_1^{(n)}$ Łoś theorem which follows.

Using the Σ_0 Łoś Theorem 1.29 one has that if $\pi : M \longrightarrow_E^* N$ then the map from $\mathbb{D}$ to N given by $\langle a, f\rangle \mapsto \pi(f)(a)$ is structure preserving, in particular on $\in$: $\langle a, f\rangle \mathrel{\dot\in} \langle b, g\rangle \iff \pi(f)(a) \in \pi(g)(b)$. We then have that $\mathbb{D}$ is a model of extensionality, and $\sim$ is a congruence relation. We thus, as for the Σ_0 ultrapower, form the equivalence classes, written as $[a, f]$, for $\langle a, f\rangle \in \mathbb{D}$. We shall assume from now on that $\dot\in$ is well founded and we thus have an onto factor map $[\quad] : \mathbb{D} \longrightarrow \langle N, \in, A', B'\rangle$ satisfying $[x] = / \in [y]$ iff $x \sim / \dot\in\, y$ and $\dot C(x) \iff C'([x])$. Using 1.29 again we have the map $\pi : M \longrightarrow_{\Sigma_0} N$ is defined by the usual constant functions c_x (with $\mathrm{dom}(c_x) = [\kappa]^0$): $\pi(x) = [0, c_x]$.

It will be useful to have some notation to stratify $\Gamma(\kappa, M)$.

1.30 Definition. Suppose $f \in \Gamma(\kappa, M)$.

$$\Gamma = \Gamma_n(\kappa, M) =_{\mathrm{df}} \begin{cases} \{f \in \Gamma(\kappa, M) \mid \mathrm{ran}(f) \subseteq H_M^n\} & \text{if } \rho_M^{n+1} > \kappa; \\ \{f \in \Gamma(\kappa, M) \mid \mathrm{ran}(f) \in H_M^n\} & \text{if } \omega\rho_M^{n+1} \leq \kappa < \omega\rho_M^n. \end{cases}$$

1.31 Lemma. *Suppose n is such that $\omega\rho_M^{n+1} \leq \kappa < \omega\rho_M^n$. Set $\overline{H} = H_M^n$, and $H = \bigcup \pi``\overline{H}$ with π as above. Then:*

(i) $\pi{\restriction}\overline{H} : \overline{H} \longrightarrow_E H$, *i.e.* $\pi{\restriction}\overline{H}$ *is the coarse ultrapower map;*

(ii) $\mathrm{crit}(\pi) = \kappa$, *and* $[a, f] = \pi(f)(a)$ *for* $a \in [\nu]^{<\omega}$ *and* $f \in {}^\kappa\overline{H} \cap \overline{H}$.

(iii) $\mathcal{P}(\kappa) \cap \overline{H} = \mathcal{P}(\kappa) \cap H$.

Proof. (i) Let $x \in H$; then $x = [a, f]$ for some $\langle a, f\rangle \in \mathbb{D}$. Suppose $f \in \Gamma_m(\kappa, M)$. We just need to know that f could have been chosen from $\overline{H}$, i.e. as $\pi \restriction \bar{H}$ is cofinal into H we can find $y \in \overline{H}$ with $x \in \pi(y)$. We can assume that $\mathrm{ran}(f) \subseteq y$, by intersecting f with $\kappa \times y$ if need be, and now the latter is essentially a $\mathbf{\Sigma}_1^{(n-1)}(M)$ bounded subset of ρ_M^n, and thus is in M. By acceptability it must be in $\overline{H}$. (ii) is the usual argument for such ultrapowers. For (iii) ($\subseteq$) is straightforward. For ($\supseteq$) suppose $x \subseteq \kappa, x = [a, f] \in N$. Then $x = \{\xi < \kappa \mid \xi \in \pi(f)(a)\} = \{\xi < \kappa \mid \{u \mid \xi \in f(u)\} \in E_a\}$. Amenability says precisely that then $x \in H$. ⊣

We thus have some sort of agreement with the notion of a Σ_0 ultrapower at the "crossing structure" $\overline{H}$ where the projectum crosses over the measurable cardinal. It is more work to establish that we are getting the correct kind of embedding to ensure Σ^*-elementarity.

1.32 Theorem (Fine Structural Ultrapower Theorem).

(i) *N is an acceptable J-model.*

(ii) (a) $\pi : M \longrightarrow_{\Sigma_0^{(n)}} N$ *if* $\omega\rho_M^n > \kappa$,

(b) $\pi : M \longrightarrow_{\Sigma_2^{(n)}} N$ *if* $\omega\rho_M^{n+1} > \kappa$.

(iii) *(Łoś Theorem) Let φ be in $\Sigma_0^{(n)}$ (or in $\Sigma_1^{(n)}$, if $\kappa < \omega\rho_M^{n+1}$) and $b \supseteq \bigcup_i a_i$. Then*

$$N \models \varphi(\pi(f_1)(a_1), \ldots, \pi(f_n)(a_n)) \iff \{u \mid M \models \varphi(\overrightarrow{f_i^{a_i b}(u)})\} \in E_b$$

Additionally if we assume E close to M:

(iv) $\pi : M \longrightarrow_{\Sigma^*} N$,

(v) $\mathcal{P}(\kappa) \cap \mathbf{\Sigma}^*(M) = \mathcal{P}(\kappa) \cap \mathbf{\Sigma}^*(N)$.

Proof. The proof of the theorem is in two stages. The main difficulty is in showing that the maps are between the relevant reducts. What this amounts to is showing that the elements in the natural "strata" that one defines from the functions in Γ are actually those obtained by using the iterated definition of projectum over N. The "strata" referred to arise as the H_n in the following definition: Set $\omega\rho_n = H_n \cap \mathrm{On}$ where:

$$H_n = \begin{cases} \{[a, f] \mid f \in \Gamma_n \wedge \langle a, f\rangle \in D\} & \text{if } \rho_M^n > \kappa, \\ H_M^n & \text{otherwise}. \end{cases}$$

(1) *H_n is transitive.*

Proof of (1). Trivial if $\rho_M^n < \kappa$. Suppose $[b, g] \in [a, f] \in H_n$. If $\omega\rho_M^{n+1} \leq \kappa < \omega\rho_M^n$ then transitivity follows in the usual manner for Σ_0 ultrapowers. We may assume then that $g \in \Gamma_m$ for an $m \leq n$ and $\rho_M^{n+1} > \kappa$. Now show there is a $[b', g'] \in H_n$ with $[b', g'] = [b, g]$ by appealing to Lemma 1.28. $\dashv$ (1)

We thus have $N = H_0 \supseteq H_1 \supseteq H_2 \supseteq \ldots$ and that if $x \in H_M^n$, then $\pi(x) \in H_n$; hence if we interpret on the N-side the variables v_j^i as varying over H_i, then π respects types. Thus $\langle N, \in, A', B', H_0, H_1, H_2 \ldots \rangle$ is an $\mathcal{L}^*$-stucture, the *pseudo-interpretation* of $\mathcal{L}^*$. Our task is then two-fold: to show that $\pi : M \longrightarrow_{\Sigma^*} N$ in this pseudo-interpretation, and then to show that it is actually the *correct* interpretation, that is, $H_n = H_N^n$ for any n.

(2)(Łoś Theorem for Pseudo-Interpretations). *Let $\varphi(v^{i_1}, \ldots, v^{i_k})$ be a $\Sigma_1^{(n)}$ formula, where $\omega\rho_M^{n+1} > \kappa$, or a $\Sigma_0^{(n)}$ formula if $\omega\rho_M^{n+1} \leq \kappa < \omega\rho_M^n$. Let $[a, f_j] \in H_{i_j}$ with $f_j \in \Gamma_{i_j}$, with $i_j \leq n$ for $1 \leq j \leq k$. Then*

$$N \models \varphi(\overrightarrow{[a, f_j]}) \Longleftrightarrow \{u \mid M \models \varphi(f_1(u), \ldots, f_k(u))\} \in E_a.$$

Proof of (2).

Case 1. $\rho_M^{n+1} > \kappa$.

By induction on the complexity of φ. Note that $\{u \mid M \models \varphi(f_1(u), \ldots, f_k(u))\} \in M$ by Lemma 1.27. We shall only check here the quantifier step $\varphi \equiv \exists v^m\, \psi(v^m, v^{i_1}, \ldots, v^{i_k})$ where $m \leq n$, $\psi \in \Sigma_0^{(m)}$ and (2) is assumed to hold for ψ. The forward direction is quite straightforward: we assume $N \models \varphi(\overrightarrow{[a, f_j]})$; then there is a $[b, f_0] \in H_m$ with $a \subseteq b$, $f_0 \in \Gamma_m$ and $N \models \psi([b, f_0], [b, f_1^{ab}], \ldots, [b, f_k^{ab}])$. By the inductive hypothesis:

$$\{u \mid M \models \psi(f_0^{ab}(u), f_1^{ab}(u), \ldots, f_k^{ab}(u))\} \in E_b\,.$$

As $\mathrm{ran}(f_0) \subseteq H_M^m$ we have:

$$\{u \mid M \models \exists v^m \psi(v^m, f_1^{ab}(u), \ldots, f_k^{ab}(u))\} \in E_b$$

and so:

$$\{u \mid M \models \exists v^m \psi(v^m, f_1(u), \ldots, f_k(u))\} \in E_a\,.$$

Conversely, assume that $\{u \mid M \models \exists v^m \psi(v^m, f_1(u), \ldots, f_k(u))\} \in E_a$. By Lemma 1.28 there is a $g \in \Gamma, g : [\kappa]^{\mathrm{card}(a)} \longrightarrow H_M^m$ such that:

$$\{u \mid M \models \psi(g(u), f_1(u), \ldots, f_k(u))\} \in E_a\,.$$

By the inductive hypothesis, $N \models \psi([a, g], \overrightarrow{[a, f_j]})$. As $\omega\rho_M^{m+1} > \kappa$, and $\mathrm{ran}(g) \subseteq H_M^m$ we have an $[a, g] \in H_m$ and hence $N \models \exists v^m \psi(v^m, \overrightarrow{[a, f_j]})$. $\dashv$ *Case 1*

Case 2. $\omega\rho_M^{n+1} \leq \kappa < \omega\rho_M^n$.

Note that the only difficulty in *Case 1* was in the converse direction, when we had to appeal to Lemma 1.28. Again we look only at a representative (now bounded) quantifier step: $\varphi \equiv \exists v^n \in u^n \psi(v^n, u^n, v^{i_1}, \ldots, v^{i_k})$ with (2) assumed proven for ψ. The forward direction is just as in the previous case, so we omit it. So suppose for some $f_0 \in \Gamma_n$

$$\{u \mid M \models \exists v^n \in f_0(u)\psi(v^n, f_0(u), f_1(u), \ldots, f_k(u))\} \in E_a\,.$$

We define a witnessing function g another way, making use of the fact, that by the definition of Γ_n in this case, that $f_0 \in H^n_M$:

$$g(u) = \begin{cases} \text{the } <_M \text{-least } w \in f_0(u) \text{ so that } (\psi(w, f_0(u), f_1(u), \ldots, f_k(u)))_M \\ 0 \quad \text{ if this } w \text{ does not exist.} \end{cases}$$

We may assume $n > 0$ (for otherwise (2) is trivially true in this case). Then, using the fact that the canonical wellorder of M is Δ^M_1, we have that g is a $\boldsymbol{\Sigma}^{(n)}_0(M)$ subset of $\kappa \times \bigcup \mathrm{ran}(f_0) \in H^n_M$, and thus is an element of H^n_M. (This is true even if $n = 0$.) Then for some $X \in H^n_M$ we have $\{u \mid M \models \psi(g(u), f_0(u), f_1(u), \ldots, f_k(u))\} \supseteq X \in E_a$ and we can finish as before. $\dashv$ (2)

(3) a) *If* $\rho^{n+1}_M > \kappa$ *then* $\pi : M \longrightarrow_{\Sigma^{(n)}_2} N$ *in the pseudo-interpretation.*
b) *If* $\omega\rho^{n+1}_M \leq \kappa < \omega\rho^n_M$ *then* $\pi : M \longrightarrow_{\Sigma^{(n)}_1} N$ *in the pseudo-interpretation.*

Proof of (3). Suppose $\psi \equiv \exists v^n \varphi(v^n, \vec{u})$ where φ is $\Sigma^{(n)}_0$ (or $\Pi^{(n)}_1$ for part a)). Let $\vec{x} \in M$, and assume that $N \models \exists v^n \varphi(v^n, \pi(\vec{x}\,))$. Let then $[a, f] \in H_n, f \in \Gamma_n$ be such that $N \models \varphi([a, f], \vec{x}\,)$. By *Case 1* (or *2* for b)) above $\{u \mid M \models \varphi(f(u), \vec{x}\,)\} \supseteq X \in E_a$. Hence $M \models \exists v^n \varphi(v^n, \vec{x}\,)$ as $\mathrm{ran}(f) \subseteq H^n_M$. $\dashv$ (3)

We now briefly consider what happens below the critical point κ.

(4) (i) *Let* $\kappa \geq \rho^n_M$. *Then* $\pi : M \longrightarrow_{\Sigma^{(n)}_\omega} N$ *in the pseudo-interpretation.*
(ii) $\pi : M \longrightarrow_{\Sigma^*} N$ *in the pseudo-interpretation.*

Proof of (4). (i) This is essentially because we have defined $H_n = H^n_M$ for such n, and $\pi{\restriction}H^n_M = \mathrm{id}{\restriction}H^n_M$ and thus the variables of type n are interpreted in the same domains. Suppose (4) failed for some least such n, as witnessed by some $\Sigma^{(n)}_k$ formula φ of least complexity. Note that φ is not the 'atomic' $\Sigma^{(n-1)}_1$ part of a $\Sigma^{(n)}_0$ formula (by definition of n, or else by (3)b) if $\omega\rho^{n-1}_M > \kappa$). Elementary considerations show we can assume φ is then not atomic, nor of the form $\neg\psi, (\psi \wedge \chi)$ but is $\varphi \equiv \exists v^n \psi(v^n, \vec{u})$. Suppose $N \models \exists v^n \psi(v^n, \pi(\vec{x}\,))$ for some $\vec{x} \in H^n_M$. Let $y \in H_n = H^n_M$ witness this. $N \models \psi(y, \pi(\vec{x}\,))$. As ψ is a simpler formula, and $\pi(y) = y$, we have $M \models \psi(y, \vec{x})$. (ii) then follows from (i) and (3). $\dashv$ (4)

(5) *N is an acceptable J-structure, π is Q-preserving, and $H_n = J^{A'}_{\rho_n}$ for all $n < \omega$.*

Proof of (5). The property of being an acceptable J-structure is a Q-condition (cf. [46] 1.21). If $\pi : M \longrightarrow N$ is a standard Σ_0 ultrapower, then in fact π is Σ_0 and cofinal into N and such maps preserve Q properties. However otherwise π is at least $\Sigma^{(0)}_2$ (and *a fortiori* Σ_2) -preserving by (3)a) which suffices. The same reasoning works level-by-level: if $\rho^{n+1}_M > \kappa$ then $\pi{\restriction}H^n_M : \langle H^n_m, H^n_m \cap A, H^n_m \cap B\rangle \longrightarrow_{\Sigma_2} \langle H_n, A' \cap H_n, B' \cap H_n\rangle$; thus H_n is a J-structure constructed from A'. If $\omega\rho^{n+1}_M \leq \kappa < \omega\rho^n_M$ then $\pi{\restriction}H^n_M : \langle H^n_m, H^n_m \cap A, H^n_m \cap B\rangle \longrightarrow_{\Sigma_0} \langle H_n, A' \cap H_n, B' \cap H_n\rangle$ which is moreover cofinal, again, as it is a standard Σ_0 ultrapower map (Lemma 1.31), so the same is true at this level. For $\rho^n_M \leq \kappa$ it is trivial. $\dashv$ (5)

If we can now show the pseudo-interpretation is the correct one, we shall have fulfilled all the clauses (i)-(iv) of the Theorem. The following computation of the size of the relevant ordinals finishes the task: it only remains to show that the projecta of N correspond to the ordinals ρ_n. We divide into the cases: above and below the measurable κ.

(6) $\rho_m = \rho^m_N$ *if* $\kappa \leq \omega\rho^m_M$

Proof of (6). By induction on m. This is trivial for $m = 0$.

(6a) *$\langle H_m, B\rangle$ is amenable for $B \in P(\omega\rho_m) \cap \mathbf{\Sigma}^{(m-1)}_1(N)$. Hence $\rho_m \leq \rho^m_N$.*

Proof of (6a). Suppose $B \subseteq \omega\rho_m$ and is $\Sigma^{(m-1)}_1(N)$ in the parameter $[a, f]$, say $B(x) \iff N \models \varphi(x, [a, f])$ with $\varphi \in \Sigma^{(m-1)}_1$. Let $w = [b, g] \in H_m$. (Without loss of generality we shall assume $a = b$ here.) We require $B \cap w \in H_m$. Define h by $h(y^m, v^0) = \{t^m \in y^m \mid M \models \varphi(t^m, v^0)\}$. Thus $h(y^m, v^0)$ is a $\Sigma^{(m)}_0$ function of value type m and thus is good. Hence it is in Γ. h is defined for all y^m, v^0 and $k : [\kappa]^{\mathrm{card}(a)} \longrightarrow H^m_M$ where $k(u) = h(g(u), f(u))$ (note that $\mathrm{ran}(k)$ is indeed contained in H^m_M, as each element of the form $h(y^m, v^0)$ is *bounded* $\Sigma^{(m)}_0$). k is in Γ being a composition of such.

If $\kappa < \omega\rho^{m+1}_M$ we may conclude that $k \in \Gamma_m$ as $\mathrm{ran}(k) \subseteq H^m_M$. If $\omega\rho^{m+1}_M \leq \kappa < \omega\rho^m_M$ we need to see that $k \in H^m_M$ to infer this. Notice that $k(u) \subseteq g(u)$ and thus $K = \{\langle u, z\rangle \mid u \in k(z)\}$ is a $\Sigma^{(m)}_0(M)$ of $\{\langle u, z\rangle \mid u \in g(z)\}$ and the latter is in H^n_M. Hence so is K and we have that k is thus rudimentary over $M^{m,p}$ where p is a suitable choice of parameters including those used to define f, g. Hence $k \in H^m_M$. As

$$[\kappa]^{\mathrm{card}(a)} = \{u \mid M \models \forall t^m \in g(u)[\varphi(t^m, f(u)) \longleftrightarrow t^m \in k(u)]\} \in E_a$$

by Łoś we have $N \models \forall t^m \in [a, g](\varphi(t^m, [a, f]) \longleftrightarrow t^m \in [a, k])$. Hence

$$[a, k] = [a, g] \cap \{t^m \mid N \models \varphi(t^m, [a, f])\} = w \cap A \in H_m\,.$$

$\dashv$ (6a)

(6b) *There is an* $A \subseteq \rho_m \cap \mathbf{\Sigma}_1^{(m-1)}(N)$ *such that* $A \notin N$. *Hence* $\rho_m \geq \rho_N^m$.

Proof of (6b).

Case 1. $\kappa < \omega\rho_M^{m+1}$.

Suppose $\bar{A}$ is $\Sigma_1^{(m-1)}(M)$-definable in a parameter $\overline{p}$, but with $\bar{A} \cap \rho_M^m \notin M$, for example we can find such using $\bar{p} \in P_M^m$.

Let A be $\Sigma_1^{(m-1)}(N)$ in $\pi(\bar{p}) = p$ by the same formula φ say. We show that $A \cap \omega\rho_m \notin M$. Suppose otherwise. Let $A \cap \omega\rho_m = [a, f] \in N$. Then $N \models \forall x^m(x^m \in [a, f] \longleftrightarrow \varphi(x^m, p))$.

This is a $\Pi_1^{(m)}$, in p, formula, and by the Łoś Theorem (2):

$$\{u \mid M \models \forall x^n(x^n \in f(u) \longleftrightarrow \varphi(x^n, p))\} \in E_a.$$

So there is an $u \in [\kappa]^{\mathrm{card}(a)}$ with $f(u) \cap \rho_M^n = \overline{A} \cap \rho_M^n$ – a contradiction.

Case 2. $\omega\rho_M^{m+1} \leq \kappa < \omega\rho_M^m$.

We alert the reader to this more problematic case. Here we explicitly use the weak amenability property of the extender to ensure that the map is *cofinal* at the level of this crossing projectum structure.

Let $p \in P_M^{m+1}$, and set $\overline{A} = A^{p(m-1)}_{M^{m-1,p\restriction m-1}}$. Then $\bar{A}$ is $\Sigma_1^{(m-1)}(M)$ in p; suppose that it is defined by φ. Let $\widetilde{A}$ be defined over N in the same way using $\pi(p)$. $M^{m,p\restriction m} = \langle H, \overline{A}\rangle$ is amenable and $\pi\restriction H_M^m$ is Σ_0 and cofinal into H_m. As π is $\Sigma_0^{(m)}$-preserving into N, we conclude that for any $x \in H_M^m$ $\pi(x \cap \overline{A}) = \pi(x) \cap \widetilde{A}$. Hence:

(i) $\langle H_m, \widetilde{A}\rangle$ is amenable; (ii) $\pi\restriction H_M^m : M^{m,p\restriction m} \longrightarrow_{\Sigma_0} \langle H_m, \widetilde{A}\rangle$ cofinally, and hence is Σ_1-elementary.

However now we see that ρ_m must equal ρ_N^m; for suppose $\rho_m < \rho_N^m$: then $\widetilde{A} \cap H_m \in N$. As $p \in P_M^{m+1}$ we can pick $B \in \mathcal{P}(\omega\rho_M^{m+1}) \cap \Sigma_1(M^{m,p\restriction m})$ in $p(m)$, with $B \notin M$. As $\pi\restriction\kappa = \mathrm{id}\restriction\kappa$ we have that $B \in \mathcal{P}(\kappa) \cap \mathbf{\Sigma}_1(\langle H_m, \widetilde{A}\rangle)$. Since $\langle H_m, \widetilde{A}\rangle \in N$, then so is B. By the amenability of our extender E (and acceptability of our structures) $\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap N$; hence $B \in M$ – a contradiction. $\dashv$ (6b)&(6)

In the course of proving (6b) we also showed:

(7) $\pi``P_M^m \subseteq P_N^m$ if $\omega\rho_M^m > \kappa$.

1.33 Remark. The argument of *Case 2*, where $\omega\rho_M^{m+1} \leq \kappa < \omega\rho_M^m$, shows that the weak amenability of E implies that $\omega\rho_N^{m+1} \leq \rho_{m+1}$. Without the requirement of amenability we should only be able to prove the inequality $\omega\rho_N^m \geq \rho_m$ and not its reverse. (Further only that (7) holds for those

m with $\omega\rho_M^{m+1} > \kappa$.) The assumption of amenability then ensures the $\Sigma_0^{(m)}$-*cofinality* of the ultrapower map (and hence that it is in fact $\Sigma_1^{(n)}$-preserving). The assumption of Σ_1-amenability of M will enable us to prove (i) the equality $\omega\rho_N^m = \rho_m$; (ii) as well as that of the projecta below the measurable; (iii) finally, the full Σ^*-elementarity of the map π.

1.34 Remark. Still considering the above *Case 2* where $\omega\rho_M^{m+1} \leq \kappa < \omega\rho_M^m$, we can vary assumptions on E, or M and still get that $\pi : M \longrightarrow_{\Sigma_0^{(n)}} N$ cofinally: if we assume E is Σ_1-amenable this suffices (see [62] Lemma 3.2.6); similarly if $R_M^m \neq \varnothing$ then we can get the same conclusion (as well as the bonus that $\pi``R_M^m \subseteq R_N^m$ see [62] Lemma 3.2.4.)

First we shall need a preliminary lemma:

1.35 Lemma. *Suppose E is close to M, i.e., E is weakly amenable and Σ_1-amenable over M, and $\omega\rho_M^{n+1} \leq \kappa < \omega\rho_M^n$. Let $B \in \mathbf{\Sigma}_1^n(N) \cap \mathcal{P}(\kappa)$. Then $B \in \mathbf{\Sigma}_1^n(M)$.*

Proof. Suppose for some $\varphi \in \Sigma_0^{(n)}$ and all ξ,

$$\xi \in B \Longleftrightarrow \exists v^n \varphi(v^n, \xi, \pi(f)(a)) .$$

By weak amenability,

$$\xi \in B \Longleftrightarrow \exists u \in H_M^n \exists v^n \in \pi(u) \varphi(v^n, \xi, \pi(f)(a)) ,$$

and by Łoś for $\Sigma_0^{(n)}$,

$$\xi \in B \Longleftrightarrow \exists u \in H_M^n (\{w \mid \exists v^n \in u \varphi(v^n, \xi, f(w))\} \in E_a) .$$

Notice that the set $X_\xi = \{w \mid \exists v^n \in u \ \varphi(v^n, \xi, f(w))\}$ is a $\Sigma_0^{(n)}$ subset of $\mathrm{dom}(f) \in H_M^n$ (whichever n satisfies $f \in \Gamma_n$) and hence $X_\xi \in H_M^n$. By the Σ_1-amenability assumption, "$X \in E_a$" is $\mathbf{\Sigma}_1(M)$. Thus $B \in \mathbf{\Sigma}_1^{(n)}(M)$. $\dashv$

(8) *Suppose $\kappa \geq \omega\rho_M^m$. Then* (i) *$\rho_m = \rho_N^m$, and hence $H_M^m = H_N^m$; and* (ii) *$\mathbf{\Sigma}_1^{(m)}(M) \cap \mathcal{P}(H_M^m) = \mathbf{\Sigma}_1^{(m)}(N) \cap \mathcal{P}(H_N^m)$.*

Proof of (8). Let n be such that $\omega\rho_M^{n+1} \leq \kappa < \omega\rho_M^n$. We prove (8) by induction on $m \geq n+1$. By the last lemma if $m = n+1$, or the inductive hypothesis otherwise:

$$\mathbf{\Sigma}_1^{(m-1)}(M) \cap \mathcal{P}(\omega\rho_M^m) = \mathbf{\Sigma}_1^{(m-1)}(N) \cap \mathcal{P}(\omega\rho_M^m) .$$

For (i), that $\rho_m = \rho_N^m$, now follows directly from the fact that a) $\forall \gamma < \omega\rho_M^m a \in \mathbf{\Sigma}_1^{(m-1)}(N) \cap \mathcal{P}(\gamma)$ then $a \in H_M^m = H_N^m$; and b) if $A \in \mathbf{\Sigma}_1^{(m-1)}(M) \cap \omega\rho_M^m, A \notin M$ then $A \notin N$.

For (ii), by Lemma 1.6 we have, substituting either M or N for T, that $C \in \mathbf{\Sigma_1^{(m)}}(T) \cap \mathcal{P}(H^m_T)$ if and only if $C \in \mathbf{\Sigma}_1(\langle H^m_T, Q\rangle$ for some $Q \in \mathbf{\Sigma}_1^{(m-1)}(T) \cap \mathcal{P}(H^m_T)$. By the inductive hypothesis (or again Lemma 1.35 if $m = n+1$), we have such a $Q \in \mathbf{\Sigma}_1^{(m-1)}(M) \cap \mathbf{\Sigma}_1^{(m-1)}(N)$. Since (i) shows $H^m_M = H^m_N$, we have, for $C \subseteq H^m_M$:

$$C \in \mathbf{\Sigma}_1^{(m)}(M) \Longleftrightarrow C \in \mathbf{\Sigma}_1(\langle H^m_M, Q\rangle \Longleftrightarrow C \in \mathbf{\Sigma}_1^{(m)}(N).$$

$\dashv$ (6&8)

We can now conclude that the ultrapower embedding is properly Σ^*-preserving.

(9) $\pi : M \longrightarrow_{\Sigma^*} N$ *in the standard interpretation.*

Proof of (9). (4) shows that $\pi : M \longrightarrow_{\Sigma^*} N$ in the pseudo-interpretation. (6) and 8(i) together show that this interpretation is the correct one. $\dashv$ (9)

1.36 Lemma. π "$P^*_M \subseteq P^*_{N.}$

Proof. Given (7) it suffices to show π"$P^m_M \subseteq P^m_N$ if $\omega\rho^m_M \leq \kappa$: however if $p \in P^m_M$ is chosen, there is a $C \in \Sigma_1^{(m-1)}(M)$ in p with $C \cap \omega\rho^m_M \notin M$. We have that π is Σ^*-elementary, and $\pi{\restriction}\kappa = \mathrm{id}{\restriction}\kappa$, hence if $\widetilde{C}$ has the same $\Sigma_1^{(m-1)}$ definition over N in $\pi(p)$, we shall have $C \cap \omega\rho^m_M = \widetilde{C} \cap \omega\rho^m_N \notin N$. Hence $\pi(p) \in P^m_N$. $\dashv$

(10) *For* $\langle a, f\rangle \in D$, $[a, f] = \pi(f)(a)$.

Proof of (10). All that we have left to show is (iii) in Definition 1.23, that N is the closure of $\mathrm{ran}(\pi) \cup \nu$ under good $\Sigma_1^{(n)}$ functions, for $\omega\rho^{n+1}_M > \kappa$. We've extended π to such functions so it is enough to show $[a, f] = \pi(f)(a)$, for $f \in \Gamma$. If $f \in M$ this would be a standard argument (see, e.g., [46] 8.4 Claim 4). So suppose $f \in \Gamma_n$, with $\mathrm{dom}(f) = [\kappa]^k$, for some n with $\rho^{n+1}_M > \kappa$, and let G be a functionally absolute $\Sigma_1^{(n)}$ definition of the graph of f in some parameter $p \in M$. Let $\widetilde{G}$ be given by the same definition over N in $\pi(p)$. Then $[\kappa]^k = \{u \mid G(f(u), u, p)\} \in E_a$. Hence $\widetilde{G}([a, f], a, \pi(p))$. Then $\pi(f)(a) = [a, f]$ as $\widetilde{G}$ is the graph of $\pi(f)$. $\dashv$ (10)

We are only left with proving the last clause (v) of the theorem that $\mathcal{P}(\kappa) \cap \mathbf{\Sigma}^*(M) = \mathcal{P}(\kappa) \cap \mathbf{\Sigma}^*(N)$. That $\subseteq$ holds is given by (9). For $\supseteq$ we argue as follows: if for all n $\rho^n_M > \kappa$ then $\mathcal{P}(\kappa) \cap \mathbf{\Sigma}^*(N) \subseteq \mathcal{P}(\kappa) \cap N = \mathcal{P}(\kappa) \cap M$. Otherwise for some n we have $\omega\rho^{n+1}_M \leq \kappa < \omega\rho^n_M$. Lemma 1.35 is not quite sufficient, as we need to prove a result for further $m \geq n+1$.

1.37 Lemma. *Let n be such that $\omega\rho_M^{n+1} \le \kappa < \omega\rho_M^n$. Let $\varphi(v^m, v^{m-1}, \dots, v^n, v^{n-1}, \dots, v^0)$ be a $\Sigma_1^{(m)}$ formula for some $m \ge n$. Let $a \in [\nu]^{<\omega}$ and $f^{n-1}, \dots, f^0$ be such that $f^j \in \Gamma_j, f^j : [\kappa]^{|a|} \longrightarrow H_M^j$ for $j < n$. Then there is a $\Sigma_1^{(m)}$ formula $\varphi'(v^m, v^{m-1}, \dots, v^n)$ which is effectively obtainable from φ, the $\boldsymbol{\Sigma_1}(M)$ definition of E_a, and the $\boldsymbol{\Sigma_1^{(n-1)}}(M)$ definitions of the f^j, with the property that $(\forall x^m \cdots \forall x^{n+1} \in M)(\forall f \in \Gamma_n$, with $\mathrm{dom}(f) = [\kappa]^{|a|})$*

$$N \models \varphi(x^m, \dots, x^{n+1}, [a,f], [a, f^{n-1}], \dots, [a, f^0]) \\ \iff M \models \varphi'(x^m, \dots, x^{n+1}, f)\,.$$

Proof. This is by induction on $m \ge n$. To simplify notation, we shall assume the $\boldsymbol{\Sigma_1^{(n-1)}}(M)$ definitions of the f^j, and the $\boldsymbol{\Sigma_1}(M)$ definition of E_a, are in face lightface, so parameter free. (It is routine to carry these parameters along otherwise.) We first consider the case that $m = n$. (This means that the variables $v^m, \dots, v^{n+1}$ are absent from φ.) Let $\varphi(v^n, v^{n-1}, \dots, v^0) \equiv \exists w^n \psi(w^n, v^n, v^{n-1}, \dots, v^0)$, where $\psi \in \Sigma_0^{(n)}$. Now for the requisite kind of f and f^j:

$$N \models \exists w^n \psi(w^n, [a,f], [a, f^{n-1}], \dots, [a, f^0]) \\ \iff (\exists y^n \in H_M^n) N \models \exists w^n \in \pi(y^n) \psi(w^n, [a,f], [a, f^{n-1}], \dots, [a, f^0])$$

(as $\pi{\restriction}H_M^n$ is cofinal into H_n); now by Łoś we have that:

$$(\exists y^n \in H_M^n)\{u \mid M \models \exists w^n \in y^n \psi(w^n, f(u), f^{n-1}(u), \dots, f^0(u))\} \in E_a \\ \iff (\exists y^n \in H_M^n)(\exists z^n \in H_M^n)(z^n \in E_a \\ \wedge z^n = \{u \mid M \models \exists w^n \in y^n \psi(w^n, f(u), f^{n-1}(u), \dots, f^0(u))\})\,.$$

This last line can be seen to be a $\Sigma_1^{(n)}$ formula after substituting the $\Sigma_1^{(n-1)}$ definitions of the good $f^j(u)$ for v^j, and is our φ'.

For $m \ge n+1$ the argument is similar, but simpler: let $\varphi \equiv \exists w^m \psi$ with $\psi \in \Sigma_0^{(m)}$. We suppose that we have proven the effectiveness of the translation procedure for $\Sigma_1^{(n-1)}$ and hence for the atomic components of ψ; further that the result works inductively through the connectives, trivially enough. The argument for bounded quantifiers is similar to that for unbounded; we thus concentrate here on this one step, and assume as inductive hypothesis that the result holds for ψ. Now, again assuming we have the requisite functions f and f^j and:

$$N \models \varphi(x^m, \dots, x^{n+1}, [a,f], [a, f^{n-1}], \dots, [a, f^0]) \\ \iff (\exists w^m \in H_M^m) N \models \psi(w^m, x^m, \dots, x^{n+1}, [a,f], [a, f^{n-1}], \dots, [a, f^0]) \\ \iff (\exists w^m \in H_M^m) M \models \psi'(w^m, x^m, \dots, x^{n+1}, f) \\ \iff M \models \exists w^m \psi'(w^m, x^m, \dots, x^{n+1}, f)\,,$$

using that $H^n_M = H_n$ at the first equivalence, and the inductive hypothesis at the second. $\dashv$

This now can be used to show that $\mathcal{P}(\kappa)\cap\boldsymbol{\Sigma}_{\mathbf{1}}^{(\boldsymbol{m})}(M) \supseteq \mathcal{P}(\kappa)\cap\boldsymbol{\Sigma}_{\mathbf{1}}^{(\boldsymbol{m})}(N)$. Hence the last clause (v) is now proven and the proof of the Ultrapower Theorem is complete. $\dashv$ 1.32.

1.3. Pseudo-Ultrapowers

We develop a theory of *pseudo-ultrapowers*. Ordinary ultrapowers (both coarse and fine) can be considered as special types of pseudo-ultrapowers. The latter are used in many combinatorial constructions, and particularly come into their own in the proof of the Covering Lemma for fine structural inner models, or for Global Square, $\square$, or indeed many areas where combinatorial arguments involve fine structure.

One can think of pseudo-ultrapowers as an attempt to develop an answer to the following question. Suppose we have acceptable structures $\overline{Q}$, Q with a map $\sigma : \overline{Q} \longrightarrow_{\Sigma_0} Q$ cofinally between them. Suppose further that $\overline{Q} = J^{\overline{A}}_{\overline{\nu}}$, $Q = J^A_\nu$and moreover that $\overline{Q}$ is extended by an acceptable $\overline{M} = \langle J^{\overline{A}}_{\overline{\beta}}, \overline{B}\rangle$ where either $\overline{\nu} = \overline{\beta}$ or else $\overline{\nu}$ is a cardinal in $\overline{M}$. Can we find a structure M and an extension of σ to a structure preserving map $\widetilde{\sigma} \supseteq \sigma, \widetilde{\sigma} : \overline{M} \longrightarrow M$, where $M = \langle J^A_\beta, B\rangle$? We should hope that the extended map $\widetilde{\sigma}$ would be sufficiently fine-structural. What we should like is that $\widetilde{\sigma}$ should be $\Sigma_1^{(n)}$ for any n with $\omega\rho^n_{\overline{M}} \geq \overline{\nu}$. This may not be always possible for every such n, but with certain conditions on $\overline{M}$ we can achieve this. Weakening these conditions could allow us to weaken the conclusion: for example, we could have this level of elementarity if $\omega\rho^{n+1}_{\overline{M}} \geq \overline{\nu}$.

Requiring additionally that $\overline{\nu}$ be regular in $\overline{M}$ will ensure that the extended map takes $\overline{\nu}$ cofinally into its image. This requirement is a feature of the Global Square proof where we use a pseudo-ultrapower that takes just enough functions to preserve the regularity of $\overline{\nu}$ in the sense of $\overline{M}$ whilst at the same time preserving the fact that it is $\boldsymbol{\Sigma}_{\mathbf{1}}^{(\boldsymbol{n})}(\overline{M})$ singularized for some $n < \omega$.

1.38 Definition. $\Gamma = \Gamma_{\overline{M},\overline{\nu}} =_{\mathrm{df}}$

$$\{f \mid (\mathrm{dom}(f) \in \overline{Q} \wedge \mathrm{ran}(f) \subseteq \overline{M}) \\ \wedge ((f \in \boldsymbol{\Sigma}_1^{(n)}(\overline{M}) \wedge \omega\rho^{n+1}_{\overline{M}} \geq \overline{\nu}) \vee (f \in H^n_{\overline{M}} \wedge \omega\rho^n_{\overline{M}} \geq \overline{\nu}))\}\,.$$

We now define a term model, whose domain is:

$$D = \{\langle a, f\rangle \mid f \in \Gamma_{\overline{M},\overline{\nu}} \wedge\ a \in \sigma(\mathrm{dom}(f))\}\,.$$

We define a pseudo-epsilon relation e and a congruence I by:

$$\begin{aligned}\langle a,f\rangle\, e\, \langle b,g\rangle &\Longleftrightarrow \langle a,b\rangle \in \sigma(\{\langle u,v\rangle \mid f(u)\in g(v)\})\\ \langle a,f\rangle\, I\, \langle b,g\rangle &\Longleftrightarrow \langle a,b\rangle \in \sigma(\{\langle u,v\rangle \mid f(u)= g(v)\})\\ \dot{C}(\langle a,f\rangle) &\Longleftrightarrow a\in\sigma(\{u \mid f(u)\in C\}) \text{ for } C=\overline{A},\ \overline{B}.\end{aligned}$$

We set $\mathbb{D}=\mathbb{D}_{\bar{Q},\bar{\sigma},\bar{M}}=\langle D,I,e,\dot{A},\dot{B}\rangle$.

1.39 Lemma. *Let $\varphi(v_0,\ldots,v_k)$ be a Σ_0 formula. Let $\langle a_0,f_0\rangle,\ldots,\langle a_k,f_k\rangle\in D$,and let $b=\langle a_0,\ldots a_k\rangle$. Then*

$$\begin{aligned}&\mathbb{D}\models\varphi(\langle a_0,f_0\rangle,\ldots,\langle a_k,f_k\rangle)\\ &\qquad\qquad\Longleftrightarrow b\in\sigma(\{\langle u_0,\ldots u_k\rangle \mid M\models\varphi(f_0(u_0),\ldots,f_k(u_k))\})\,.\end{aligned}$$

There is very little difference between the proof of this and the previous Lemma 1.29. By this, we can see that I is an identity for $\mathbb{D}$ and $\mathbb{D}$ models Extensionality. We shall for the purposes of this development assume that e is wellfounded. We thus can define a transitivisation map

$[\ \ ]: D\longrightarrow M;$

$[x]=[y]\Longleftrightarrow x\ \ I\ y;\ [x]\in[y]\Longleftrightarrow x\ \ e\ y;$

$A_M([x])\Longleftrightarrow\dot{A}(x); B_M([x])\Longleftrightarrow\dot{B}(x).$

With Σ_0-elementarity guaranteed by the last lemma we define:

1.40 Definition. $\widetilde{\sigma}$, the canonical extension of σ, is defined by: $\widetilde{\sigma}(x)=[\langle 0,\{\langle 0,x\rangle\}\rangle]$; M is the pseudo-ultrapower of $\overline{M}$ by σ.

1.41 Lemma. $\widetilde{\sigma}{\restriction}\overline{Q}=\sigma$.

Proof. Set $\widetilde{Q}=\bigcup\widetilde{\sigma}``\overline{Q}$. Then $\widetilde{Q}$ is transitive. We shall verify: $\widetilde{Q}=\{[\langle a,f\rangle]\mid f\in\overline{Q}\}$. For $(\supseteq)$, pick a,f as specified, and set $b=\mathrm{ran}(f)$. As $f\in\overline{Q}$, so is b. Then $[a,f]\in[0,\langle 0,b\rangle]$ as can be directly verified from the Łoś lemma above. For $(\subseteq)$, let $x\in\widetilde{\sigma}(y)$ with $y\in\overline{Q}$. If $x=[a,f]$ with $f\in\Gamma, a\in\sigma(\mathrm{dom}(f))$, then $a\in\sigma(\{u\in\mathrm{dom}(f)\mid f(u)\in y\})$. However if we set $f'=_{\mathrm{df}} f\cap(\mathrm{dom}(f)\times y)$, (where $\mathrm{dom}(f)\times y$ is a member of $\overline{Q}$) when f itself is not in $\overline{M}$, then this is at most $\mathbf{\Sigma}_1^{(n)}(\overline{M})$ where $\omega\rho_{\overline{M}}^{n+1}\geq\bar{\nu}$; it thus lies in $\overline{M}$ and hence by acceptability, in $\overline{Q}$. It is easy to verify by Łoś again that $[a,f]=[a,f']$, and we have thus verified $(\subseteq)$.

Define $\sigma':\widetilde{Q}\longrightarrow Q$ by $\sigma'([a,f])=\sigma(f)(a)$. We have just shown that $\mathrm{dom}(\sigma')=\widetilde{Q}$. σ' is well-defined:

$$\begin{aligned}[][a,f]=[b,g]\ &\Longleftrightarrow \langle a,b\rangle\in\sigma(\{\langle u_0,u_1\rangle\mid f(u_0)=g(u_1)\})=\\ &\qquad\{\langle u_0,u_1\rangle\in\sigma(\mathrm{dom}(f)\times\mathrm{dom}(g))\mid\sigma(f)(u_0)=\sigma(g)(u_1)\}\\ &\Longleftrightarrow\sigma(f)(a)=\sigma(g)(b)\,.\end{aligned}$$

A similar pair of equivalences show that $[a,f] \in [b,g] \iff \sigma(f)(a) \in \sigma(g)(b)$, and hence σ' is $\in$-preserving.

σ' is onto Q: this is clear, since if $v \in Q$ is arbitrary and $v \in \sigma(x)$ for some $x \in \overline{Q}$ (as σ is cofinal) then let $f \in \overline{Q}$ be the constant function $f : \gamma \longrightarrow x$ with constant value x with $\gamma < \bar{\nu}$; then clearly $v = \sigma(f)(\alpha)$ for some $\alpha < \sigma(\gamma)$. Thus σ' can only be the identity and $\widetilde{\sigma}(x) = [0, \{\langle 0, x\rangle\}] = \sigma(x)$. $\dashv$

The following gives the basic preservation properties of pseudo-ultrapowers. The reader should compare this with Theorem 1.32.

1.42 Theorem (Pseudo-Ultrapower Theorem). *Let $\widetilde{\sigma} : \bar{M} \longrightarrow_{\Sigma_0} M$ be the canonical extension of $\sigma : \overline{Q} \longrightarrow_{\Sigma_0} Q$. Then*

(i) *$\widetilde{\sigma}$ is Q-preserving, M is an acceptable end extension of Q, and $M = \{\widetilde{\sigma}(f)(u) \mid u \in \sigma(\mathrm{dom}(f)), f \in \Gamma\}$.*

(ii) (a) *$\widetilde{\sigma}$ is $\Sigma_0^{(n)}$-preserving for n with $\omega\rho^n_{\bar{M}} \geq \overline{\nu}$;*

(b) *$\widetilde{\sigma}$ is $\Sigma_2^{(n)}$-preserving for n with $\omega\rho^{n+1}_{\bar{M}} \geq \overline{\nu}$.*

Proof. We stratify the functions that will be responsible for the various projectum levels of the target structure M, which we shall name as H_n at first, later demonstrating that they have the correct interpretation. The reader will thus see that the tactic is analogous to that of the previous fine-structural ultrapower result.

Define

$$\Gamma_n = \begin{cases} \{f \in \Gamma \mid \mathrm{ran}(f) \subseteq H^n_{\bar{M}}\} \text{ if } \omega\rho^{n+1}_{\bar{M}} \geq \bar{\nu} \\ \{f \in \Gamma \mid \mathrm{ran}(f) \in H^n_{\bar{M}}\} \text{ if } \omega\rho^{n+1}_{\bar{M}} < \bar{\nu} \leq \omega\rho^n_{\bar{M}} \end{cases}$$

$$H_n = \{[\langle a, f\rangle] \mid f \in \Gamma_n \wedge \langle a, f\rangle \in D\} \text{ if } \Gamma_n \text{ is defined.}$$

$$\omega\rho_n = H_n \cap \mathrm{On}.$$

Note (i) If $\omega\rho^{n+1}_{\bar{M}} < \bar{\nu} \leq \omega\rho^n_{\bar{M}}$ and $f \in \Gamma_n$ then $f \in \bar{M}$.

Then as for standard ultrapowers:

(1) *H_n is transitive.*

(2) *Let $\langle\overrightarrow{\langle a_i, f_i\rangle}\rangle \in {}^m D$, where $f_i \in \Gamma_{j_i}$ and $j_i \leq n$, let $\varphi(v_1^{j_1}, \ldots v_m^{j_m}) \in \Sigma_0^{(n)}$ where $\omega\rho^n_{\bar{M}} \geq \bar{\nu}$ (or $\Sigma_1^{(n)}$ if $\omega\rho^{n+1}_{\bar{M}} \geq \bar{\nu}$). Then*

$$M \models \varphi[\overrightarrow{[a_i, f_i]}] \iff \vec{a_i} \in \sigma(\{\vec{u} \mid M \models \varphi[\overrightarrow{f_i(\vec{u})}]\}) \,.$$

This directly yields as for fine ultrapowers (but only in the pseudo-interpretation):

(3) *$\widetilde{\sigma} : \bar{M} \longrightarrow_{\Sigma_1^{(n)}} M$ for any n with $\omega\rho^n_{\bar{M}} \geq \bar{\nu}$, and $\widetilde{\sigma} : \bar{M} \longrightarrow_{\Sigma_2^{(n)}} M$ for any n with $\omega\rho^{n+1}_{\bar{M}} \geq \overline{\nu}$.*

(4) *M is acceptable. $\widetilde{\sigma}$ is Q-preserving, and $H_n = |J^{A'}_{\rho_n}|$ where $|M| = |J^{A'}_{\alpha'}|$, for n with $\omega\rho^n_{\bar{M}} \geq \bar{\nu}$.*

Proof of (4). If $\rho^1_{\bar{M}} \geq \bar{\nu}$ then by (3) $\widetilde{\sigma}$ is at least Σ_2-preserving; if $\rho^1_{\bar{M}} < \bar{\nu}$ then $\widetilde{\sigma}$ is actually a cofinal map (it is using only functions f *in*: M see the Note above). In either case Q properties are preserved, and acceptability is such. The last part follows just as in (5) of Theorem 1.32, again noting that $\widetilde{\sigma}$ is cofinal into H_n if $\omega\rho^n_{\bar{M}} \geq \overline{\nu} > \omega\rho^{n+1}_{\bar{M}}$, since only functions $f \in \Gamma_n \subseteq |\bar{M}|$ are used. $\dashv$ (4)

(5) $\quad \rho_m = \rho^m_M$ if $\bar{\nu} \leq \omega\rho^{m+1}_{\bar{M}}$; $\quad \rho_m \leq \rho^m_M$ if $\nu \leq \omega\rho^m_{\bar{M}}$.

Proof of (5). This is by induction on m; for $m = 0$ it is trivial. We imitate the corresponding proofs of (6) in 1.32.

(5a) *$\langle H_m, A\rangle$ is amenable for $A \in P(\omega\rho_m) \cap \mathbf{\Sigma}^{(m-1)}_1(M)$. Hence $\rho_m \leq \rho^m_M$.*

Proof of (5a). As before suppose $A \subseteq \omega\rho_m$ and is $\Sigma^{(m-1)}_1(M)$ in the parameter $[a, f]$, say $A(x) \iff N \models \varphi(x, [a, f])$ with $\varphi \in \Sigma^{(m-1)}_1$. Let $w = [b, g] \in H_m$ (again without loss of generality we shall assume $a = b$ here, and similarly by enlarging domains if need be, that $\mathrm{dom}(f) = \mathrm{dom}(g)$ (it suffices given f, g, a, b as here to define functions f', g' with $\mathrm{dom}(f') = \mathrm{dom}(g') = \mathrm{dom}(f) \times \mathrm{dom}(g)$ and further amalgamate to get $c = \langle a, b\rangle$ etc.). We require $A \cap w \in H_m$. This time define

$$h(y^m, v^0) = \{t^m \in y^m \mid \bar{M} \models \varphi(t^m, v^0)\}\,.$$

Thus $h(y^m, v^0)$ is a $\Sigma^{(m)}_0$ function of value type m and thus is in Γ, being good. We may now define $k : \mathrm{dom}(g) \longrightarrow H^m_M$ where $k(u) = h(g(u), f(u))$.

If $\bar{\nu} \leq \omega\rho^{m+1}_M$ we may conclude that $k \in \Gamma_m$. If $\omega\rho^{m+1}_M < \bar{\nu} \leq \omega\rho^m_M$ we again have that k is $\Sigma^{(m)}_0$-definable in parameters (note now that $g \in H^m_M$) and we conclude that $k \in H^m_M$ for the same reasons. If

$$\mathrm{dom}(g) = \{u \mid \overline{M} \models \forall t^m \in g(u)[\varphi(t^m, f(u)) \longleftrightarrow t^m \in k(u)]\}$$

and as $a \in \sigma(\mathrm{dom}(g))$, by the Łoś theorem we have:

$$M \models \forall t^m \in [a, g](\varphi(t^m, [a, f]) \longleftrightarrow t^m \in [a, k])\,.$$

Hence $[a, k] \cap w = [a, g] \cap \{t^m \mid M \models \varphi(t^m, [a, f])\} = A \cap w \in H_m$. $\dashv$ (5a)

(5b) *Assume $\bar{\nu} \leq \omega\rho^{m+1}_{\bar{M}}$. Then there is an $A \subseteq \omega\rho_m \cap \mathbf{\Sigma}^{(m-1)}_1(M)$ such that $A \notin M$. Hence $\rho_m \geq \rho^m_M$.*

Proof of (5b). Suppose $\bar{A}$ is $\Sigma^{(m-1)}_1(\overline{M})$-definable in a parameter $\overline{p}$, but with $\bar{A} \cap \omega\rho^m_{\overline{M}} \notin \overline{M}$, for example we can find such using $\bar{p} \in P^m_{\overline{M}}$. Let A be $\Sigma^{(m-1)}_1(M)$ in $\pi(\bar{p}) = p$ by the same formula φ, say. We show that

$A \cap \omega\rho_m \notin M$. Suppose otherwise. Let $A \cap \omega\rho_m = [a, f] \in M$. Then, in the pseudo-interpretation, $M \models \forall x^m(x^m \in [a, f] \longleftrightarrow \varphi(x^m, p))$. By the Łoś theorem $a \in \sigma(w)$ where

$$w = \{u \mid \overline{M} \models \forall x^m(x^m \in f(u) \longleftrightarrow \varphi(x^m, \bar{p}))\}.$$

So $w \neq \varnothing$ so there is a u with $f(u) = \bar{A} \cap \rho^m_{\bar{M}} = A \cap \rho^m_{\bar{M}}$ – a contradiction. $\dashv$ ((5)&1.42

It is possible to improve the elementarity of the embedding where $\omega\rho^n_{\bar{M}} \geq \bar{\nu} > \omega\rho^{n+1}_{\bar{M}}$, under certain conditions.

1.43 Lemma. *$\widetilde{\sigma}$ is $\Sigma^{(n)}_0$-preserving and cofinal (hence $\rho_n = \rho^n_M$), and thus $\Sigma^{(n)}_1$-preserving, if (a) n satisfies $\omega\rho^n_{\bar{M}} \geq \bar{\nu} > \omega\rho^{n+1}_{\bar{M}}$, and (b) $R^n_{\bar{M}} \neq \varnothing$. Further $\widetilde{\sigma}$ "$R^n_{\bar{M}} \subseteq R^n_M$.*

For a proof of this we refer the reader to [62] Lemma 3.2.4. For our purposes in proving Global Square, we shall need a refinement of the above construction. We should like the canonical extension map $\tilde{\sigma}$ to be sufficiently fine structure preserving to ensure that if $\bar{\nu}$ is regular in $\bar{M}$, but $\Sigma^{(n)}_1$ singularized over $\bar{M}$ (by this we mean that for some $\beta < \bar{\nu}$ there is a good $\Sigma^{(n)}_1$ map of a subset of β cofinally into $\bar{\nu}$), then whilst M still thinks that $\nu (= \tilde{\sigma}(\bar{\nu}))$ is regular, there is nevertheless also a good $\Sigma^{(n)}_1(M)$-singularizing map. This may require us to take fewer functions than the Definition 1.38 allows.

We now describe the small modification to do this. Assume then that we are in the situation under discussion:

1.44 Definition. Suppose that $\bar{\nu}$ is regular in $\bar{M}$, but is $\Sigma^{(k)}_1$-singularized over $\bar{M}$ for some k: we define $k = k(\bar{M})$ to be least $k < \omega$ for which this is true.

Note that this implies that $\omega\rho^{k+1}_{\bar{M}} \leq \bar{\nu}$. Moreover:

1.45 Lemma. $\omega\rho^{k+1}_{\bar{M}} \leq \bar{\nu} \leq \omega\rho^k_{\bar{M}}$

Proof. By the last remark we have to show that $\bar{\nu} \leq \omega\rho^k_{\bar{M}}$. Suppose this failed. Then this would imply that $k = m+1$ for some $m \in \omega$. Let $r \in \bar{M}$ be such that there is a good $\Sigma^{(k)}_1(\bar{M})$ function f singularizing $\bar{\nu}$ with domain of f contained in some $\gamma < \bar{\nu}$ and with $\omega\rho^k_{\bar{M}} < \gamma$. Let $X = \widetilde{h}^{m+1}_{\bar{M}}(\gamma \cup \{r, p\})$ be the $\Sigma^{(m)}_1(\bar{M})$ hull of the objects displayed where p has been chosen from $P^*_{\bar{M}}$. This is of a lower degree of elementarity, and hence $X \cap \bar{\nu}$ is bounded in $\bar{\nu}$. Let $\pi : N \longrightarrow \bar{M}$ be the inverse of the transitive collapse map on X. As $\pi{\restriction}\omega\rho^{k+1}_{\bar{M}} = \mathrm{id}{\restriction}\omega\rho^{k+1}_{\bar{M}}$ and $p \in \mathrm{ran}(\pi) \cap P^*_{\bar{M}}$ we map appeal to 1.20 and see that π is Σ^*-preserving. However the parameter r is in the $\mathrm{ran}(\pi)$ and thus $\mathrm{ran}(f) \subseteq \mathrm{ran}(\pi) = X$. This is absurd. $\dashv$

We now simply restrict the functions used in forming the canonical extension to those $\Sigma_1^{(n)}(\bar{M})$-definable functions, for an $n < k$.

1.46 Definition. Let $\bar{M}, \bar{\nu}, k = k(\bar{M})$ be as above. Define $\Gamma^k_{\bar{M},\bar{\nu}} =_{\rm df}$:

$$\{f \mid \mathrm{dom}(f) \in \overline{Q} \wedge \mathrm{ran}(f) \subseteq \overline{M}) \wedge (n < k \wedge f \in \mathbf{\Sigma}_1^{(n)}(\overline{M}) \wedge \omega\rho^{n+1}_{\overline{M}} \geq \overline{\nu})\}.$$

1.47 Definition. Suppose we construct the pseudo-ultrapower using functions from $\Gamma^k_{\bar{M},\bar{\nu}}$ where $k = k(\bar{M})$. Then the resulting map $\tilde{\sigma}$ is the *canonical k-extension* of σ and M the *canonical k-pseudo-ultrapower.*

1.48 Theorem (k-Pseudo-Ultrapower Theorem). *Let $\bar{M}, \bar{\nu}, k = k(\bar{M})$ be as above. Let $\widetilde{\sigma} : \bar{M} \longrightarrow_{\Sigma_0} M$ be the canonical k-extension of $\sigma : \overline{Q} \longrightarrow_{\Sigma_0} Q$. Then*

(i) *$\widetilde{\sigma}$ is Q-preserving, M is an acceptable end extension of Q, and $M = \{\widetilde{\sigma}(f)(u) \mid u \in \sigma(\mathrm{dom}(f)), f \in \Gamma\}$.*

(ii) (a) *$\widetilde{\sigma}$ is $\Sigma_2^{(n)}$-preserving for $n < k$;*

(b) *$\widetilde{\sigma}$ is $\Sigma_0^{(k)}$-preserving.*

1.49 Lemma. *Suppose $\bar{\nu}$, $\bar{M}, n = k(\bar{M}), \widetilde{\sigma}, M$, are as above. Then*

(i) *$\rho_n = \rho^n_M$, and $\widetilde{\sigma}$ is $\Sigma_0^{(n)}$-preserving and cofinal (thus $\Sigma_1^{(n)}$-preserving);*

(ii) *$\widetilde{\sigma}(\bar{\nu}) = \nu$ and the latter is regular in M;*

(iii) *n is least so that there is a $\mathbf{\Sigma_1^{(n)}}(M)$ map cofinalising ν.*

Note: As in Lemma 1.45 the hypothesis implies here too that $\omega\rho^n_{\bar{M}} \geq \overline{\nu} \geq \omega\rho^{n+1}_{\bar{M}}$.

Proof. From what we have already done, (i) will follow if we can show $\rho_n \geq \rho^n_M$. We first note that $\widetilde{\sigma}(\bar{\nu})$ is regular in M. For $\widetilde{\sigma}(\bar{\nu})$ this is either because $n = 0$ and $\widetilde{\sigma}$ is then Σ_0 and cofinal - hence Σ_1-preserving, or because $n > 0$ and by the above $\widetilde{\sigma}$ is Σ_2-preserving. As one would hope $\widetilde{\sigma}(\bar{\nu}) = \nu$: suppose $\eta = \widetilde{\sigma}(f)(a) < \widetilde{\sigma}(\bar{\nu})$ for some $f \in \Gamma^{k(\bar{M})}$. Then $f \in \bar{M}$. Set $\delta = \sup(\mathrm{ran}(f) \cap \bar{\nu})$. As $\mathrm{dom}(f) \in \bar{M}$ and $\bar{\nu}$ is regular in $\bar{M}$, $\delta < \bar{\nu}$. Then $\mathrm{ran}(\widetilde{\sigma}(f)) \subset \widetilde{\sigma}(\delta)$. Thus $\eta < \widetilde{\sigma}(\delta)$. This proves (ii).

By hypothesis there is $\bar{f}$ a good function $\Sigma_1^{(n)}(\bar{M})$ in a parameter $\bar{p}$ say, that maps a subset of some $\gamma < \bar{\nu}$ cofinally into $\bar{\nu}$. (We'll assume $\mathrm{ran}(f) \subseteq \bar{\nu}$.) Let $p = \widetilde{\sigma}(\bar{p})$. Let $\widetilde{A} =_{\rm df} A^{n,p}_M \cap J^{A^{n,p}_M}_{\rho_n}$; $\widetilde{M} = \langle J^{\widetilde{A}}_{\rho_n}, \widetilde{A}\rangle$. Then $\widetilde{\sigma}{\restriction}\bar{M}^{n,\bar{p}} : \bar{M}^{n,\bar{p}} \longrightarrow_{\Sigma_0} \widetilde{M}$ cofinally. $\bar{f}$ is rudimentary over $\bar{M}^{n,\bar{p}}$ and we can let f be rudimentary over $\widetilde{M}$ by the same definition. Then $\mathrm{dom}(f)$ is bounded in ν. As $\widetilde{\sigma}\,{\restriction}\bar{\nu}$ is cofinal into ν (because σ is) we have $\widetilde{\sigma}``\mathrm{ran}(\bar{f}) \subseteq \mathrm{ran}(f)$ and the latter is thus cofinal in ν and hence f singularises ν. However if $\rho_n < \rho^n_M$ we should have $\widetilde{M}$, and thus f, both members of M. This contradicts the regularity of ν in M which we have just remarked upon above.

Thus $\widetilde{\sigma}$ is $\Sigma^{(n)}_0$-preserving and cofinal, hence $\Sigma^{(n)}_1$-preserving; $\widetilde{\sigma}(\bar{f})$ is thus well-defined, and is a good $\boldsymbol{\Sigma^{(n)}_1}(M)$ map cofinalising ν. $\dashv$

1.50 Remark. We can conclude from the above existence of the good cofinalising map that $\omega\rho^n_M \geq \nu \geq \omega\rho^{n+1}_M$ but *not* that $\nu > \omega\rho^{n+1}_M$ even if $\overline{\nu} > \omega\rho^{n+1}_{\bar{M}}$.

In the above the $\in$-relation, e, of the term model $\mathbb{D}$ was assumed wellfounded. The Interpolation Lemma to follow is again not in the most general form, but will be useful later. The hypotheses imply that $\omega\rho^n_{\bar{M}} \geq \overline{\nu} \geq \omega\rho^{n+1}_{\bar{M}}$ for the named n; we thus may form the term model using the functions in Γ_n. Since $\overline{M}$ is $\Sigma^{(n)}_0$-embeddable into the wellfounded model M by the map g below, we can then embed the term model into M *via* $g'([a, G]) = g(G)(a)$ and g' is then a $\Sigma^{(n)}_0$ map. Hence it is $\in$-preserving on its domain, and so demonstrates that e is wellfounded. The structure $\widetilde{M}$ is that formed by the pseudo-ultrapower construction as the canonical $k(\bar{M})$-extension of $\sigma = g{\restriction}J^{\overline{A}}_{\overline{\nu}} : J^{\overline{A}}_{\overline{\nu}} \longrightarrow J^A_\nu$ from above.

1.51 Lemma (Interpolation Lemma). *Suppose $\overline{M} = \langle J^{\overline{A}}_{\bar{\beta}}, \overline{B}\rangle$ is a structure such that $\overline{\nu}$ is regular in $\overline{M}$, but that $k(\bar{M})$ is defined, i.e. for some least $n < \omega$, there is a $\boldsymbol{\Sigma^{(n)}_1}(\overline{M})$ function singularising $\overline{\nu}$. Suppose further that $g : \overline{M} \longrightarrow_{\Sigma^{(n)}_1} M = \langle J^A_\beta, B\rangle$. Let $\widetilde{\nu} = \sup g``\overline{\nu}$. Let $\sigma_1 =_{\mathrm{df}} \mathrm{id}{\restriction}J^{A'}_{\widetilde{\nu}}$. Then there is a structure $M' = \langle J^{A'}_{\beta'}, B'\rangle$, a map $\widetilde{g} : \overline{M} \longrightarrow M'$ with (a) $\widetilde{g} \supseteq g{\restriction}J^{\overline{A}}_{\overline{\nu}}$ and (b) $\widetilde{g}$ is $\Sigma^{(n)}_0$ and cofinal at the n'th level (and hence $\Sigma^{(n)}_1$-preserving); further there is a unique $g' : M' \longrightarrow_{\Sigma^{(n)}_0} M$, with $g = g' \circ \widetilde{g}$ and $g'{\restriction}\widetilde{\nu} = \sigma_1$.*

$$\begin{tikzcd}
\langle J^{\bar{A}}_{\bar{\beta}}, \bar{B}\rangle \arrow[r, "\widetilde{\sigma} = \widetilde{g}\quad \Sigma^{(n)}_1"'] \arrow[rr, bend left, "g\quad \Sigma^{(n)}_1"] & \langle J^{A'}_{\beta'}, B'\rangle \arrow[r, "g'\quad \Sigma^{(n)}_0"'] & \langle J^A_\beta, B\rangle \\
\langle J^{\widetilde{A}}_{\bar{\nu}}, \widetilde{B}\rangle \arrow[u, hook] \arrow[r, "\sigma = g{\restriction}J^{\bar{A}}_{\bar{\nu}}"] & \langle J^{A'}_{\widetilde{\nu}}, B'\rangle \arrow[u, hook] \arrow[r, "g'{\restriction}J_{\widetilde{\nu}} = \sigma_1"] & \langle J^A_\nu, B\rangle \arrow[u, hook]
\end{tikzcd}$$

Figure I.1: The Interpolation Lemmata

Proof. As intimated, M' and $\widetilde{g}$ come from the construction of this section as the canonical $k(\bar{M})$-extension of the cofinal mapping $g\restriction J^{\overline{A}}_{\overline{\nu}} : J^{\overline{A}}_{\overline{\nu}} \longrightarrow_{\Sigma_0} J^{A}_{\widetilde{\nu}}$ playing the role of $\sigma : \overline{Q} \longrightarrow_{\Sigma_0} Q$. That $\widetilde{g}$ has the requisite properties has been established. g' is defined through: $g'(\widetilde{g}(G)(a)) = g(G)(a)$. Then g' has to be the unique such map with $g = g' \circ \widetilde{g}$ and $g'\restriction\widetilde{\nu} = \mathrm{id}\restriction\widetilde{\nu}$, since $g'(\widetilde{g}(G)(a)) = g' \circ \widetilde{g}(G)(g'(a)) = g(G)(a)$. (Note that trivially $g'\restriction\widetilde{\nu} = \mathrm{id}\restriction\widetilde{\nu} \longleftrightarrow g'\restriction J^{\widetilde{A}}_{\widetilde{\nu}} = \mathrm{id}\restriction J^{\widetilde{A}}_{\widetilde{\nu}}$.) $\dashv$

A slightly more generalised form of the above will be useful. This allows σ_1 to be different from the identity function.

1.52 Lemma (Generalized Interpolation Lemma). *Suppose $\overline{M} = \langle J^{\overline{A}}_{\overline{\beta}}, \overline{B}\rangle$ is a structure such that $\overline{\nu}$ is regular in $\overline{M}$, but that $k(\bar{M})$ is defined, i.e. for some least $n < \omega$, there is a $\boldsymbol{\Sigma}^{(n)}_{\mathbf{1}}(\overline{M})$ function singularising $\overline{\nu}$. Suppose further for $n = k(\bar{M})$ that $g : \overline{M} \longrightarrow_{\Sigma^{(n)}_1} M = \langle J^{A}_{\beta}, B\rangle$ and that $\sigma : J^{\overline{A}}_{\overline{\nu}} \longrightarrow J^{A'}_{\widetilde{\nu}}$, and $\sigma_1 : J^{A'}_{\widetilde{\nu}} : J^{A'}_{\widetilde{\nu}} \longrightarrow J^{A}_{\nu}$ are both Σ_0 with σ cofinal, and finally that $\sigma_1 \circ \sigma = g\restriction J^{\overline{A}}_{\overline{\nu}}$.*

Then there is a structure $M' = \langle J^{A'}_{\beta'}, B'\rangle$, a map $\widetilde{g} : \overline{M} \longrightarrow M'$ with (a) $\widetilde{g} \supseteq \sigma$ and (b) $\widetilde{g}$ is $\Sigma^{(n)}_0$ cofinal at the n'th level (and hence $\Sigma^{(n)}_1$-preserving); further there is a unique $g' : M' \longrightarrow_{\Sigma^{(n)}_0} M$, with $g = g' \circ \widetilde{g}$ and $g'\restriction\widetilde{\nu} = \sigma_1$.

Proof. The argument is just as before except now g' is defined through: $g'(\widetilde{g}(G)(a)) = g(G)(\sigma_1(a))$. Again g' has to be the unique such map with $g = g' \circ \widetilde{g}$ and $g'\restriction\widetilde{\nu} = \sigma_1\restriction\widetilde{\nu}$, since $g'(\widetilde{g}(G)(a)) = g' \circ \widetilde{g}(G)(g'(a)) = g(G)(\sigma_1(a))$. $\dashv$

2. Global $\Box$

We derive a global $\Box$ sequence in L, the constructible hierarchy. We assume now $V = L$. Our template is basically that of the proof of Jensen [1] here, although we use ideas from [30]. We shall build our C_ν sequence along a closed and unbounded class $S \subseteq \mathrm{Sing}$ of singular ordinals. The definition of the C_ν sets themselves we shall present is extremely uniform, and quite simple. They will be preserved by Σ_0 embeddings between levels of the J-hierarchy of a sufficient ordinal height to see that ν is singular. The choice of the closed and unbounded class S is also rather flexible: we require S again be simply defined so that membership in it is preserved downwards into Σ_1-elementary substructures. To this end we could take S to be the class of all limits of admissible ordinals. (Recall that α is *admissible* if $\langle L_\alpha, \in\rangle$ is a model of Kripke-Platek set theory.) Although this is fine from a technical point of view, it appears to require more closure on the structures J_α than it really needs. For most purposes it will be useful to assume that $\omega.\nu = \nu$ as a minimum, (this will have the effect of tying up C_ν with the ordinal height of J_ν), and that $\omega\nu$ is sufficiently closed under some further basic ordinal theoretic operations. We shall accordingly take S to be the class of *primitive recursively (p.r.) closed ordinals.*

2.1 Definition.

(i) The *primitive recursive set functions* are obtained by adding to the rudimentary functions the following schema:

$$f(\vec{x}, y) = g(\vec{x}, y, \langle f(\vec{x}, z) \mid z \in y\rangle)\,.$$

f is *primitive recursive (p.r.) in* $A \subseteq V^n$ iff f is p.r. in χ_A.

(ii) X is *p.r. closed* (*p.r. closed in* A) iff X is closed under the p.r. functions (the p.r. in A functions, respectively).

(iii) α is *p.r. closed* if L_α is p.r. closed.

2.2 Lemma.

(i) *ω is p.r. closed.*

(ii) *Let $\alpha > \omega$. Then α is p.r. closed iff α is closed under the functions f_i $(i < \omega)$ where:*

$$f_0(\nu) = \omega^\nu; \quad f_{i+1}(\nu) = \text{ the } \nu\text{th fixed point of } f_i.$$

If the reader desires then the above can be taken as a definition of a p.r. closed ordinal. It follows relatively easily that:

(1) $\alpha > \omega \wedge \alpha$ p.r. closed $\Longrightarrow \omega\alpha = \alpha$ and hence $J_\alpha = L_\alpha$.

(2) $\omega\alpha$ p.r. closed $\Longleftrightarrow$ J_α p.r. closed.

(3) $\{\alpha \mid \alpha$ is p.r. closed$\}$ is closed and unbounded in On.

(4) If α is p.r. closed but not a limit point of the p.r. closed ordinals, then $\alpha = \sup_i\{f_i(\beta) \mid \beta < \alpha\}$ and hence $\mathrm{cf}(\alpha) = \omega$.

Noting that p.r. closure is essentially a Π_2 property of a structure we obtain the relevant property for our subsequent definitions:

2.3 Lemma. *Let $\omega\alpha$ be p.r. closed, and let $f : J_{\overline{\alpha}} \longrightarrow_{\Sigma_1} J_\alpha$. Then $\omega\overline{\alpha}$ is p.r. closed.*

(As we commented above, the property of α being a limit of admissibles, would have satisfied the consequent of the above lemma too.)

2.4 Theorem ($V = L$). *Let $S \subseteq \mathrm{Sing}$ be as above. There is a uniformly definable class $\langle C_\nu \mid \nu \in S\rangle$ so that:*

(i) *$C_\nu \subseteq \nu$ is a set of ordinals unbounded in ν and closed beneath it;*

(ii) *$\mathrm{ot}(C_\nu) < \nu$;*

(iii) *$\overline{\nu} \in (C_\nu)^* \Longrightarrow \overline{\nu} \in S \wedge C_{\bar{\nu}} = \overline{\nu} \cap C_\nu$;*

(iv) *If $f : \langle J_{\overline{\nu}}, \overline{C}\rangle \longrightarrow_{\Sigma_1} \langle J_\nu, C_\nu\rangle$ then $\overline{\nu} \in S$ and $\overline{C} = C_{\overline{\nu}}$.*

We think of C_ν as a canonical singularising sequence for ν, i.e. it provides a sequence cofinal in ν, and of order type less than ν. It will be defined using the least level of the J-hierarchy, over which we can define a map from a bounded subset of ν cofinally into ν. Our desire for short sequences is expressed by (ii). The important clause is (iii) the *Coherence property*: the limit points $\bar{\nu}$ of C_ν, are themselves singular, and the initial segment $C_\nu \cap \bar{\nu}$ gives the canonical singularising sequence for $\bar{\nu}$ itself. (i)-(iii) are often referred to as (one form of) Global $\Box$. Clause (iv) is also a strong one, providing as it does information on condensation-like coherency. We shall first give a construction of a global sequence satisfying most of the above: it simply will not always be unbounded in ν – this may happen when $\mathrm{cf}(\nu) < \omega$. The theorem below contains the heart of the argument. Obtaining the theorem above from it, is relatively speaking, at least over L, a minor adjustment.

2.5 Theorem ($V = L$). *There is a uniformly definable class $\langle C_\nu \mid \nu \in S\rangle$ so that:*

(i) *C_ν is a set of ordinals ν and closed beneath it; if $\mathrm{cf}(\nu) > \omega$ then C_ν is unbounded in ν.*

(ii) *$\mathrm{ot}(C_\nu) < \nu$;*

(iii) *$\overline{\nu} \in (C_\nu) \Longrightarrow \overline{\nu} \in S \wedge C_{\bar{\nu}} = \overline{\nu} \cap C_\nu$;*

(iv) *If $f : \langle J_{\overline{\nu}}, \overline{C}\rangle \longrightarrow_{\Sigma_1} \langle J_\nu, C_\nu\rangle$ then $\overline{\nu} \in S$ and $\overline{C} = C_{\overline{\nu}}$.*

As indicated if ν is a singular ordinal, then there will be a least level $J_{\beta(\nu)}$ of the J-hierarchy over which ν is *definably singularised*, i.e. there will be a function (possibly partial) that is $\Sigma_\omega(J_{\beta(\nu)})$-definable (possibly in some parameters) mapping (a subset of) some γ cofinally into ν. However any such function is also good $\Sigma_1^{(n)}(J_{\beta(\nu)})$ for some n. This level $J_{\beta(\nu)}$ will be our *singularising structure* M_ν, and n is the level of complexity at which we shall work. We set out a formal definition of our subject matter. Notice, that given $J_{\beta(\nu)}$ all the objects defined below are Σ_0-definable from it.

2.6 Definition. Let $\nu \in S$. Then we associate the following to ν:

(a) $n_\nu =_{\mathrm{df}}$ the least $n \in \omega$ so that there is a good $\boldsymbol{\Sigma_1^{(n)}}(M_\nu)$ function singularising ν, where $M_\nu = J_{\beta(\nu)}$.

(b) $M_\nu^k =_{\mathrm{df}} M_\nu^{k,p_{M_\nu}\restriction k}$ for $k \leq n_\nu$.

(c) $h_\nu^k =_{\mathrm{df}} h_{M_\nu}^{k,p_{M_\nu}\restriction k}$; $h_\nu =_{\mathrm{df}} h_\nu^{n_\nu}$; $\widetilde{h}_\nu =_{\mathrm{df}} \widetilde{h}_{M_\nu}^{n_\nu+1}$.

(d) (i) $\omega\rho_\nu =_{\mathrm{df}} \mathrm{On} \cap M_\nu^{n_\nu}$; (ii) $\kappa_\nu \simeq$ the largest cardinal of J_ν, if such exists.

(e) $p_\nu =_{\mathrm{df}} p_{M_\nu} \backslash \nu$ if ν is a limit cardinal of J_ν; $p_\nu =_{\mathrm{df}} p_{M_\nu} \backslash \kappa_\nu$ otherwise; $q_\nu =_{\mathrm{df}} p_\nu \cap \omega\rho_{M_\nu}^{n_\nu}$.

(f) $\alpha_\nu =_{\mathrm{df}} \max\{\alpha < \nu \mid \nu \cap \widetilde{h}_\nu(\alpha \cup \{p_\nu\}) = \alpha\}$, setting $\max \varnothing = 0$.

(g) $\gamma_\nu =_{\mathrm{df}} \min\{\gamma < \nu \mid$ there exists a $\Sigma_1^{(n)\, M_\nu}(\{p_\nu\})$ partial function

$$F : a \longrightarrow \nu, \text{ cofinal, with } a \subseteq \gamma\}.$$

Thus if $\nu = \kappa_\nu^+$, note then we may possibly have κ_ν in p_ν (if κ_ν happens to be in p_ν). We also note that $p_\nu \backslash \nu$ ($= p_\nu$ unless κ_ν is defined) is the least parameter p so that $\widetilde{h}_\nu(\nu \cup p) = J_{\beta(\nu)}$. It is easy to see that α_ν in f) is always defined, as the set of α's specified is closed; note also that α_ν is perforce strictly less than the first ordinal γ partially mapped by $\widetilde{h}_\nu$(with parameter p_ν) cofinally into ν. We set that first ordinal γ at g) to be γ_ν. Such a γ_ν exists, since by assumption there is some good $\boldsymbol{\Sigma_1^{(n)}}(M_\nu)$ function singularising ν, using some parameter r. In fact as we are in L we may take $r = p_{M_\nu}$. If $\gamma' = \max(p_{M_\nu} \cap \nu)+1$ ($\max(p_{M_\nu} \cap \kappa_\nu)+1$ if κ_ν is defined), then clearly $\widetilde{h}_\nu(\gamma' \cup p_\nu)$ is cofinal in ν (as we have enough parameters in the domain of this hull to define our cofinalising map); hence $\gamma_\nu \leq \gamma'$ exists.

2.7 Lemma. $\omega\rho_{M_\nu}^{n_\nu} \geq \nu \geq \omega\rho_{M_\nu}^{n_\nu+1}$

Proof. Let $n = n_\nu$. If the second inequality failed, then the partial function $\boldsymbol{\Sigma_1^{(n)}}(J_{\beta(\nu)})$-singularising ν would be a subset of ν and thus a bounded subset of $\omega\rho_{M_\nu}^{n+1}$ belonging to $J_{\beta(\nu)}$. Suppose the first inequality failed. Then $n > 0$, and $\widetilde{h}_{M_\nu}^n$ maps $H_{M_\nu}^n = J_{\rho_{M_\nu}^n}$ onto $J_{\beta(\nu)} = M_\nu$. However $\widetilde{h}_{M_\nu}^n$ has a $\Sigma_1^{(n-1)}$ definition, which can be easily amended to provide a $\Sigma_1^{(n-1)}$ map from $\omega\rho_{M_\nu}^n$ onto ν – thus contradicting our definition of n. $\dashv$

2.8 Definition. For $\nu, \overline{\nu} \in S$:

(i) We set $f : \bar{\nu} \Longrightarrow \nu$ if $|f| : J_{\overline{\nu}} \longrightarrow_{\Sigma_1} J_\nu$, and $|f|$ is the restriction of $f^* : J_{\beta(\overline{\nu})} \longrightarrow_{\Sigma_1^{(n)}} J_{\beta(\nu)}$ where $n = n_\nu, \nu = f^*(\overline{\nu})$ if $\nu < \beta(\nu); \kappa_\nu \in \mathrm{ran}(f)$ (if κ_ν is defined); α_ν, p_ν are both in $\mathrm{ran}(f^*)$.

(ii) $\mathbb{F} =_{\mathrm{df}} \{\langle\bar{\nu}, |f|, \nu\rangle \mid f : \bar{\nu} \Longrightarrow \nu\}$. We sometimes write if $f : \bar{\nu} \Longrightarrow \nu$ that $\bar{\nu} = d(f)$ and $\nu = r(f)$.

(iii) If $\nu < \beta$, we set $p(\nu) =_{\mathrm{df}} p_\nu \cup \{\alpha_\nu, \nu, \kappa_\nu\}$ (the latter if κ_ν is defined); if $\nu = \beta$ then $p(\nu) =_{\mathrm{df}} p_\nu \cup \{\alpha_\nu, \kappa_\nu\}$ (again the latter only if it is defined).

(iv) $f^*_{(\delta,v,\nu)}$ is the inverse of the transitive collapse of the hull $\widetilde{h}_\nu(\delta \cup \{p(\nu), v\})$ in $J_{\beta(\nu)}$.

2.9 Lemma. *If $\exists\bar{\nu}(f : \bar{\nu} \Longrightarrow \nu)$ then $|f|$ and f^* are uniquely determined by* $\mathrm{ran}(|f|) \cap \nu$.

Proof. As $J_{\beta(\nu)}$ is sound, we have by our definitions, and the Fact (i) before Lemma 1.19, that $\widetilde{h}_\nu(\omega\nu \cup \{p_\nu\}) = J_{\beta(\nu)}$ (and a similar statement with $\bar{\nu}$ replacing ν throughout). We have a $\Delta_1(J_\nu)$ onto map $g : \omega\nu \twoheadrightarrow J_\nu$. Thus, if $Y = \widetilde{h}_\nu((\omega\nu \cap \mathrm{ran}(|f|)) \cup \{p_\nu\})$, then $Y = \widetilde{h}_\nu(\mathrm{ran}(|f|) \cup \{p_\nu\}) = \mathrm{ran}(f^*)$: if $x \in \mathrm{ran}(f^*)$, then by soundness of $J_{\bar{\beta}}$ above $\bar{\nu}$, $x = f^*(h_{\bar{\nu}}(\bar{\xi}, p_{\bar{\nu}}))$ for some $\bar{\xi} < \omega\bar{\nu}$; thus $x \in Y$. However $f^*(h_{\bar{\nu}}(\bar{\xi}, p_{\bar{\nu}})) = h_\nu(|f|(\bar{\xi}), p_\nu)$ since f^* is $\Sigma_1^{(n)}$-preserving, we have by Lemma 1.22 that $f^*(p_{\bar{\nu}}) = p_\nu$. The converse inclusion is immediate. ⊣

2.10 Remark. The next lemma will show that in clause (iv) there is some μ and a restriction map $|f_{(\delta,q,\nu)}|$ of $f^*_{(\delta,q,\nu)}$ so that $\langle\mu, |f_{(\delta,q,\nu)}|, \nu\rangle \in \mathbb{F}$.

In the sequel we shall not carefully distinguish $|f|$ (or f), from its *canonical extension* f^*, (observe Lemma 2.9), and sometimes write $f : J_{\bar{\nu}} \longrightarrow_{\Sigma_1} J_\nu$ where more correctly we should write $f^* {\upharpoonright} J_{\bar{\nu}}$ or $|f| : J_{\bar{\nu}} \longrightarrow_{\Sigma_1} J_\nu$ (as in the conclusion of the next Lemma). Or conversely if $f : \bar{\nu} \Longrightarrow \nu$ then we should properly write "$f^* : J_{\beta(\bar{\nu})} \longrightarrow_{\Sigma_1^{(n)}} J_{\beta(\nu)}$" but sometimes we are slip-shod and again simply substitute f for f^* by virtue of the last lemma.

We shall first prove a pair of lemmata concerning relationships between singularising structures, and associated maps between them. These will then facilitate the definition of our C_ν sequences.

2.11 Lemma. *Let $f : J_{\bar{\beta}} \longrightarrow_{\Sigma_1^{(n)}} J_{\beta(\nu)}$ where $n = n_\nu$; suppose $f(\bar{\alpha}, \bar{p}) = \alpha_\nu, p_\nu$, and (where appropriate) $f(\bar{\kappa}, \bar{\nu}) = \kappa_\nu, \nu$. (The latter if $\nu < \beta(\nu)$; if $\nu = \beta(\nu)$ then we take $\bar{\nu} = \bar{\beta}$.) Then $\bar{\nu} \in S$, $\bar{\beta} = \beta(\bar{\nu})$ (thus $M_{\bar{\nu}} = J_{\bar{\beta}}$), $f : \bar{\nu} \Longrightarrow \nu$; moreover $n, \bar{\alpha}, \bar{p}, \bar{\kappa}$ (the latter when defined) are $n_{\bar{\nu}}, \alpha_{\bar{\nu}}, p_{\bar{\nu}}, \kappa_{\bar{\nu}}$.*

(Remark: we shall drop the qualification "when defined" from now on and will also let the reader include or exclude the ordinal ν as appropriate.)

Proof. We verify that $J_{\bar{\beta}}$ is the singularising structure for $\bar{\nu}$ and the other parameters are as shown, and thus are moved correctly by f. That $\bar{\nu} \in S$ is trivial if $\nu \in \mathrm{ran}(f)$ and follows from Lemma 2.3 otherwise.

(1) $\bar{p} = p_{J_{\overline{\beta}}} \backslash \overline{\nu}$.

Proof of (1). By the Condensation Lemma 1.22 $\dashv$ (1)

Setting $\beta = \beta(\nu)$, we let $\overline{h}$ have the same functionally absolute definition over $J_{\bar{\beta}}$ as $\widetilde{h}_\nu$ does over J_β. $\overline{h}$ is thus $\mathbf{\Sigma_1^{(n)}}(J_{\bar{\beta}})$.

(2) *$\overline{\alpha}$ is defined from $J_{\bar{\beta}}$ as α was defined from $J_{\beta(\nu)}$.*

Proof of (2). Set

$$H(\xi^n, \zeta^n) \Longleftrightarrow \widetilde{h}_\nu(\omega\xi^n \cup \{p_\nu\}) \cap \nu \subseteq \zeta^n$$
$$\overline{H}(\xi^n, \zeta^n) \Longleftrightarrow \bar{h}(\omega\xi^n \cup \{\bar{p}\}) \cap \bar{\nu} \subseteq \zeta^n .$$

The former H is then $\Pi_1^{(n)J_\beta}(\{p_\nu, \nu\})$, ($\nu$ is omitted if $\nu = \omega\rho^n_{J_\beta}$); the latter $\overline{H}$ is $\Pi_1^{(n)J_{\overline{\beta}}}(\{\bar{p}, \bar{\nu}\})$, by the same definition (again, we include $\bar{\nu}$ when $\bar{\nu} < \omega\rho^n_{J_{\overline{\beta}}}$, which occurs by the $\Sigma_1^{(n)}$-elementarity of f^* when $\nu = \omega\rho^n_{J_\beta}$). As $J_\beta \models H(\alpha, \alpha)$ we have $J_{\bar{\beta}} \models \overline{H}(\bar{\alpha}, \bar{\alpha})$. However for any ξ with $\bar{\alpha} < \xi^n < \bar{\nu}$ we must have $J_{\bar{\beta}} \models \neg\overline{H}(\xi^n, \xi^n)$, because $J_\beta \models \neg H(f(\xi^n), f(\xi^n))$ since $\alpha < f(\xi^n) < \nu$. These properties describe α and $\bar{\alpha}$. $\dashv$ (2)

(3) $\exists \overline{\xi}^n < \bar{\nu}(\widetilde{h}_\nu(f(\bar{\xi}^n) \cup \{p_\nu\})$ *is unbounded in* ν).

If (3) holds for some $\overline{\xi}$ then, in fact $\bar{h}(\xi \cup \{\bar{p}\})$ is cofinal in $\bar{\nu}$, since

$$J_\beta = M_\nu \models (\forall\zeta^n < \nu)(\exists\delta^n < f(\bar{\xi}^n))(\exists i < \omega)(\zeta^n < \widetilde{h}_\nu(i, \langle\delta^n, p_\nu\rangle) < \nu)$$

is a $\Pi_2^{(n)}$ expression and hence goes down to $J_{\bar{\beta}}$ in the objects $\bar{p}, \bar{h}$, $\bar{\xi}$ and $\bar{\nu}$ (the latter if $\nu = f(\bar{\nu}) < \omega\rho^n_{J_\beta}$) It thus witnesses that $\bar{\beta} = \beta(\bar{\nu})$ and $n \geq n_{\bar{\nu}}$. Thus also $\bar{h}$ is a suitable singularising function for $\bar{\nu}$. So suppose (3) fails. Note that then we must have $\tau =_{\mathrm{df}} \sup f``\bar{\nu} \leq \gamma_\nu < \nu$. However, then we have:

(4) $\tau \neq \nu \cap \widetilde{h}_\nu(\tau \cup \{p_\nu\})$.

We cannot have equality here as that would imply that $\tau \leq \alpha_\nu$. However the latter is in $\mathrm{ran}(f{\restriction}\bar{\nu})$!

Hence the following true in J_β:

$$\exists i \in \omega \exists \xi^n < \tau(\nu > \widetilde{h}_\nu(i, \langle\xi^n, p_\nu\rangle) \geq \tau) .$$

Let i, ξ^n witness this, and pick $\bar{\delta} < \bar{\nu}$ so that $f(\bar{\delta}) > \xi^n$. Then for any $\bar{\mu} < \bar{\nu}$ as $f(\bar{\mu}) < \tau$:

$$J_\beta \models (\exists\zeta^n < f(\bar{\delta}))(\nu > \widetilde{h}_\nu(i, \langle\zeta^n, p_\nu\rangle) \geq f(\bar{\mu})) .$$

This is $\Sigma_1^{(n)}$ and hence, for all $\bar{\mu} < \bar{\nu}$, goes down to $J_{\bar{\beta}}$. Now:

$$J_{\bar{\beta}} \models \forall \bar{\mu} < \bar{\nu} \exists \zeta^n < \bar{\delta}(\bar{\nu} > \bar{h}(i, \langle \zeta^n, \bar{p} \rangle \geq \bar{\mu}).$$

Again, this says that $\bar{h}$ is a suitable singularising function for $\bar{\nu}$. The only possibility left is that (4) could hold. However that would imply that $\tau \leq \alpha_\nu$ which we have already ruled out.

(5) $\bar{\beta} = \beta(\bar{\nu})$ *and* $n = n_{\bar{\nu}}$.

We have just seen that $\bar{\beta} = \beta(\bar{\nu})$ and $n \geq n_{\bar{\nu}}$. Suppose $m < n$ and that $\bar{g}$ is a $\Sigma_1^{(m)}(J_{\bar{\beta}})$ good function in the parameter $\bar{r}$. Let g be $\Sigma_1^{(m)}(J_\beta)$ using the same functionally absolute definition and the parameter $f(\bar{r})$. Suppose $\bar{\delta} < \bar{\nu}$. By the $\Sigma_1^{(n)}$-elementarity of f we have the following $\Sigma_1^{(n)}$ statement holds in J_β (as $\operatorname{ran}(g \restriction f(\bar{\delta}))$ is bounded in ν):

$$(\exists \xi^n < \nu)(\forall \zeta^m < f(\bar{\delta}))(\forall \eta^m < \nu)(g(\zeta^m) = \eta^m \longrightarrow \eta^m < \xi^n)$$

(assuming $\nu < \beta$; otherwise drop the bound ν.) As f is actually $\Sigma_1^{(n)}$-preserving, we have in $J_{\bar{\beta}}$:

$$(\exists \xi^n < \bar{\nu})(\forall \zeta^m < \bar{\delta})(\forall \eta^m < \bar{\nu})(\bar{g}(\zeta^m) = \eta^m \longrightarrow \eta^m < \xi^n)$$

As $\bar{\delta}$ was arbitrary, we conclude that $\operatorname{ran}(\bar{g} \restriction \xi)$ is bounded on any $\xi < \bar{\nu}$. Hence $n \leq n_{\bar{\nu}}$. $\dashv$

2.12 Definition. If $f : \bar{\nu} \Longrightarrow \nu$, let $\lambda(f) =_{\mathrm{df}} \sup f``\overline{\nu}$; $\rho(f) =_{\mathrm{df}} \sup f``\rho_{\bar{\nu}}$.

2.13 Lemma. *Suppose $f : \bar{\nu} \Longrightarrow \nu$, and let $\lambda = \lambda(f)$. Then $\lambda \in S$ and there exists a unique $f_0 : \bar{\nu} \Longrightarrow \lambda$ with $f \restriction \overline{\nu} = f_0 \restriction \overline{\nu}$.*

Proof. With $J_{\beta(\bar{\nu})} = M_{\bar{\nu}}$ and $J_{\beta(\nu)} = M_\nu$ a direct application of the Interpolation Lemma (1.51) with λ as $\widetilde{\nu}$, $M_{\bar{\nu}}$, M_ν as $\overline{M}, M$ respectively, and using $f^* : M_{\overline{\nu}} \longrightarrow_{\Sigma_1^{(n)}} M_\nu$ (where f^* is the canonical extension of f) we have the structure $\widetilde{M} = J_{\widetilde{\beta}}$ and maps $\widetilde{f}, f'$ as specified.

(1) $\lambda \in S, n = n_\lambda$.

Proof of (1). As $\widetilde{h}_{\bar{\nu}}(\gamma_{\bar{\nu}} \cup \{p_{M_{\overline{\nu}}}, r\})$ is cofinal in $\overline{\nu}$ for some parameter r then $\lambda \cap \widetilde{h}^{n+1}_{\widetilde{M}}(\widetilde{f}(\gamma_{\bar{\nu}}) \cup \{p', \widetilde{f}(r)\})$ is cofinal in λ (setting $p' = \widetilde{f}(p_{\overline{\nu}}) = f'^{-1}(p_\nu)$). Thus λ is p.r. closed, but not $\Sigma_1^{(n)}$-regular in $J_{\widetilde{\beta}}$. Hence $n \geq n_\lambda$. We need to show that λ is $\Sigma_1^{(n-1)}$-regular in $J_{\widetilde{\beta}}$. Suppose this fails and there is a good $\boldsymbol{\Sigma_1^{(n-1)}}(\widetilde{M})$ function $\widetilde{g}$, mapping a subset of some $\widetilde{\delta} < \lambda$ cofinally into λ. $\widetilde{g}(\xi)$ can be taken to be of the form $\widetilde{G}(\xi, s)$ in some parameter s

$$
\begin{tikzcd}
\overline{M} = J_{\beta(\bar{\nu})} \arrow[r, "\widetilde{f}"'] \arrow[rr, bend left, "f^*"] & \widetilde{M} = J_{\widetilde{\beta}} \arrow[r, "f'"'] & J_{\beta(\nu)} = M \\
J_{\bar{\nu}} \arrow[u, hook] \arrow[r, "f^* \restriction J_{\bar{\nu}}"] & J_{\lambda} \arrow[u, hook] \arrow[r, "f'^* \restriction J_{\lambda}", "id"'] & J_{\nu} \arrow[u, hook]
\end{tikzcd}
$$

Figure I.2: Lemma 2.13

where the latter is, by the construction of the pseudo-ultrapower, of the form $\widetilde{f}(\bar{g}_0)(\eta)$ for some good $\mathbf{\Sigma}_1^{(n-1)}(M_{\overline{\nu}})$ function $\overline{g}_0$ and some $\eta < \lambda$. We now carry back via $\widetilde{f}$ the supposed singularisation of λ to one of $\overline{\nu}$ which is a contradiction. Let $\overline{G}$ be the good $\Sigma_1^{(n-1)}(M_{\overline{\nu}})$ function by the same functionally absolute definition as $\widetilde{G}$. Define a good $\mathbf{\Sigma}_1^{(n-1)}(M_{\overline{\nu}})$ partial function by $\overline{g}(\xi) \simeq \overline{G}((\xi)_0, \overline{g}_0((\xi)_1))$. Then $\overline{g}$ will singularise $\overline{\nu}$: pick a $\overline{\tau} < \overline{\nu}$ sufficiently large so that $\widetilde{f}(\overline{\tau}) > \max\{\widetilde{\delta}, \eta\}$; we show that $\bar{g}$ takes a subset of $\bar{\tau}$ cofinally into $\bar{\nu}$. Let $\zeta < \overline{\nu}$ be arbitrary. We find a $\xi < \bar{\tau}$ with $\overline{g}(\xi) > \zeta$, and this will be our contradiction. As $\widetilde{g}$ is assumed cofinalising, there is an $\iota < \widetilde{\delta}$ so that $\widetilde{f}(\zeta) < \widetilde{g}(\iota) = \widetilde{G}(\iota, \widetilde{f}(\bar{g}_0)(\eta)) < \lambda$.
Thus the following is $\Sigma_1^{(n)}$:

$$\exists u^n[u^n < \widetilde{f}(\overline{\tau}) \wedge \widetilde{f}(\zeta) < \widetilde{G}((u^n)_0, \widetilde{f}(\overline{g}_0)((u^n)_1)) < \lambda]$$

Hence in $M_{\overline{\nu}}$ we have as required:

$$\exists u^n[u^n < \overline{\tau} \wedge \zeta < \overline{G}((u^n)_0, \overline{g}_0((u^n)_1)) = \overline{g}(u^n) < \bar{\nu}].$$

$\dashv$ (1)

(2) $p' = p_\lambda$.

Proof of (2). By the pseudo-ultrapower construction we have $\widetilde{M} = \widetilde{h}^{n+1}_{\widetilde{M}}(\lambda \cup \{p'\})$ (and equals $\widetilde{h}^{n+1}_{\widetilde{M}}(\widetilde{\kappa} \cup \{p'\})$ where $\widetilde{\kappa} = \widetilde{f}(\kappa_{\overline{\nu}})$ if $\kappa_{\overline{\nu}}$ is defined). Because of this we have that $p' \in R^n_{\widetilde{M}}$ and can thus be lengthened to a $p'' \in P^{n+1}_{\widetilde{M}}(= R^{n+1}_{\widetilde{M}}$ by the soundness of the L-hierarchy). By the minimality of the standard parameter and the definition of p_λ we thus have $p_\lambda \leq^* p'$. However if $p_\lambda <^* p'$ held, we should have for some $i \in \omega, \vec{\xi} < \lambda$ that $p' = \widetilde{h}^{n+1}_{\widetilde{M}}(i, \langle \vec{\xi}, p_\lambda \rangle)$, and thus $p_\nu = \widetilde{h}^{n+1}_\nu(i, \langle f'(\vec{\xi}), f'(p_\lambda) \rangle)$ whence $M_\nu = \widetilde{h}^{n+1}_\nu(\nu \cup f'(p_\lambda))$; this is a contradiction as $f'(p_\lambda) <^* p_\nu$. $\dashv$ (2)

We may set $f_0^* = \widetilde{f}$ once we have shown:

(3) *If* $\widetilde{\alpha} =_{\mathrm{df}} \widetilde{f}(\alpha_{\overline{\nu}})$ *then* $\widetilde{\alpha} = \alpha_\lambda$.

We note that $\alpha_\nu = \widetilde{\alpha}$.
That $h_\lambda(\widetilde{\alpha} \cup \{p_\lambda\}) \cap \lambda \subseteq \widetilde{\alpha}$ is proven using:

$$\overline{H}(\xi^n, \zeta^n) \longleftrightarrow h_{\bar{\nu}}(\omega\xi^n \cup \{p_{\bar{\nu}}\}) \cap \bar{\nu} \subseteq \zeta^n .$$

Then $\overline{H}(\alpha_{\bar{\nu}}, \alpha_{\bar{\nu}})$ is a $\Pi_1^{(n)}$ fact about $\alpha_{\bar{\nu}}$ that is preserved up by $\widetilde{f}$ to yield $h_\lambda(\widetilde{\alpha} \cup \{p_\lambda\}) \cap \lambda \subseteq \widetilde{\alpha}$.

This shows that $\widetilde{\alpha}$ is sufficiently closed. To show that no larger ordinal γ with $\widetilde{\alpha} < \gamma < \lambda$ satisfies $h_\lambda(\gamma \cup \{p_\lambda\}) \cap \lambda \subseteq \gamma$; as $\widetilde{f}{\restriction}\bar{\nu}$ is cofinal into λ we may set $\bar{\gamma} = \widetilde{f}^{-1}{}``\gamma$ and then have $\alpha_{\bar{\nu}} < \bar{\gamma}$ and $\widetilde{f}(\bar{\gamma}) \geq \gamma$. So for some $i \in \omega$ and $\bar{\zeta} < \bar{\gamma}$ we have $\bar{\gamma} \leq h_{\bar{\nu}}(i, \langle\bar{\zeta}, p_{\bar{\nu}}\rangle) < \bar{\nu}$. Applying $\widetilde{f}$ we have that $\gamma \leq h_\lambda(i, \langle\widetilde{f}(\bar{\zeta}), p_\lambda\rangle) < \lambda$. As $\widetilde{f}(\bar{\zeta}) < \gamma$ we see that γ is not a closed ordinal. $\dashv$ (2.13)

2.14 Lemma. *Suppose* $f : \bar{\nu} \Longrightarrow \nu$. *Then* $\lambda(f) < \nu \longleftrightarrow \rho(f) < \rho_\nu$.

Proof. Let $\lambda = \lambda(f)$. We prove both directions by contraposition. So in the forward direction, suppose $\rho(f) = \rho_\nu$. Then, in the diagram of the previous Lemma the map f' is not only $\Sigma_0^{(n)}$ but is cofinal at the n'th level, and thus $\Sigma_1^{(n)}$-preserving. This together with $f'(\lambda, p_\lambda) = \nu, p_\nu$, implies that $\nu \cap f'{}``h_\lambda(\lambda \cup \{p_\lambda\}) \subseteq \nu \cap h_\nu(\lambda \cup \{p_\nu\}) = \lambda$. Were $\lambda < \nu$ this would contradict the fact that $\lambda > \alpha_\nu$ as the latter is in $\operatorname{ran}(f)$.

Conversely, suppose $\lambda = \nu$. Let $\bar{k}$ be the good $\Sigma_1^{(n)}(J_{\beta(\nu)})$ partial function in a parameter $\bar{q}$, cofinalising $\bar{\nu}$. Applying the $\Sigma_1^{(n)}$-preserving f^* we have a good function, k, with the same definition as $\bar{k}$ (in the parameter $f^*(\bar{q})$). Now suppose $\rho' =_{\mathrm{df}} \rho(f) < \rho_\nu$ for a contradiction. Then q would be in $J_{\rho(f)}$, but moreover all the witnesses to the existential quantifiers of type n needed to see that k is cofinal in $\lambda = \nu$ would be in $J_{\rho(f)}$, and those quantifiers could thus be replaced by bounded ones: if, say, $\bar{k}(\xi) = \zeta \leftrightarrow \exists x^n \varphi(x^n, \xi, \zeta, \bar{q})$ with $\varphi \in \Sigma_0^{(n)}$ we then have

$$k(f^*(\xi)) = f^*(\zeta) \Longleftrightarrow \exists x^n \in J_{\rho(f)} \varphi(x^n, f^*(\xi), f^*(\zeta), q) .$$

Hence over $J_{\beta(\nu)}$ we have a $\boldsymbol{\Sigma_0^{(n)}}$ definition of a singularizing function for ν. However as $\nu = \lambda \leq \rho(f) < \rho_\nu$ we have that this function is in $J_{\beta(\nu)}$. This is a contradiction. $\dashv$

2.1. Defining C_ν

We define outright what the C_ν-sequences will be:

2.15 Definition. Let $\nu \in S$; $C_\nu^+ =_{\mathrm{df}} \{\lambda(f) \mid f \Longrightarrow \nu\}$; $C_\nu =_{\mathrm{df}} C_\nu^+ \backslash \{\nu\}$.

2.16 Definition. Let $f : \bar{\nu} \Longrightarrow \nu$. Then:

$$\beta(f) =_{\mathrm{df}} \max\{\beta \le \nu \mid f{\upharpoonright}\beta = \mathrm{id}{\upharpoonright}\beta\}.$$

Simple closure says that $\beta(f)$ is defined. Note that $\beta(f) = \nu$ iff $f = \mathrm{id}_\nu$ iff $f(\beta(f)) \not> \beta(f)$ (if $\nu < \beta(f)$). Further, it is easy to see that $M_{\bar{\nu}} \models$"$\beta(f)$ is a regular cardinal" (this last is a standard argument: if $\beta(f) < \nu$ were singular in $M_{\bar{\nu}}$ then $\beta(f) > \beta$ and Σ_1-elementarity of f would be a contradiction).

The next lemma lists some properties of $f_{(\gamma,q,\nu)}$ which were defined at 2.8 (and see Remark 2.10.) The first of these is a *minimality property* of $f_{(\gamma,q,\nu)}$.

2.17 Lemma.

(i) *If $\gamma \le \nu$ then $f_{(\gamma,q,\nu)}$ is the least $f \Longrightarrow \nu$ such that $f{\upharpoonright}\gamma = \mathrm{id}{\upharpoonright}\gamma$ with $q, p(\nu) \in \mathrm{ran}(f^*)$, in that if g is any other such with these two properties, (meaning that if $g \Longrightarrow \nu$ with extension g^* is so that $\gamma \cup \{q, p(\nu)\} \subseteq \mathrm{ran}(g^*)$), then $g^{-1}f_{(\gamma,q,\nu)} \in \mathbb{F}$.*

(ii) $f_{(\gamma,q,\nu)} = f_{(\beta,q,\nu)}$ *where* $\beta = \beta(f_{(\gamma,q,\nu)})$.

(iii) $f_{(\nu,0,\nu)} = \mathrm{id}_\nu$;

(iv) *Let $f : \bar{\nu} \Longrightarrow \nu$ with $\bar{\gamma} \le \bar{\nu}, f``\bar{\gamma} \subseteq \gamma \le \nu, \bar{q} \in J_{\bar{\nu}}, f^*(\bar{q}) = q$, then*

$\mathrm{ran}(f^* f^*_{(\bar{\gamma},\bar{q},\bar{\nu})}) \subseteq \mathrm{ran}(f^*_{(\gamma,q,\nu)})$.

With (i) *this implies: if $\beta(f) \ge \gamma$ then $f f_{(\bar{\gamma},\bar{q},\bar{\nu})} = f_{(\gamma,q,\nu)}$.*

(v) *Set $g = f_{(\gamma,q,\nu)}$; $\lambda = \lambda(g)$ and $g_0 = \mathrm{red}(g)$. Then $q \in J_\lambda$ and $g_0 = f_{(\gamma,q,\lambda)}$.*

Proof. For (i) note this makes sense since we have specified in effect that $\mathrm{ran}(g^*) \supseteq \mathrm{ran}(f^*_{(\gamma,q,\nu)})$. (i)-(iv) are easy consequences of the definitions. We verify (v). We know that $g_0 \Longrightarrow \lambda$. Set $g_0' = f_{(\gamma,q,\lambda)}$ and we shall argue that $g_0 = g_0'$. Let $k = g_0^{-1} g_0'$. The argument of Lemma 2.13 shows that $d(g_0) = d(g)$; as $g_0{\upharpoonright}\gamma = \mathrm{id}{\upharpoonright}\gamma$, and $q \in \mathrm{ran}(g_0)$ by (i) the minimality of $g_0' \Longrightarrow \lambda$ implies we have such a k defined. Thus $k \in \mathbb{F}$. But $k \Longrightarrow d(g_0)$ so we conclude, as $d(g_0) = d(g)$, that $gk \in \mathbb{F}$. But $\mathrm{ran}((gk)^*) \cap \lambda = \mathrm{ran}(g^*) \cap \lambda$. So, using that $gk{\upharpoonright}\gamma = \mathrm{id}{\upharpoonright}\gamma$, and $q, p(\nu) \in \mathrm{ran}(gk)$, and then (i) again, we have $(gk)^{-1}g = k^{-1} \in \mathbb{F}$. Hence $k = \mathrm{id}_{d(g_0')}$ and thus $g_0 = g_0'$. ⊣

When we have a *cofinal* map f, meaning that $|f|$ is cofinal into $r(f)$, as in the next lemma, then our definitions are preserved through $\Longrightarrow$:

2.18 Lemma. *Let $f : \bar{\nu} \Longrightarrow \nu$ with $\lambda(f) = \nu$. Let $\bar{\gamma} < \bar{\nu}$, and $\overline{q} \in J_{\bar{\nu}}$, with $\gamma = f(\bar{\gamma}), f(\bar{q}) = q$. Set $\bar{g} = f_{(\bar{\gamma},\bar{q},\bar{\nu})}$; $g = f_{(\gamma,q,\nu)}$. Then*

(i) $\lambda(\bar{g}) < \bar{\nu} \Longleftrightarrow \lambda(g) < \nu$;

(ii) *If* $\lambda(\bar{g}) < \bar{\nu}$ *then* $f(\lambda(\bar{g})) = \lambda(g)$ *and* $f(\beta(\bar{g})) = \beta(g)$.

Proof. We first assume $\lambda(\bar{g}) < \bar{\nu}$. Set $\lambda' = f(\lambda(\bar{g}))$. The following is $\Pi_1^{(n)M_{\bar{\nu}}}(\{\lambda(\bar{g}), \bar{\gamma}, p(\bar{\nu})\})$:

$$\forall x^n \forall \xi^n < \bar{\gamma} \forall i < \omega (x^n = \tilde{h}_{\bar{\nu}}(i, \langle \xi^n, \overline{q}, p(\bar{\nu})\rangle) \wedge x^n < \bar{\nu} \longrightarrow x^n < \lambda(\bar{g}))\,.$$

(if $\bar{\nu} = \mathrm{On} \cap M_{\bar{\nu}}$ the conjunct $x^n < \bar{\nu}$ is omitted). Hence

$$\forall x^n \forall \xi^n < \gamma \forall i < \omega (x^n = \tilde{h}_{\nu}(i, \langle \xi^n, q, p(\nu)\rangle) \wedge x^n < \nu \longrightarrow x^n < \lambda')$$

as f is $\Pi_1^{(n)}$-preserving. Hence $\lambda' \geq \lambda(g)$.

Claim 1. $\lambda' \leq \lambda(g)$.

As $\lambda(\bar{g}) < \bar{\nu}$ we have $\omega\rho(\bar{g}) < \omega\rho_{\bar{\nu}}$ by Lemma 2.14. Hence if we set $\bar{N} = \langle J_{\rho(\bar{g})}, A^{n,p_{\bar{\nu}}\restriction n} \cap J_{\rho(\bar{g})}\rangle$ we have that $\overline{N} \in M_{\bar{\nu}}$ and is an amenable structure, with $\lambda(\bar{g}) = \sup(\bar{\nu} \cap h_{\overline{N}}(\bar{\gamma} \cup \{\bar{q}, p(\bar{\nu}) \cap \omega\rho_{\bar{\nu}}\})$.

Applying f^*, and with $N = f(\overline{N})$, we have (noting that $f(\{\bar{q}, p(\bar{\nu}) \cap \omega\rho_{\bar{\nu}}\}) = \{q, p(\nu) \cap \omega\rho_{\nu}\})$

$$\lambda' = \sup(\nu \cap h_N(\gamma \cup \{q, p(\nu) \cap \omega\rho_\nu\})\,.$$

However for amenable structures (including N) we have a uniform definition of the canonical $\Sigma_1(N)$-Skolem function h_N. As $\langle N, A_N\rangle$ is a Σ_0 substructure of $\langle M_\nu^n, A_\nu^n\rangle$, we have that $h_N \subseteq h_\nu$. Hence

$$\lambda' = \sup(\nu \cap h_\nu(\gamma \cup \{q, p(\nu) \cap \omega\rho_\nu\})) = \sup(\nu \cap \tilde{h}_\nu(\gamma \cup \{q, p(\nu)\}))\,.$$

Thus $\lambda' \leq \lambda(g)$. This finishes *Claim 1*.

Claim 2. $f(\beta(\bar{g})) = \beta(g)$.

Let $\beta = f(\beta(\bar{g}))$. Note that $\bar{g} = f_{(\beta(\bar{g}), \bar{q}, \bar{\nu})}$. Consequently $\beta(\bar{g}) \notin \mathrm{ran}(\overline{g})$.

$$\begin{aligned}\beta = f(\beta(\bar{g})) &= f(\sup\{\bar{\delta} < \bar{\nu} \mid \bar{\delta} \subseteq \mathrm{ran}(\overline{g})\}) \\ &= f(\sup\{\bar{\delta} < \bar{\nu} \mid \bar{\delta} \subseteq h_{\overline{N}}(\bar{\gamma} \cup \{\bar{q}, p(\bar{\nu}) \cap \omega\rho_{\bar{\nu}}\})\}) \\ &= \sup\{\delta < \nu \mid \delta \subseteq h_{\bar{N}}(\gamma \cup \{q, p(\nu) \cap \omega\rho_\nu\})\}\,.\end{aligned}$$

And by the argument above is less than or equal to $\sup\{\delta < \nu \mid \delta \subseteq h_\nu(\delta \cup \{q, p(\nu) \cap \omega\rho_\nu\}) = \beta(g)$. Suppose however $\beta < \beta(g)$. Then in M_ν we have:

$$\exists \xi^n < \gamma \exists i < \omega (\beta = \tilde{h}_\nu(i, \langle \xi, q, p(\nu)\rangle))$$

However f is $\Sigma_1^{(n)}$-preserving, so this goes down to $M_{\bar{\nu}}$ as:

$$\exists \bar{\xi}^n < \bar{\gamma} \exists i < \omega (\beta(\bar{g}) = \tilde{h}_{\bar{\nu}}(i, \langle \bar{\xi}^n, \bar{q}, p(\bar{\nu})\rangle)\,.$$

But this implies $\beta(\bar{g}) \in \operatorname{ran}(\bar{g})$ after all which is a contradiction! This finishes *Claim 2* and (ii). Finally, just note for ($\Longleftarrow$) of (i) as $\rho(f) = \rho_\nu$, if $\lambda(g) < \nu$ then by Lemma 2.14 there is an $\eta = f(\bar{\eta}) < \rho(f)$ with $\tilde{h}_\nu(\gamma \cup \{q, p(\nu)\}) \cap \omega\rho_\nu \subseteq \eta$. As before this is $\Pi_1^{(n)}$ and goes down to $M_{\bar{\nu}}$ as $\tilde{h}_{\bar{\nu}}(\bar{\gamma} \cup \{\bar{q}, p(\bar{\nu})\}) \cap \omega\rho_{\bar{\nu}} \subseteq \bar{\eta}$. Hence $\lambda(\bar{g}) < \lambda$. $\dashv$

2.19 Definition. Let $\nu \in S, q \in J_\nu$. $B(q,\nu) =_{\mathrm{df}} B^+(q,\nu)\backslash\{\nu\}$ where $B^+(q,\nu) =_{\mathrm{df}} \{\beta(f_{(\gamma,q,\nu)}) \mid \gamma \leq \nu\}$.

Then $B(q,\nu)$ is the set of those $\beta < \nu$ so that $\beta = \beta(f)$ where $f = f_{(\beta,q,\nu)}$. Recall that B^* is always the class of limit points of B for any set $B \subseteq \mathrm{On}$.

2.20 Lemma. *Let $f = f_{(\gamma,q,\nu)}$ where $q \in J_\nu$.*

(i) *Suppose $\gamma \in B(q,\nu)^*$. Then* $\operatorname{ran}(f) = \bigcup_{\beta \in B(q,\nu)\cap\gamma} \operatorname{ran}(f_{(\beta,q,\nu)})$.

(ii) *Let $\gamma \leq \nu$. Suppose $\bar{\nu}$ is such that $f : \bar{\nu} \Longrightarrow \nu$ with $f(\bar{q}) = q$. Then* $\gamma \cap B(q,\nu) = B(\bar{q},\bar{\nu})$.

(iii) *Let $\lambda = \lambda(f)$; $f_0 = \mathrm{red}(f)$. Then* $\gamma \cap B(q,\lambda) = \gamma \cap B(q,\nu)$.

Proof. (i) is clear; (ii) follows from Lemma 2.17(iv), and (iii) from (ii) and Lemma 2.17(v). $\dashv$

2.21 Definition. Let $\nu \in S, q \in J_\nu$.

$$\Lambda^+(q,\nu) =_{\mathrm{df}} \{\lambda(f_{(\gamma,q,\nu)}) \mid \gamma \leq \nu\}$$
$$\Lambda(q,\nu) =_{\mathrm{df}} \Lambda^+(q,\nu)\backslash\{\nu\}\,.$$

Notice that $\Lambda(q,\nu) \subseteq C_\nu$ and we can think of these as first approximations to C_ν as q varies. We proceed to analyse these sets.

2.22 Lemma. *Let $\nu \in S, q \in J_\nu$.*

(i) *$\Lambda(q,\nu)$ is closed below ν;*

(ii) *$\mathrm{ot}(\Lambda(q,\nu)) \leq \nu$;*

(iii) *if $\lambda \in \Lambda(q,\nu)$ then $q \in J_\lambda$ and $\Lambda(q,\lambda) = \lambda \cap \Lambda(q,\nu)$.*

Proof. Set $\Lambda = \Lambda(q,\nu)$.

(i) Let $\eta \in \Lambda^*$. We are claiming that $\eta \in \Lambda^+(q,\nu)$. For each $\lambda \in \Lambda(q,\nu) \cap \eta$ pick $\beta_\lambda \in B(q,\nu)$ with $\lambda(f_{(\beta,q,\nu)}) = \lambda$. Clearly $\lambda \leq \lambda' \longrightarrow \beta_{\lambda'} \leq \beta_\lambda$. Let γ be the supremum of these β_λ, and then using the closure of $B(q,\nu)$ from (i) of the last lemma, we have $\lambda(f_{(\gamma,q,\nu)}) = \sup_\lambda \lambda(f_{(\beta_\lambda,q,\nu)}) = \eta$.

(ii) is clear.

(iii) Let $\lambda \in \Lambda$, and $g = \lambda(f_{(\beta,q,\nu)})$, where we take $\beta = \beta(g)$. Suppose $g : \bar{\nu} \Longrightarrow \nu$. Let $g(\overline{q}) = q$ and set $g_0 = \mathrm{red}(g)$. Then by Lemma 2.17(v) $g_0 = \lambda(f_{(\beta,q,\nu)}))$. If $\gamma \geq \beta$ then $\lambda = \lambda(f_{(\gamma,q,\lambda)}) \leq \lambda(f_{(\gamma,q,\nu)})$. If $\gamma \leq \beta$ then

$$|f_{(\gamma,q,\lambda)})| = |g_0||f_{(\gamma,\bar{q},\bar{\nu})}| = |g||f_{(\gamma,\bar{q},\bar{\nu})}) = |f_{(\gamma,q,\nu)}|$$

where the first equality is justified by Lemma 2.17(v). $\dashv$

2.23 Lemma. *If* $f : \bar{\nu} \Longrightarrow \nu$, $\mu = \lambda(f), \overline{q} \in J_{\overline{\nu}}, f(\bar{q}) = q$, *then:*

(i) $\Lambda(\overline{q}, \bar{\nu}) = \varnothing \longrightarrow \mu \cap \Lambda(q, \nu) = \varnothing$,

(ii) f "$\Lambda(\overline{q}, \bar{\nu}) \subseteq \Lambda(q, \mu)$,

(iii) *If* $\overline{\lambda} = \max(\Lambda(\overline{q}, \bar{\nu}))$ *and* $\lambda = f(\bar{\lambda})$ *then* $\lambda = \max(\mu \cap \Lambda(q, \nu))$.

Proof. (i) By its definition, if $\Lambda(\overline{q}, \bar{\nu}) = \varnothing$ then $f_{(0,\overline{q},\bar{\nu})}$ is cofinal into $\bar{\nu}$. Hence $\mathrm{ran}(ff_{(0,\overline{q},\bar{\nu})})$ is both cofinal in μ, and contained in $\mathrm{ran}(f_{(0,q,\nu)})$ by Lemma 2.17(iv), thus $\mu \cap \Lambda(q, \nu) = \varnothing$. This finishes (i). Note that By 2.22 (iii) $\Lambda(q, \mu) = \mu \cap \Lambda(q, \nu)$. Let $f_0 = \mathrm{red}(f)$.

(ii) Let $\overline{\lambda} = \lambda(f_{(\bar{\beta},\overline{q},\bar{\nu})}) \in \Lambda(\overline{q}, \bar{\nu})$, and let $f(\overline{\beta}, \overline{\lambda}) = \beta, \lambda = f_0(\overline{\beta}, \overline{\lambda})$. Then $f_0(\lambda(f_{(\bar{\beta},\bar{q},\bar{\nu})})) = \lambda(f_{(\beta,q,\mu)}) \in \Lambda(q, \mu)$.

(iii) Let $\overline{\beta} = \sup\{\gamma \mid \lambda(f_{(\gamma,\bar{q},\bar{\nu})}) \leq \overline{\lambda}\}$. Then $\lambda(f_{(\bar{\beta},\bar{q},\bar{s})}) = \overline{\lambda}$, and by the assumed maximality of $\overline{\beta}$ we have $\lambda(f_{(\bar{\beta}+1,\bar{q},\bar{\nu})})) = \bar{\nu}$. Set $\beta = f(\bar{\beta}) = f_0(\bar{\beta})$, then $\lambda = f_0(\bar{\lambda}) = \lambda(f_{(\beta,q,\mu)})$ using Lemma 2.18 for second equality. However $\lambda(f_{(\beta+1,q,\mu)}) \geq \mu$, since, again by Lemma 2.17(iv), $\mathrm{ran}(f_0 f_{(\bar{\beta}+1,}\bar{q}, \bar{\nu})) \subseteq \mathrm{ran}(f_{(\beta+1,q,\mu)})$. Hence $\lambda = \max(\Lambda(q, \mu)) = \max(\mu \cap \Lambda(q, \nu))$. $\dashv$

We now note that our definitions of $\lambda(f)$, $B(q, \nu)$, $\Lambda(q, \nu)$, etc. are extremely uniform (and indeed are primitive recursively defined) in the appropriate parameters. In particular if $\mu \in S$, then we can define $F_\mu = \{f_{(\gamma,q,\nu)} \mid \nu \in S \cap \mu, q \in J_\nu, \gamma \leq \nu\}, E_\mu = \{\langle \nu, M_\nu, p(\nu), \tilde{h}_\nu \rangle \mid \nu \in S \cap \mu\}$, $G_\mu = \{\langle \langle \nu, q \rangle, \Lambda(q, \nu) \rangle \mid q \in J_\nu, \nu \in S \cap \mu\}$.

2.24 Remark. E_μ, F_μ, G_μ are uniformly $\Delta_1(J_\mu)$ for $\mu \in S$, with, for $\mu' < \mu, E_{\mu'}, F_{\mu'}, G_{\mu'} \in J_\mu$.

2.25 Lemma. *Let* $f : \bar{\nu} \Longrightarrow \nu$ *with* $\overline{q} \in J_{\overline{\nu}}, f(\bar{q}) = q$. *Then*

(i) *If* $\lambda(f) = \nu$ *then* $|f| : \langle J_{\bar{\nu}}, \Lambda(\bar{q}, \bar{\nu}) \rangle \longrightarrow_{\Sigma_1} \langle J_\nu, \Lambda(q, \nu) \rangle$;

(ii) *Otherwise:* $|f| : \langle J_{\bar{\nu}}, \Lambda(\bar{q}, \bar{\nu}) \rangle \longrightarrow_{\Sigma_0} \langle J_\nu, \Lambda(q, \nu) \rangle$

Proof. (i) It suffices to show that $|f|(\Lambda(\bar{q}, \bar{\nu}) \cap \bar{\tau}) = \Lambda(q, \nu) \cap |f|(\tau)$ for arbitrarily large $\overline{\tau} < \bar{\nu}$. This will follow easily from the last lemma.

If $\bar{\lambda} \in \Lambda(\bar{q}, \bar{\nu})$, then $\Lambda(\bar{q}, \bar{\nu}) \cap \bar{\lambda} = \Lambda(\bar{q}, \bar{\lambda})$ by Lemma 2.22, and by the Remark, if $f(\bar{\lambda}) = \lambda$, we have $f(\Lambda(\bar{q}, \bar{\lambda})) = \Lambda(q, \lambda) = \lambda \cap \Lambda(q, \nu)$ (with the latter equality by Lemma 2.22 again). If $\Lambda(\bar{q}, \bar{\nu})$ is unbounded in $\bar{\nu}$, this suffices; if it is empty or bounded, then the last lemma takes care of these cases.

For non-cofinal maps (ii) we still have, if $\lambda(f) = \mu$, that

$$|f_0| : \langle J_{\bar{\nu}}, \Lambda(\bar{q}, \bar{\nu})\rangle \longrightarrow_{\Sigma_1} \langle J_\mu, \Lambda(q, \mu)\rangle$$

where $f_0 = \mathrm{red}(f)$. However $\Lambda(q, \mu) = \mu \cap \Lambda(q, \nu)$, and $|f_0| = |f|$. ⊣

We turn to a decomposition of the C_ν sets into a finite sequence of sets of the form $\Lambda(l^i_\nu, \nu)$. This will enable us to extend the results of the last few lemmas from the $\Lambda(q, \nu)$ sets to C_ν sets, whilst at the same time giving us a handle on other data about the C_ν.

2.26 Definition. Let $s \in S, \eta \leq \nu$. $l^i_{\eta\nu} < \nu$ is defined for $i < m_{\eta\nu} \leq \omega$ by induction on i :

$$l^0_{\eta\nu} = 0; \quad l^{i+1}_{\eta\nu} \simeq \max(\eta \cap \Lambda(l^i_{\eta\nu}, \nu))\,.$$

We also write l^i for $l^i_{\eta\nu}$ if the context is clear; also we set $l^i_\nu \simeq l^i_{\nu\nu}$; $m_\nu = m_{\nu\nu}$.

We note some easily checked facts about this definition (which we shall use without comment; the last here is by induction on i):

Facts.

- $l^i_{\eta\nu} \leq l^{i+1}_{\eta\nu}$ $(i < m_{\eta\nu})$.
- $i > 0$ implies $l^i_{\eta\nu} \in \eta \cap C_\nu$.
- Let $l^i_{\eta\nu}$ be defined, and suppose $l^i_{\eta\nu} < \mu \leq \eta$. Then $l^i_{\eta\nu} = l^i_{\mu\nu}$.

2.27 Lemma. *Let $f : \bar{\nu} \Longrightarrow \nu$.*

(i) *If $\lambda = \lambda(f)$ then $l^i_{\lambda\nu} \simeq f(l^i_{\bar{\nu}})$;*

(ii) *let $\overline{\eta} < \bar{\nu}, f(\overline{\eta}) = \eta$; then $l^i_{\eta\nu} \simeq f(l^i_{\bar{\eta}\bar{\nu}})$.*

Proof. (i) By induction on i. If $i = 0$ this is trivial. Suppose $i = j+1$. Then, as inductive hypothesis $l^j_{\lambda\nu} = f(l^j_{\bar{\nu}})$, and thus $|f| : \langle J_{\overline{\bar{\nu}}}, \Lambda(l^j_{\bar{\nu}}, \bar{\nu})\rangle \longrightarrow_{\Sigma_1} \langle J_\lambda, \Lambda(l^j_{\lambda\nu}, \lambda)\rangle$, by the last lemma, as $|\,\mathrm{red}(f)| = |f|$. However $\Lambda(l^j_{\lambda\nu}, \lambda) = \lambda \cap \Lambda(l^j_{\lambda\nu}, \nu)$, by 2.22. Hence, $f(l^i_{\bar{\nu}}) \simeq f(\max \Lambda(l^j_{\bar{\nu}}, \bar{\nu})) \simeq \max(\lambda \cap \Lambda(l^j_{\lambda\nu}, \nu)) \simeq l^i_{\lambda\nu}$ with the middle equality holding by the inductive hypothesis and Lemma 2.23(iii). (ii) is proved similarly. ⊣

2.28 Corollary.

(i) *Let $f : \bar{\nu} \Longrightarrow \nu$ cofinally. Then $l^i_\nu \simeq f(l^i_{\bar{\nu}})$.*

(ii) *Let $\lambda \in C_\nu$. Then $l^i_{\lambda\nu} \simeq l^i_\lambda$.*

Proof. (i) is immediate. For (ii) choose $f : \bar{\nu} \Longrightarrow \nu$ with $\lambda = \lambda(f)$, and set $f_0 = \mathrm{red}(f)$. Then $l^i_{\lambda\nu} \simeq f(l^i_{\bar{\nu}}) \simeq f_0(l^i_{\bar{\nu}}) \simeq l^i_\lambda$ with the last equality holding from (i). ⊣

2.29 Lemma. *Let $\eta \leq \nu, \lambda = \min(C^+_\nu \backslash \eta)$. Then $l^i_\nu \simeq l^i_{\lambda\nu} \simeq l^i_{\eta\nu}$ (for any $i < \omega$ for which any term is defined).*

Proof. Induction on i, again $i = 0$ is trivial. Suppose $l^j_\nu = l^j_{\eta\nu} = l^j_{\lambda\nu}$ and $i = j + 1$. Set $l = l^j_{\eta\nu}$, then we have: $\Lambda(l, \nu) \cap \eta = \Lambda(l, \nu) \cap \lambda$, since $\Lambda(l, \nu) \subseteq C_\nu$ and $C_\nu \cap [\eta, \lambda) = \varnothing$. Suppose, without loss of generality that $l^i_{\eta\nu}$ is defined. Then $l^i_{\eta\nu} = \max(\eta \cap \Lambda(l, \nu)) = \max(\lambda \cap \Lambda(l, \nu)) = l^i_{\lambda\nu} = l^i_\lambda$. ⊣

2.30 Lemma. *Let $j \leq i < m_\nu$. Set $l = l^i_\nu$. Then $l^j_\nu \in \mathrm{ran}(f_{(0,l,\nu)})$.*

Proof. Set $f = f_{(0,l,\nu)}$. Suppose $f : \bar{\nu} \Longrightarrow \nu$, and $\lambda = \lambda(f)$. Then $l^j_{\lambda\nu} \simeq f(l^j_{\bar{\nu}})$ by Lemma 2.27*(i)*. But l^j_ν exists, and $l^j_\nu < \lambda \leq \nu$. Hence $l^j_\nu = l^j_{\lambda\nu} = f(l^j_{\bar{\nu}})$. ⊣

We now prove a key lemma showing that the $\langle l^j_{\lambda\nu} \rangle$ sequences are finite.

2.31 Lemma. *Let $\nu \in S, \eta \leq \nu$. Then $m_{\eta\nu} < \omega$.*

Proof. Suppose this fails. Then for some $\eta \leq \nu$ we have that $l^i_{\eta\nu}$ is defined for $i < \omega$. Let $\lambda = \min(C^+_\nu \backslash \eta)$. Then $l^i_{\lambda\nu} = l^i_{\eta\nu}$ by Lemma 2.29. Choose $f : \bar{\nu} \Longrightarrow \nu$ with $\lambda = \lambda(f)$. Then $l^i_{\lambda\nu} = l^i_\lambda = f(l^i_{\bar{\nu}})$ for $i < \omega$ by 2.28(ii) and 2.27(i). Hence, taking ν as λ, we may assume, without loss of generality, that l^i_ν is defined for $i < \omega$ for some $\nu \in S$. This will yield a contradiction. We obtain an infinite descending chain of ordinals by showing that as i increases, and thus l^i_ν does so strictly too, the maximal β^i that must be contained in the range of any $f :\Longrightarrow \nu$ together with l^i_ν in order for $\mathrm{ran}(f)$ to be unbounded in ν strictly *decreases.* This is absurd.

Set $l = l^i_\nu$. Define: $\beta^i = \beta^i_\nu =_{\mathrm{df}} \max\{\beta \mid \lambda(f_{(\beta,l,\nu)}) < \nu\}$. By the definition of l^{i+1}_ν we have that $\lambda(f_{(\beta,l,\nu)}) < \nu_s \longleftrightarrow \lambda(f_{(\beta,l,\nu)}) \leq l^{i+1}_\nu$. Furthermore, by the definition of β^i :

(1) $\lambda(f_{\beta^i,l,\nu}) \leq l^{i+1}_\nu$;

(2) $\lambda(f_{\beta^i+1,l,\nu}) = \nu$.

Claim. $\beta^{i+1} < \beta^i$ for $i < \omega$.

Proof of Claim. Set $f = f_{(\beta^{i+1},l^{i+1},\nu)}$. Then $\lambda(f) = l^{i+2}$. (We omit the subscript ν from now on.) Let $f : \bar{\nu} \Longrightarrow \nu$. Then $l^j_{\bar{\nu}}$ exists and $f(l^j_{\bar{\nu}}) = l^j_{l^{i+1},\nu} = l^j_\nu$ for $j < i + 1$ since $l^j < l^{i+1} < \nu$. (The first equality comes from Lemma 2.27*(i)* and (1), the second from Lemma 2.29). ⊣ *Claim*

(3) $\beta^i \geq \beta^{i+1}$.

Proof of (3). Suppose not. Then $(\beta^i + 1) \cup \{l^i\} \subseteq \text{ran}(f)$. Hence we have $\text{ran}(f_{(\beta^i+1,l^i,\nu)}) \subseteq \text{ran}(f)$. Consequently by (2), $\lambda(f) = \nu > l^{i+2}$. Contradiction! $\dashv$ (3)

(4) $\beta^i \neq \beta^{i+1}$.

Proof of (4). Suppose not. As β^{i+1} is the first ordinal moved by f we conclude that $f(\beta^i) > \beta^i$. Set $g = f_{(\beta^i,l,\nu)}, \bar{g} = f_{(\beta^i,\bar{l},\bar{\nu})}$ where $\bar{l} = l^i_{\bar{\nu}}$. Then $g = f\bar{g}$, since $f{\restriction}\beta^i = \text{id}, f(\bar{l}) = l (= l^i_\nu)$. Hence $l^{i+1} = \lambda(g) = \lambda(f\bar{g}) < l^{i+2} = \lambda(f)$. Hence $\lambda(\bar{g}) < \bar{\nu}$. Now we set: $g_0 = f_{(\beta^i,l,l^{i+2})}$. If further $f_0 = \text{red}(f)$, then we have also $g_0 = f_0\bar{g}$ by 2.17(iv). As $l^{i+1} = \lambda(g) < l^{i+2}$, Lemma 2.18(ii) applies and:

$$f(\beta(\bar{g})) = f_0(\beta(\bar{g})) = \beta(g_0) = \beta(g) = \beta^i\,.$$

Hence $\beta^i \in \text{ran}(f)$ which is a contradiction. This proves the *Claim* and hence the Lemma. $\dashv$

We now set $l_{\eta\nu} = l^{m-1}_{\eta\nu}$, where $m = m_{\eta\nu}$. Again we write l_ν for $l_{\nu\nu}$. We note that then $\Lambda(l_{\eta\nu}, \nu) \cap \eta$ is either unbounded in η or is empty. We analyse the latter case further.

2.32 Lemma. *Suppose $\Lambda(l_{\eta\nu}, \nu) \cap \eta = \varnothing$. Set $l = l_{\eta\nu}$. Then:*

(i) $l = 0 \Longrightarrow C_\nu \cap \eta = \varnothing$,

(ii) $l > 0 \Longrightarrow l = \max(C_\nu \cap \eta)$,

(iii) $\eta \in C^+_\nu \Longrightarrow \eta = \lambda(f_{(0,l,\nu)})$.

Proof. Set $\rho = \min(C^+_\nu \setminus (l + 1)$.

(1) $l = l_{\rho\nu}$.

Proof of (1). Set $n = m_{\eta\nu} - 1$. Then $l = l^n_{\eta\nu} < l + 1 < \eta$. Hence (by Fact after 2.1) $l = l^n_{l+1,\nu}$. But $\Lambda(l, \nu) \cap (l + 1) = \varnothing$. Hence $l^{n+1}_{l+1,\nu}$ is undefined and $l = l_{l+1,\nu}$. Hence $l = l_{\rho,\nu}$ by Lemma 2.29. $\dashv$ (1)

(2) $\lambda(f_{(0,l,\nu)}) = \rho$.

Proof of (2). Choose $f : \bar{\nu} \Longrightarrow \nu$, with $\lambda(f) = \rho$ witnessing that $\rho \in C_\nu$. Then, by Lemma 2.27(i), $f(l_{\bar{\nu}}) = l_{\rho\nu} = l$. Set $\bar{l} = l_{\bar{\nu}}$. Now note that we must have that $\lambda(f_{(0,\bar{l},\bar{\nu})}) = \bar{\nu}$. For, if this failed then $f(\lambda(f_{(0,\bar{l},\bar{\nu})})) = \lambda(f_{(0,l,\nu)}) < \rho$ by Lemma 2.18, and so the latter is in $C^+_\nu \cap (l, \rho)$, which is absurd! Then $\lambda(f_{(0,l,\nu)}) = \lambda(ff_{(0,\bar{l},\bar{\nu})}) = \lambda(f) = \rho$. $\dashv$ (2)

From (2) it follows that $\rho \geq \eta$ (for otherwise this contradicts the definition of l as $l_{\eta\nu}$). Hence we have the two alternatives:

If $l = 0$ then (i) holds: $\rho = \min(C^+_\nu \setminus 1) = \min(C^+_\nu) \geq \eta$. If $l > 0$ then $l = \max(C_\nu \cap \eta)$ since $(C_\nu \cap \eta)\setminus(l + 1) \subseteq (C_\nu \cap \rho)\setminus(l + 1) = \varnothing$ and thus we have (ii).

Finally for (iii) if $\eta \in C^+_\nu \longrightarrow \eta = \max(C^+_\nu \setminus (l + 1) = \rho = \lambda(f_{(0,l,\nu)})$. $\dashv$

We now get the closure of the sets C_ν^+ as well as a precise characterisation of the points they contain.

2.33 Lemma. *Let λ be an element or a limit point of C_ν^+. Let $l = l_{\lambda\nu}$. Then there is a β such that $\lambda = \lambda(f_{(\beta,l,\nu)})$. Hence C_ν is closed in ν, and*

$$C_\nu^+ = \{\lambda(f_{(\beta,l,\nu)}) \mid \beta \leq \nu, l < \nu\}\,.$$

Proof. The last sentence is immediate from the penultimate one.

Case 1. $\lambda \cap \Lambda(l, \nu) = \varnothing$

Then $C_\nu \cap \lambda = \varnothing$ or $l = \max(C_\nu \cap \lambda)$ by the last lemma. Hence λ is not a limit point of C_ν^+. Hence $\lambda \in C_\nu^+$, and thus $\lambda = \lambda(f_{(0,l,\nu)})$ by *(iii)* of that lemma.

Case 2. $\lambda \cap \Lambda(l, \nu)$ is unbounded in λ.

Given $\mu \in \Lambda(l, \nu) \cap \lambda$, let β_μ be such that $\lambda(f_{(\beta_\mu,l,\nu)}) = \mu$. Then $\lambda(f_{(\beta,l,\nu)}) = \lambda$ where $\beta = \sup_\mu \beta_\mu$. $\dashv$

The following is (iii) of Theorem 2.5.

2.34 Lemma. $\lambda \in C_\nu \longrightarrow \lambda \cap C_\nu = C_\lambda$.

Proof. Assume inductively the Lemma proven for all ν' with $\nu' < \nu$, and we prove the result for ν by induction on λ. Let $l = l_{\lambda\nu}$. Hence by Cor.2.28 $l = l_\lambda$. By Lemma 2.33 $\lambda \in \Lambda(l, \nu)$. Set $\Lambda = \lambda \cap \Lambda(l, \nu)$. Then by Lemma 2.22(ii) $\Lambda = \Lambda(l, \lambda)$.

Case 1. $\Lambda = \varnothing$.

If $l = 0$, then $C_\lambda \subseteq \lambda \cap C_\nu = \varnothing$ (the latter by Lemma 2.32). If $l > 0$, then $l = l_\lambda = \max(C_\lambda \cap \lambda) = \max(C_\lambda) = l_{\lambda\nu} = \max(\lambda \cap C_\nu)$ with the last equality also by the same lemma. As $l < \lambda$, we use the inductive hypothesis on λ: $l \cap C_\nu = C_l$ However$C_l = l \cap C_\lambda$by the overall inductive hypothesis taking λ as a $\nu' < \nu$. Hence $C_\lambda = \lambda \cap C_\nu = C_l \cup \{l\}$.

Case 2. Λ is unbounded in λ.

Then $\mu \in \Lambda \longrightarrow \mu \in C_\nu \cap C_\lambda$. Hence by the overall inductive hypothesis $C_\mu = \mu \cap C_\lambda$ and (as $\mu < \lambda$) $C_\mu = \mu \cap C_\nu$. Hence $C_\lambda = \lambda \cap C_\nu = \bigcup_{\mu \in \Lambda} C_\mu$. $\dashv$

The following completes (i) of Theorem 2.5:

2.35 Lemma. $\sup(C_\nu) < \nu \longrightarrow \mathrm{cf}(\nu) = \omega$.

Proof. Let $l = \sup(C_\nu) = l_\nu$. Then $X = \mathrm{On} \cap \mathrm{ran}(f_{(0,l,\nu)})$ is countable, and cofinal in ν. $\dashv$

We now observe that we readily have (ii) of Theorem 2.5, since $\mathrm{ot}(C_\nu) \leq \Sigma_{i=0}^{m_\nu - 1} \beta_\nu^i < \nu$ where the β_ν^i are from Lemma 2.31.

2.36 Lemma. *Let* $f : \bar{\nu} \Longrightarrow \nu$*. Then* $|f| : \langle J_{\bar{\nu}}, C_{\bar{\nu}} \rangle \longrightarrow_{\Sigma_0} \langle J_\nu, C_\nu \rangle$.

Proof. It suffices to show that for arbitrarily large $\tau < \bar{\nu}$ that $|f|(C_{\bar{\nu}} \cap \tau) = C_\nu \cap |f|(\tau)$. (As usual we shall write "f" for "$|f|$".)

Case 1. $\Lambda(l_{\bar{\nu}}, \bar{\nu})$ is unbounded in $C_{\bar{\nu}}$.

If $\bar{\lambda} \in C_{\bar{\nu}}$ and $\lambda = f(\bar{\lambda})$ then by 2.23 (and 2.22) $\lambda \in \Lambda(f(l_{\bar{\nu}}), \nu) \subseteq C_\nu$. By 2.24 we have $E_{\bar{\lambda}} \in J_{\bar{\nu}}$ and $f(E_{\overline{\lambda}}) = E_\lambda$. By Lemma 2.32 $C_{\bar{\lambda}} = \{\lambda(f_{(0,l,\nu)}) < \bar{\lambda} \mid l < \bar{\lambda}\} \in J_{\bar{\nu}}$ and is uniformly Σ_0 definable from $E_{\bar{\lambda}}$ over $J_{\bar{\nu}}$. Consequently $|f|(C_{\overline{\lambda}}) = C_\lambda$, by Σ_1-elementarity of $|f|$. But $C_{\overline{\lambda}} = \overline{\lambda} \cap C_{\bar{\nu}}$, $C_\lambda = \lambda \cap C_\nu$.

Case 2. $\Lambda(l_{\bar{\nu}}, \bar{\nu}) = \varnothing$.

Set $l_{\bar{\nu}} = \bar{l}$ and let $f(l_{\bar{\nu}}) = l$. Then $l = l_{\lambda\nu}$ where $\lambda = \lambda(f)$. However $\lambda(f_{0,\bar{l},\bar{\nu}}) = \bar{\nu}$ by our case hypothesis. Thus $\lambda(f_{(0,l,\nu)}) = \lambda(f f_{(0,\bar{l},\bar{\nu})}) = \lambda$. Hence $\Lambda(l, \nu) \cap \lambda = \varnothing$. So we have one of the following two cases holding, by Lemma 2.32:

Case 2.1. $\bar{l} = l = 0$. Then, $C_{\bar{\nu}} = C_\nu \cap \lambda = \varnothing$, and the result is trivial.

Case 2.2. $\bar{l} = \max C_{\bar{\nu}}$. Then $l > 0$ and thus $l = \max(C_\nu \cap \lambda)$. Hence for sufficiently large $\bar{\tau} > \bar{l}$:

$$\begin{aligned} f(\bar{\tau} \cap C_{\bar{\nu}}) = f(C_{\bar{\nu}}) &= f(C_{\bar{\nu}} \cap \bar{l} \cup \{\bar{l}\}) \\ &= (C_\nu \cap l) \cup \{l\} = C_\nu \cap \lambda = f(\bar{\tau}) \cap C_\nu. \end{aligned}$$

$\dashv$

The following completes (iv) of 2.4 on the condition that C_ν is cofinal in ν. We shall see afterwards how to tweak a C'_ν to get the result for the non-cofinal C_ν.

2.37 Lemma. *If* $f : \langle J_{\overline{\nu}}, \overline{C} \rangle \longrightarrow_{\Sigma_1} \langle J_\nu, C_\nu \rangle$ *and* $\sup(C_\nu) = \nu$ *then* $\overline{\nu} \in S$ *and* $\overline{C} = C_{\overline{\nu}}$.

Proof. Suppose $f, \overline{C}, \overline{\nu}$ are as in the antecedent. For $\tau \in C_\nu$, let $f_\tau : \overline{\nu}_\tau \Longrightarrow \nu$ with $\lambda(f_\tau) = \tau$. By the Interpolation Lemma we have (a) $f_{\overline{\nu}_\tau \nu_\tau} : M_{\overline{\nu}_\tau} \longrightarrow_{\Sigma_1^{(n)}} M_\tau$, (b) $\pi_\tau : M_\tau \longrightarrow_{\Sigma_0^{(n)}} M_\nu$; (c) $\pi_\tau \restriction \tau = \mathrm{id} \restriction \tau$, and (d) $\pi_\tau(p_\tau) = p_\nu$. Notice that the map $f_{\overline{\nu}_\tau \nu_\tau}$ can be defined from $M_{\overline{\nu}_\tau}$ and M_τ inside J_ν, and both of them in turn are there definable just from τ, since $\beta(\tau) < \nu$.

(1) π_τ, M_τ depend only on τ, and π_τ is the unique $\Sigma_0^{(n)}$-preserving map satisfying (b)-(d).

For such τ let $X = \bigcup_{\tau \in C_\nu} \mathrm{ran}(\pi_\tau)$.

(2) $M_\nu = X$.

Proof of (2). Let $x \in M_\nu$. Clearly X is a $\Sigma_0^{(n)}$ substructure of M_ν. Suppose $x = \widetilde{h}_\nu(i, \xi, p_\nu)$ for a $\xi < \nu$. Suppose we take for "$y = \widetilde{h}_\nu(u, v)$" a functionally absolute $\Sigma_1^{(n)}$ definition of the form $\exists z^n H(z^n, y, u, v)$ with $H \in \Sigma_0^{(n)}$. It will suffice to find $\tau > \xi$ and some $x_\tau \in M_\tau$ with $x_\tau = \widetilde{h}_\tau(i, \xi, p_\tau)$. We then would have $\pi_\tau(x_\tau) = x$. For this we need to be able to find such a suitable witnessing $z^n \in \operatorname{ran}(\pi_\tau)$ for some sufficiently large $\tau \in C_\nu$. For this it will suffice to know:

(3) $\sup_{\tau \in C_\nu}(\operatorname{ran}(\pi_\tau) \cap \omega\rho_\nu) = \omega\rho_\nu$.

Proof of (3). Let $l = l_\nu$. Then $\Lambda(l, \nu)$ is unbounded in ν. Let $\gamma = \sup\{\beta \mid \lambda(f_{(\beta,l,\nu)}) < \nu\}$. Then $\lambda(f_{(\gamma,l,\nu)}) = \nu$ and $\sup(\operatorname{ran}(f_{(\gamma,l,\nu)}) \cap \omega\rho_\nu) = \omega\rho_\nu$ by Lemma 2.14. For $\beta < \gamma$ let $\lambda_\beta = \lambda(f_{(\beta,l,\nu)})$. Then $\sup\{\lambda_\beta \mid \beta < \gamma\} = \nu$. However by considering $k_\beta =_{\text{df}} \operatorname{red}(f_{(\beta,l,\nu)})$ we see that $\operatorname{ran}(k_\beta) \subseteq M_{\lambda_\beta}$ for $\beta < \gamma$. Hence $\operatorname{ran}(f_{(\beta,l,\nu)}) \subseteq \operatorname{ran}(\pi_{\lambda_\beta})$. Thus $\operatorname{ran}(f_{(\beta,l,\nu)}) \subseteq X$ and so $\sup(X \cap \omega\rho_\nu) = \omega\rho_\nu$. $\dashv$ (3)

If $Y = \operatorname{ran}(f)$ let $\widetilde{Y} = \widetilde{h}_\nu(Y \cup \{p_\nu\})$.

(4) If $y \in \widetilde{Y} \cap J_\nu$ then $\exists \tau \in C_\nu(y \in \widetilde{h}_\tau((Y \cap J_\tau) \cup \{p_\tau\})$, and hence $y \in Y$.

Proof of (4). This is because if "$y = \widetilde{h}_\nu(i, \xi, p_\nu)$" then by (2) this holds relativised to $\operatorname{ran}(\pi_\tau)$ for some $\tau \in C_\nu$. Hence $y = \widetilde{h}_\tau(i, \xi, p_\tau)$. But $M_\tau, p_\tau, \widetilde{h}_\tau$ etc. are Σ_1-definable over J_ν from τ, for any $\tau \in S \cap \nu$; hence "$\exists \tau \in C_\nu(i, \xi, p_\tau \in \operatorname{dom}(\widetilde{h}_\tau))$" is a $\Sigma_1^{\langle J_\nu, C_\nu \rangle}$ statement. Hence there is such a $\tau = f(\overline{\tau})$ in $\operatorname{ran}(f)$. Hence

$$\pi_\tau \widetilde{h}_\tau(i, \xi, p_\tau) = \widetilde{h}_\nu(i, \xi, p_\nu) = y.$$

Thus $y \in Y$. $\dashv$ (4)

We may now transitivise $\widetilde{Y}$ and obtain $\widetilde{f} \supseteq f$ with $\widetilde{f} : \widetilde{M} \longrightarrow_{\Sigma_1^{(n)}} M_\nu$, and our usual arguments show that $\overline{\nu} \in S$, $\widetilde{M} = M_{\overline{\nu}}$, and $\langle \overline{\nu}, f, \nu \rangle \in \mathbb{F}$, etc. We have yet to argue that $\overline{C} = C_{\overline{\nu}}$. Clearly $\overline{C}$ is unbounded in $\overline{\nu}$. Let $\overline{l} = l_{\overline{\nu}}$. Then $\Lambda(\overline{l}, \overline{\nu})$ is unbounded in $\overline{\nu}$ (otherwise if $l =_{\text{df}} f(\overline{l})$, then $l = l_\nu$ and $\Lambda(l, \nu)$ - and C_ν - is bounded in ν).

Claim $\overline{C} \subseteq C_{\overline{\nu}}$.

Proof of Claim. If not let $\overline{\lambda}$ be least in $\overline{C} \backslash C_{\overline{\nu}}$. Suppose (for example) $\overline{\eta}_0 < \overline{\lambda}, \overline{\eta}_0 \in C_{\overline{\nu}}$ and $\overline{\eta} > \overline{\lambda}$ is $\min(\Lambda(\overline{l}, \overline{\nu}) \backslash \overline{\lambda})$. As $\overline{\eta} \in \Lambda(\overline{l}, \overline{\nu}), \eta \in \Lambda(l, \nu)$ (where $\eta =_{\text{df}} f(\overline{\eta})$ and as above $l = f(\overline{l}) = l_\nu$). So $\eta \in C_\nu$. Hence as $\overline{\lambda} \in \overline{C} \backslash (\overline{\eta}_0 + 1), f(\overline{\lambda}) \in f(\overline{C} \cap \overline{\eta}) = C_\nu \cap \eta = C_\eta$. However then $\overline{\lambda} \in C_{\overline{\eta}} \subseteq C_{\overline{\nu}}$! $\dashv$ *Claim*

However now, for $\overline{\lambda} \in \overline{C}$, we have

$$f(\overline{C} \cap \overline{\lambda}) = C_\nu \cap \lambda = C_\lambda = f(C_{\overline{\lambda}} = f(C_{\overline{\nu}} \cap \overline{\lambda}).$$

So, $\overline{C} \cap \overline{\lambda} = C_{\overline{\nu}} \cap \overline{\lambda}$ for unboundedly many $\overline{\lambda} < \overline{\nu}$, and we are done. $\dashv$

We now indicate how to modify C_ν in case it is bounded in ν. As we know, if this happens then $\mathrm{cf}(\nu) = \omega$. To deal with this situation we substitute a new C'_ν. (In all other cases we shall alter nothing and set $C'_\nu = C_\nu$.) Clearly, if we set $l = l_\nu$ we have that $\widetilde{h}_\nu$"$\omega \times \{\langle l, p_\nu \rangle\}$ is unbounded in ν, and hence by Lemma 2.14 in $\omega\rho_\nu$.

Let $u_n = \nu \cap \widetilde{h}_\nu$"$n \times \{\langle l, p_\nu \rangle\}$. Then $u_n \in J_\nu$ and is finite, with $\bigcup_n u_n$ cofinal in ν. We take $\xi_{n,\nu} =_{\mathrm{df}} \prec n, l, \alpha_\nu, u_0, \ldots, u_n \succ$ (for some suitable iterate of the gödel pairing function $\prec - \succ$.) Then $l < \xi_{n,\nu}$ and the latter is monotone as a function of n in ν.

2.38 Lemma. *Let* $\sup(C_\nu) < \nu$. *Let* $f : J_{\nu'} \longrightarrow_{\Sigma_0} J_\nu$ *with* $\xi_{n,\nu} \in \mathrm{ran}(f)$ *for every* $n < \omega$. *Then* (i) $\nu' \in S$; (ii) $\langle \nu', f, \nu \rangle \in \mathbb{F}$; *and* (iii) *if* $f(\xi'_n) = \xi_{n,\nu}$ *then* $\xi'_n = \xi_{n,\nu'}$ *and thus* $C_{\nu'} = \{\xi'_n \mid n < \omega\}$.

Proof. As f is cofinal into ν, it is Σ_1-preserving. (i) is immediate. Let $l = l_\nu$. Let $g = f_{(0,l,\nu)}$. Then $\forall n(\xi_{n,\nu} \in \mathrm{ran}(g))$ and $\lambda(g) = \nu$. Let $g : \overline{\nu} \Longrightarrow \nu$, and $\widetilde{g} : M_{\overline{\nu}} \longrightarrow_{\Sigma_1^{(n)}} M_\nu$.

Hence by the Generalised Interpolation Lemma 1.52 there is an M' (obtained by taking $f, f^{-1} \circ g, g$ here as σ_1, σ_0, g respectively there) and there is a $\widetilde{g} : M_{\overline{\nu}} \longrightarrow_{\Sigma_1^{(n)}} M'$ and $g_1 : M' \longrightarrow_{\Sigma_1^{(n)}} M_\nu$ (both being cofinal at the n'th level). Just as in the proof of Lemma 2.11 $M' = M_{\nu'}$, $n = n(\nu')$ etc. Then as $g_1 \supseteq f$, (ii) follows. If $g(\overline{l}) = l$ then $\overline{l} = l_{\overline{\nu}}$ by Cor. 2.28; taking $l' = f^{-1} \circ g(\overline{l})$ this too shows that $l' = l_{\nu'}$. Then if $f(\xi'_n) = \xi_{n,\nu}$ we have $\xi'_n = \xi_{n,\nu'}$ by the $\Sigma_1^{(n)}$-elementarity of g_1. Hence $C'_{\nu'} = \{\xi_{n,\nu'} \mid n < \omega\}$, and we are done. $\dashv$

We now mention how a more careful computation of the order types of the C_ν-sequences can be obtained. This is needed for some applications and was done using ordinal addition in [1], but the following is essentially the same calculation and is from [29]. We first generalise the definition of $\beta^i = \beta^i_\nu$.

2.39 Definition. For $\eta \leq \nu$ set : $\beta^i_{\eta\nu} =_{\mathrm{df}} \max\{\beta \mid \lambda(f_{(\beta, l^i_{\eta\nu}, \nu)}) < \eta\}$.

Many of the previous properties of the $\beta^i = \beta^i_\nu$ carry over to these $\beta^i_{\eta\nu}$. Namely:

1. $\lambda(f_{(\beta, l^i_{\eta\nu}, \nu)}) < \nu \Longleftrightarrow \lambda(f_{(\beta, l^i_{\eta\nu}, \nu)}) \leq l^{i+1}_{\eta\nu}$.

2. $\beta^i_{\eta\nu}$ is defined if and only if $l^{i+1}_{\eta\nu}$ is defined, i.e. when $i + 1 < m_{\eta\nu}$.

3. $\beta^i_{\eta\nu} \simeq \beta^i_{\lambda\nu}$ if $\lambda = \min(C^+_\nu \backslash \eta)$. (Again as before, $\lambda(f_{(\beta, l^i_{\eta\nu}, \nu)}) < \eta \Longleftrightarrow \lambda(f_{(\beta, l^i_{\eta\nu}, \nu)}) < \lambda$.)

4. $\beta^{i+1}_{\eta\nu} < \beta^i_{\eta\nu}$ when defined. (By the same argument as for $\beta^{i+1} < \beta^i$.)

Now we set $b_\eta = b_{\eta\nu} =_{\rm df} \{\beta^i_{\eta\nu} \mid i+1 < m_{\eta\nu}\}$. For $\eta \in C_\nu$ we then set $d_\eta = d_{\eta\nu} =_{\rm df} b_{\eta^+\nu}$ where $\eta^+ = \min(C^+_\nu \backslash (\eta+1))$. (For the rest of the proof we shall drop the subscript ν on ordinals, which remains unaltered throughout.) Then we shall have:

5. Let $\eta \in C_\nu$, with $l^i_{\eta^+} < \eta$. Then $l^i_{\eta^+} = l^i_\eta$ (proof by induction on i).

   Moreover:

6. Let $\eta \in C_\nu$, with $l^i_{\eta^+} < \eta$ then:

$$l^{i+1}_{\eta^+} = \eta \text{ if } \eta \in \Lambda(l^i_\eta, \nu), \text{ and equals } l^{i+1}_\nu \text{ if not}.$$

Proof of 6: $l^i_{\eta^+} = l^i_\eta$ by 5. If $\eta \in \Lambda(l^i_\eta, \nu)$ then η is maximal in this set below η^+. So the first alternative holds. Note that $i \neq m_{\eta\nu} - 1$ (otherwise by Lemma 2.33 for some β, $\eta = \lambda(f_{(\beta, l^i_{\eta\nu}, \nu)}) \in \Lambda(l^i_\eta, \nu)$). Thus $l^{i+1}_\eta$is defined and $l^{i+1}_{\eta^+}$ must equal this.

2.40 Lemma. *Let $\eta, \mu \in C_\nu$,with $\eta < \mu$. Then $d_\eta <^* d_\mu$.*

Proof. Let $\eta^+ = \min(C^+_\nu \backslash (\eta+1))$, $\mu^+ = \min(C^+_\nu \backslash (\mu+1))$. Let i be maximal so that $l^i_{\mu^+} = l^i_{\eta^+}$.Then $\beta^j_{\mu^+} = \beta^j_{\eta^+}$ for $j < i$. As $l^i_{\mu^+} \le \eta < \mu$, we have by 6. above that $l^{i+1}_{\mu^+}$ is defined and $l^{i+1}_{\mu^+} = \mu$ or l^{i+1}_μ. Moreover then $\beta^i_{\mu^+}$ is defined, and by maximality of i, $l^{i+1}_{\eta^+} \neq l^{i+1}_{\mu^+}$.

Claim $l^{i+1}_{\eta^+} < l^{i+1}_{\mu^+}$.

That $l^{i+1}_{\mu^+} < \eta^+$ is ruled out: otherwise $l^{i+1}_{\eta^+} = l^{i+1}_{\mu^+}$ again. So $l^{i+1}_{\eta^+} < \eta^+ \le l^{i+1}_{\mu^+}$, establishing the Claim.

As $\beta^i_{\mu^+}$ is defined, if $\beta^i_{\eta^+}$is undefined, then we would be finished. Set $l = l^i_{\mu^+} = l^i_{\eta^+}$. Then $\lambda(f_{(\beta^i_{\eta^+}, l, \nu)}) = l^{i+1}_{\eta^+}$ and $\lambda(f_{(\beta^i_{\mu^+}, l, \nu)}) = l^{i+1}_{\mu^+}$. Hence $\beta^i_{\eta^+} < \beta^i_{\mu^+}$ and thus $d_\eta <^* d_\mu$ as required. $\dashv$

2.41 Lemma. *Let α be p.r. closed so that for some $\alpha_0 < \alpha$, $\lambda(f_{(\alpha_0,0,\nu)}) = \nu$. Then* $\mathrm{ot}(C_\nu) < \alpha$.

Proof. First note that $\mathrm{ot}(\langle [\alpha]^{<\omega}, <^* \rangle) = \alpha$. Let $\alpha_0 < \alpha$ have the property that $\lambda(f_{(\alpha_0,0,\nu)}) = \nu$. Then $\{\beta^i_{\eta\nu} \mid \eta \le \nu, i+1 < m_{\eta\nu}\} \subseteq \alpha_0$. Thus $\mathrm{ot}\langle \{d_\eta \mid \eta \in C_\nu\}, <^* \rangle \le \mathrm{ot}(\langle [\alpha]^{<\omega}, <^* \rangle) < \alpha$. Thus $\mathrm{ot}(C_\nu) < \alpha$. $\dashv$

We make some remarks on different versions of this proof. We have obtained a rich global $\square$ sequence by taking as C_ν the set of all possible $\lambda(f)$ where $f \Longrightarrow \nu$. There are other ways: the C_ν sequences from [18] are defined by a finite recursion piecing together enumerations of initial segments C^i_ν. [30] does not construct the C_ν sets directly, but defines the "smooth category" of functions $f : \bar{\nu} \Longrightarrow \nu$ and the class of triples $\mathbb{F}$. They establish that $\mathbb{F}$ satisfies a short list of axioms of a smooth category (much as a morass can specified by a list of axioms, as is done below). From the category one can derive global $\Box$ without any reference to the model whence the category came. The argument is purely combinatorial. This is implicit in [1], and explicit in [29]. These category like objects were used by Jensen to axiomatise various structures such as premorasses, morasses, and the like, and were further used by Stanley [49] to construct morasses *via* forcing. The proof of global $\Box$ in [19] involves tying the construction to the notion of *semi-singularisation*, and is based on the account in [2] of Silver's proof of $\Box$ using Silver Machines.

In larger fine structural models global $\Box$ can also be proved. The extent of this is discussed below. We sketch here what has to be done (and overcome) in [30] in order to establish the principle in the core model built assuming that it does not have any level M with an ordinal κ with $M \models$ "the Mitchell order of κ, $o(\kappa), = \kappa^{++}$)." One problem that has to be addressed from the outset is the failure of condensation in the pure form of Lemma 1.22. (Even at the level of a single measurable cardinal this will occur). A key part of that paper is proving a form of a suitably enhanced Condensation Lemma. In such models there will be many structures that can putatively singularise an ordinal ν. One defines the class S as now a set of pairs $s = \langle \nu_s, M_s \rangle$ where $\nu_s \in$ Sing, and M_s is a mouse $\Sigma^{(n)}_1$ over which (for some n) ν_s is definably singularised. One requires that M_s be a mouse which is *sound above* ν_s. If two mice M_s and M'_s satisfy this, with both hierarchies of M_s and M'_s agreeing up to ν, then in fact a comparison argument shows they are equal. One then defines much as above category maps $f : \bar{s} \Longrightarrow s$ between $J_{\bar{s}}$ and J_s, where $J_s =_{\mathrm{df}} J^{E^{M_s}}_{\nu_s}$. f now canonically extends to an $f^* : M_{\bar{s}} \longrightarrow M_s$ at the appropriate level of $\Sigma^{(n_s)}_1$-definability. These maps f^* must carry in their range a potentially extended parameter set for the version of the Condensation Lemma to be proven (we discuss these parameters and condensation lemmas a little more below) under the smallness assumption of $\neg\exists\kappa(o(\kappa) = \kappa^{++})$. Having got a smooth category of maps between pairs, we assume a unique choice of mouse M_s and so unique $s = \langle \nu_s, M_s \rangle$, for ν_s is made, in some closed and unbounded class of ordinals D containing all singular cardinals. This should be done in some sensible way so that if $\lambda < \nu_s$ and $\lambda \in D$ then $s{\restriction}\lambda$ can be defined as an ordered pair of λ together with an initial segment of M_s which is a mouse singularizing λ, and is also our canonical choice at λ. Then one can get an appropriate form of global $\Box$.

2.2. Variants and Generalities on $\square$

Variants on $\square$: The Principles $\square_\kappa$

In [24] the principles $\square_\kappa$ for $\kappa > \omega$ a cardinal were first defined.

2.42 Definition. Let $\kappa > \omega$ be a cardinal. The principle $\square_\kappa$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha \in \mathrm{Sing} \cap \kappa^+ \rangle$ so that:

(i) C_α is closed unbounded in α;

(ii) $\mathrm{cf}(\alpha) < \kappa \Longrightarrow |C_\alpha| < \kappa$;

(iii) $\beta \in (C_\alpha)^* \Longrightarrow \beta \cap C_\alpha = C_\beta$.

Note that this implies that if $\mathrm{cf}(\alpha) = \kappa$ then $\mathrm{ot}(C_\alpha) = \kappa$. An equivalent variant on this is that with (ii) replaced by the apparently weaker:

(ii)' $\mathrm{ot}(C_\alpha) \leq \kappa$.

Note first that this is immediate for regular κ. Otherwise to see this fix $D \subseteq \kappa{+}1$ closed and unbounded with $\mathrm{ot}(D) = \mathrm{cf}(\kappa){+}1$, $0 \in D$, and $\max(D) = \kappa$. Assuming we have a sequence of C'_α's satisfying (i), (iii), and the new clause (ii)' then, if $\mathrm{ot}(C'_\alpha) \in (D)^*$, replace C'_α with $C_\alpha = \{\beta \in C'_\alpha \mid \mathrm{ot}(C'_\alpha \cap \beta) \in D\}$; otherwise there is a maximal $\delta \in D$ with $\delta < \mathrm{ot}(C'_\alpha)$; in this case set $C_\alpha = \{\beta \in C'_\alpha \mid \mathrm{ot}(C_\alpha \cap \beta) \geq \delta\}$.

A further equivalent variant adds to (ii)' a strengthened

(iii)': $\beta \in C_\alpha \Longrightarrow \beta \cap C_\alpha = C_\beta$,

but accordingly a weakened

(i)': C_α is closed below $\alpha \wedge (\mathrm{cf}(\alpha) > \omega \Longrightarrow C_\alpha$ is unbounded in $\alpha)$.

It is plausible from the definition of $\square$ that it implies $\forall\kappa > \omega(\kappa \in \mathrm{Card} \Longrightarrow \square_\kappa)$. The proof of this is a combinatorial argument (see [8] VI 6.2) which is not fine-structural, so we do not give it. In fact Jensen showed:

2.43 Theorem (Jensen [24]). *Assume V=L; then for any infinite cardinal κ, $\square_\kappa$ holds with the addition that there is a stationary $E \subseteq \kappa^+$ so that for any α, $\beta \in (C_\alpha)^* \longrightarrow \beta \notin E$.*

In the proof of the latter theorem a particular $E \subseteq \mathrm{Cof}(\omega) =_{\mathrm{df}} \{\beta \in \mathrm{On} \mid \mathrm{cf}(\beta) = \omega\}$ was designated. In fact $\square_\kappa$ implies that for any stationary $S \subseteq \kappa^+$ there is some stationary $T \subseteq S$ which does not reflect in this sense: for any $\alpha < \kappa^+$(with $\mathrm{cf}(\alpha) > \omega$ of course) $T \cap \alpha$ is not stationary. (Given any such S, by Fodor's Lemma, find $T \subseteq S$ with $\alpha \in T \longrightarrow \mathrm{ot}(C_\alpha) = \delta$ for some fixed δ. Now if $\mathrm{cf}(\beta) > \omega$, $|(C_\beta)^* \cap T| \leq 1$ and thus $T \cap \beta$ cannot be stationary.)

It is consistent relative to the existence of a Mahlo cardinal that $\square_\kappa$ fails at a regular cardinal by a result of Solovay. On the other hand:

2.44 Theorem (Jensen [24]). *Assume $V=L[A]$ where $A \subseteq \kappa^+$ and that for any ν less than κ^+, we have: $L[A \cap \nu] \models$ "ν is singular". Then $\square_\kappa$ holds.*

In L the existence of $\square_\kappa$-sequences can be proved by localised methods similar to that of global $\square$. For the $L[A]$ result above one defines singularising structures of the form $J^a_{\beta(\nu)} \models$ "$a \subseteq \nu$" for the relativised J^a-hierarchies in general. The assumption on the $A \cap \nu$ is to ensure that sufficiently often there are such singularising structures for $a = A \cap \nu$ in $L[A]$. One can prove the local $\square_\kappa$ by reworking the local proof of $\square_\kappa$; in [1] this is done in a global fashion for all possible singularising structures for arbitrary a (see Theorem 6.21 op.cit.) and then specialised results such as $\square_\kappa$, or combinatorial principles necessary for the Coding Theorem, are obtained. Thus Solovay's result is exact: if κ^+ is not Mahlo in L we may take A as a closed and unbounded set in κ^+ of L-singular cardinals and construct a $\square_\kappa(A)$ sequence which implies the existence of $\square_\kappa$ (see the next definition and discussion below).

2.45 Definition. Let $S \subseteq \mathrm{Sing}$ be a class. Then denote by $\square(S)$ the assertion that there exists a $\square$-sequence $\langle C_\alpha \mid \alpha \in S \rangle$, satisfying:

(i) C_ν is a closed subset of $\nu \cap S$; if $\mathrm{cf}(\nu) > \omega$ then it is unbounded in ν;

(ii) $\mathrm{ot}(C_\nu) < \nu$;

(iii) $\overline{\nu} \in (C_\nu)* \Longrightarrow C_{\bar\nu} = \overline{\nu} \cap C_\nu$.

Let $T \subseteq \kappa^+ \cap \mathrm{Sing}$ be closed and unbounded. $\square_\kappa(T)$ is the assertion that there exists a $\square$-sequence $\langle C_\alpha \mid \alpha \in T \rangle$, satisfying:

(i) C_ν is a closed subset of $\nu \cap T$; if $\mathrm{cf}(\nu) > \omega$ then it is unbounded in ν;

(ii) $\mathrm{ot}(C_\nu) \leq \kappa$;

(iii) $\overline{\nu} \in (C_\nu)^* \Longrightarrow C_{\bar\nu} = \overline{\nu} \cap C_\nu$.

It is a straightforward combinatorial argument in ZFC once one has a $\square_\kappa(T)$ sequence for a T closed and unbounded in κ^+ to fill in the gaps between successive members of T with suitable C_β's to enlarge the $\square_\kappa(T)$ to a $\square_\kappa(\kappa^+)$=$\square_\kappa$ sequence. For S equal to all of SingCard $=_{\mathrm{df}}$ Sing $\cap$ Card, if we have $\square(S)$ and $\forall\kappa(\square_\kappa)$ then we have global $\square$ (and conversely (Jensen)). Thus if $S \supseteq$ SingCard and moreover for all $\kappa > \omega$ $S \cap (\kappa, \kappa^+)$ is unbounded, then $\square(S) \Longrightarrow \square$ again by the "filling in of gaps" just referred to.

Variants on $\square$: Weakenings and Extensions, $\square$ with Scales

2.46 Definition. Let $0 < \lambda \leq \kappa$ be cardinals, with κ uncountable. The principle $\square^{<\lambda}_\kappa$ asserts the existence of a sequence $\langle \mathcal{F}_\alpha \mid \mathrm{Lim}(\alpha), \alpha \in (\kappa, \kappa^+) \rangle$ with, for every limit α, $0 < |\mathcal{F}_\alpha| < \lambda$, and further so that $C \in \mathcal{F}_\alpha$ implies

(i) C is closed unbounded in α;

(ii) $\mathrm{ot}(C) \leq \kappa$;

(iii) $\beta \in C^* \Longrightarrow \beta \cap C \in \mathcal{F}_\beta$.

$\Box_\kappa^{<\lambda^+}$ is abbreviated as $\Box_\kappa^\lambda$.

In [24] $\Box_\kappa^\kappa$ was formulated in an equivalent formulation called $\Box_\kappa^*$, and clearly $\Box_\kappa^1$ is $\Box_\kappa$. These intermediate so-called *weak square principles* were introduced by Schimmerling in [42]. We defer discussion of these principles until later, but Jensen had already remarked in [24] that if $2^\kappa = \kappa^+$ then $\Box_\kappa^*$ is equivalent to the existence of a special Aronszajn tree on κ^+ (also due to him is the straight equivalence of these two principles, without the assumption that $2^\kappa = \kappa^+$) ; in fact if $\kappa^{<\kappa} = \kappa$ then one can construct a $\Box_\kappa^*$ sequence; hence that if κ is regular $\Box_\kappa^*$ follows from $\mathrm{ZFC} + \mathrm{GCH}$; further notice that the case of κ singular is of particular interest for the question of the existence of $\Box_\kappa^*$ sequences. Whereas from $\Box_\kappa$ we can construct non-reflecting stationary subsets of κ^+ as above, this cannot be done from $\Box_\kappa^*$ alone (although $\Box_\kappa^{<\omega}$ does suffice; see [6]). Mitchell has shown that $\mathrm{Con}(\mathrm{ZFC} + \exists\kappa(\kappa \text{ is a Mahlo cardinal})) \Longrightarrow \mathrm{Con}(\mathrm{ZFC} + \neg\Box_{\omega_1}^*)$ by a suitable forcing collapsing κ to $\aleph_2$.

Improved-$\Box_\kappa^{<\lambda}$ *sequences* add a requirement (iv) to the above definition by insisting that at least one element $C \in \mathcal{F}_\alpha$ has $\mathrm{ot}(C) = \mathrm{cf}(\alpha)$. In general $\Box_\kappa$ does not imply Improved-$\Box_\kappa^{<\lambda}$: in the Mitchell model mentioned above, if one effects a further forcing to add back a $\Box_{\omega_2}$-sequence without adding any ω_2-sequences, one ends up with a model where Improved- $\Box_{\omega_2}^{<\omega_2}$ still fails (see [6] Sect.5). However ([6] Thm.10) Global $\Box$ implies Improved-$\Box_\kappa^{<\omega}$ for every uncountable cardinal κ.

Enhancements of $\Box$ have been developed. We detail just one here. The following is a natural property allying $\Box_{\aleph_n}$-sequences for $n < \omega$ with a scale on $\aleph_\omega$. It was defined in [5] at 3.1. Let $I_n = \mathrm{Lim} \cap (\omega_n, \omega_{n+1})$ for $n \leq \omega$. CS below will essentially be asserting the existence of $\Box_{\aleph_n}$-sequences, albeit with domains I_n rather than all of $\mathrm{Sing} \cap \omega_{n+1}$, plus a scale for $\aleph_\omega$.

2.47 Definition. CS asserts the existence of $\langle C_\alpha^n \mid 0 < n \leq \omega, \alpha \in I_n \rangle$ and $\langle f_\alpha \,|\, \alpha \in I_\omega \rangle$ so that:

(1) $\forall n \forall \alpha \in I_n$

  (a) $C_\alpha^n \subset \alpha \cap I_n$ and is closed and unbounded in α;

  (b) $\mathrm{cf}(\alpha) < \omega_n \Longrightarrow \mathrm{ot}(C_\alpha^n) < \omega_n$.

(2) $\forall \alpha \in I_\omega$ f_α is a function so that:

  (a) $\mathrm{dom}(f_\alpha) = (k, \omega]$ for some $k < \omega$ such that $\mathrm{ot}(C_\alpha^\omega) < \omega_k$;

(b) $\forall n \in \mathrm{dom}(f_\alpha), f_\alpha(n) \in I_n$;

(c) $\beta \in (C^\omega_\alpha)^* \Longrightarrow \mathrm{dom}(f_\alpha) \subseteq \mathrm{dom}(f_\beta)$;

(d) $\forall n \in \mathrm{dom}(f_\alpha)$, $(C^n_{f_\alpha(n)})^* = \{f_\beta(n) \mid \beta \in (C^\omega_\alpha)^*\}$.

(3) $\langle f_\alpha \mid \alpha \in I_\omega\rangle$ forms a scale in $\Pi_n \aleph_{n+1}$; *i.e.* it is increasing and cofinal in the eventual domination ordering.

In [5] the consistency of CS is established by forcing. (With some work the methods here of the proof of global square restricted to $\aleph_{\omega+1}$ together with a few "premorass-like" considerations from the next section, establish the existence of the scale sequence $\langle f_\alpha \mid \alpha \in I_\omega\rangle$ and hence that *CS* holds in L.) They use *CS* to demonstrate the existence of a mutually stationary sequence on $\aleph_\omega$ that is not tightly stationary. *CS* is a special case of the *condensation coherent* global $\square$-sequences of Donder, Jensen, and Stanley [13], also shown to hold in L, and can be derived from them. The authors used condensation coherent sequences to demonstrate the existence of *squared scales* of Abraham and Shelah. [13] derives their principle again from a "category" like object. A similar structure is the *Fine Scale Principle* of Friedman [17] used in his proof of the Coding Theorem.

Applications

We discuss only very briefly some positive applications of $\square$-sequences. The reader should see the chapter by Todorčević in this Handbook. We do not intend here to go extensively into applications of $\square$ and weak $\square^\lambda_\kappa$-sequences to *stationary reflection* properties but refer the reader to the extensive [6] for many results in this area, and to the chapter by Eisworth in this Handbook.

In [24] Sect.6 $\square$ arguments were originally used to prove two characterisations of weakly compact cardinals in L :

2.48 Theorem (Jensen [24]). *Assume V=L; let $\kappa \in \mathrm{Reg}$ but not weakly compact. Then there is a κ-Suslin tree.*

The proof proceeds by establishing:

2.49 Theorem (Jensen [24]). *Assume V=L; then for any $\kappa \in \mathrm{Reg}$, κ not weakly compact there is a stationary $E \subseteq \kappa$, and a sequence $\langle C_\alpha \mid \alpha \in \mathrm{Lim}\cap\kappa\rangle$ so that* (i) *C_α is closed and unbounded in α, and* (ii) *α, $\beta \in (C_\alpha)^* \Longrightarrow (\beta \notin E \wedge \beta \cap C_\alpha = C_\beta)$.*

From this, one characterisation of weakly compact cardinals in L already follows. Stationary reflection is a consequence of Π^1_1-indescribability: if $E \subseteq \kappa$ is stationary, then for some $\beta < \kappa$ we have that $E \cap \beta$ is stationary. The theorem above thus shows that in L, for non-weakly compact κ this must fail. The use of the $\langle C_\alpha \mid \alpha \in \mathrm{Lim}\cap\kappa\rangle$ sequences from the above facilitates the inductive construction of a κ-Suslin tree. Even outside of L we may obtain (see [8]):

- $\mathrm{CH} + \Box_{\omega_1} \Longrightarrow$ there exists a ω_2-Suslin tree (Gregory [22]);
- $\mathrm{GCH} + \Box_\kappa \Longrightarrow$ there exists a κ^+-Suslin tree;

The role of the C_α-sequences in all these (and other) proofs is typically to ensure that the inductive definition of the required structure can be continued at limit stages of uncountable cofinality.

A final result from Jensen's paper is a cardinal transfer theorem due to Silver (see Definition 3.8 for this notation).

2.50 Theorem (Silver [24]). *If* GCH *holds, κ is a singular cardinal, and $\Box_\kappa$ holds, then $(\omega_1, \omega) \longrightarrow (\kappa^+, \kappa)$.*

One of the major applications of $\Box$ and $\Box$-like principles is in *class forcing*: the construction of many coding conditions and coding-style conditions for forcing over L or other models, involves an extensive use of $\Box$-like (and morass-like) machinery. Jensen's original Coding Theorem (see [1]) heavily exploited such principles, especially when formulating conditions for coding information down past singular cardinals. Work of Sy Friedman has transformed notions of class forcing for a variety of constructions over L that similarly involve fine-structural arguments for defining conditions (see [18] and the chapter by Friedman in this Handbook).. Just as an example, his Fine Scale Principle alluded to above is employed in his proof of coding.

A recent use of the (proof of) global $\Box$ in $L[E]$ models is that of Koepke-Welch [33] deriving large cardinal strength from the assumptions of a mutual stationarity principle of Foreman-Magidor (see again [5]). Fine structural methods are also employed by Ishiu [23] to demonstrate the existence of *strong club guessing sequences* in L. Let $S \subseteq \mu$ be stationary with $\mu \in \mathrm{Reg}$. An S-strong club guessing sequence $\langle C_\alpha \mid, \alpha \in S\rangle$ is a sequence where each $C_\alpha \subseteq \alpha$ is closed and unbounded, and so that for any closed and unbounded $D \subseteq \mu$, there is a further closed and unbounded $E \subseteq \mu$ so that $\beta \in E \cap S \Longrightarrow \exists\delta(C_\beta \backslash \delta \subseteq D)$. Ishiu shows that in L the existence of a μ-strong club guessing sequence is equivalent to the non-ineffability of μ. The heart of the proof is to show that there is a $\mathrm{Sing} \cap \mu$-strong club guessing sequence. He directly constructs the C_α's using fine structural arguments reminiscent partly of $\diamondsuit^*$, and partly of $\Box$ methods, by looking at singularising structures. As he does not require coherence of the guessing sequence this is simpler than proving the existence of global $\Box$ up to μ.

The Failure of $\Box_\kappa$

It was an early (1974) result of Solovay that showed that the κ^+-supercompactness of a cardinal λ implied the failure of $\Box_\kappa$ [47] (this was later reduced to κ^+-compactness by Gregory; see also [48] for a discussion of this). With the advent of Jensen's Covering Lemma for L it was clear that if any cardinal κ satisfied $\kappa^+ = (\kappa^+)^L$ then the absolute nature of the defining clauses of $\square_\kappa$ implied that a $\square_\kappa$ sequence in L is a $\square_\kappa$ sequence in V. As the Covering Lemma (under the assumption of $\neg 0^\#$) implied that $\kappa^+ = (\kappa^+)^L$ for any singular, or weakly compact cardinal κ, this means that then V would have such $\square_\kappa$-sequences. In larger extender models K the Weak Covering Lemma (and thus the correct computation of cardinal successors of singulars) can be achieved for K up to a strong cardinal [27] and up to a Woodin cardinal [36]. Hence the failure of, say, $\square_{\aleph_\omega}$ expressed directly the failure of the Covering Lemma over an inner model (or at least the correct cardinal successor computation) at the cardinal $\aleph_\omega$. Indeed this failure came to be seen as a test question concerning inner models, and implies (at least if $\aleph_\omega$ is a strong limit cardinal) the existence of $\mathrm{AD}^{L[\mathbb{R}]}$; see Theorem 2.59 below.

The principle $\square_\kappa$ was seen to hold at other classes of cardinals, assuming the rigidity of a suitable canonical inner model. For Jensen's Core Model K^{MOZ} built with measures of order zero, [55] showed that successors of Jónsson cardinals were correctly computed in K^{MOZ}, hence $\square_\kappa$ held at Jónsson κ (the point here is that regular Jónsson cardinals, although weakly inaccessible need not be weakly compact, nevertheless we may show correct cardinal successor computation at κ). [59] showed the same conclusion for the core model K^{Steel}, assuming no inner model with a Woodin cardinal below the measurable cardinal Ω needed for K^{Steel}'s construction. (That $\kappa^+ = (\kappa^+)^{K^{\mathrm{Steel}}}$ under these assumptions, and hence that $\square_\kappa$ holds, for κ weakly compact is due to Schimmerling-Steel [43]). The Jónsson property in fact yields some strong reflection properties on $\square$ sequences, assuming the Weak Covering Lemma over the inner model: if there is no inner model with a Woodin cardinal, if Ω is measurable, $\kappa < \Omega$ any regular Jónsson, then the set of regular cardinals $\mu < \kappa$ such that $\square_\mu$ holds in V, is stationary below κ ([59]). Chang's Conjecture implies the failure of $\square_{\omega_1}$ as shown by Todorčević.

There are extensive results in the area of forcing and $\square$ which we shall only lightly touch on here, but we do mention that the proper forcing axiom PFA causes a dramatic failure of square: Todorčević proved ([52]) that $\mathrm{ZFC} + \mathrm{PFA} \vdash \forall\kappa\neg\square_\kappa$. Hence PFA implies the existence of an inner model with a measurable cardinal, for example. Magidor noted that Todorčević's proof actually shows that $\mathrm{ZFC} + \mathrm{PFA} \vdash \forall\kappa\neg\square_\kappa^{\aleph_1}$; however this can be contrasted with his further result that showed (unpublished 1995) that $\mathrm{Con}(\mathrm{ZFC} + \exists\kappa(\kappa \text{ a supercompact cardinal})) \Longrightarrow \mathrm{Con}(\mathrm{ZFC} + \mathrm{PFA} + \forall\kappa\square_\kappa^{\aleph_2})$. Hence the possibility arises to use the various weak square principles as a means of calibrating consistency strength. The impact of Martin's Maximum (MM) is stronger than that of PFA: Foreman and Magidor formulated a "very weak square" principle, [15], VWS_κ, weaker than $\square_\kappa^*$, and the latter showed that MM implies $\neg\,\mathrm{VWS}_\kappa$ for κ with $\mathrm{cf}(\kappa)=\omega$. By Solovay's result mentioned above, a supercompact destroys the possibility of any $\Box_\lambda$ holding for any $\lambda>\kappa$; Shelah showed from the same assumption that any $\Box^*_\lambda$ will also fail for any singular λ satisfying $\mathrm{cf}(\lambda)<\kappa<\lambda$ (see [15]). However it is consistent with a supercompact κ that for a cardinal λ satisfying: $\kappa\le\mathrm{cf}(\lambda)<\lambda$ that $\Box^{\mathrm{cf}(\lambda)}_\lambda$ holds ([6] Theorem 17). This complements the observation of Burke and Kanamori that Solovay's methods can show that if κ is λ^+ strongly compact, and $\mathrm{cf}(\lambda)<\kappa$ then $\Box^{<\mathrm{cf}(\lambda)}_\lambda$ fails.

If $\mathbb{P}$ is Prikry forcing at a measurable cardinal, then it was shown in [3] that $\Box^*_\kappa$ holds in $V^{\mathbb{P}}$. This was improved in [7] to that of $\Box^\omega_\kappa$ holding; results of [6] however show that if κ is supercompact then in $V^{\mathbb{P}}$ one has the failure of $\Box^{<\omega}_\kappa$.

Jensen [29] showed that for regular κ the principles $\Box^\lambda_\kappa$ decreased in strength as λ increases from 1 to κ. In [6] it is shown how this can be done for singulars: from a supercompact a forcing extension can be constructed in which some previously chosen cardinals $0<\nu<\mu$ (which can be finite) satisfy $\Box^\mu_{\aleph_\omega}$ but $\neg\Box^\nu_{\aleph_\omega}$.

2.3. $\Box$ in Fine-Structural Inner Models

We come now to the interesting question of the status of $\Box$-principles in canonical inner models. As for L, global $\Box$, and for all κ, $\Box_\kappa$ were shown by Welch ([57]) to hold in the first such model to go beyond L : the Dodd-Jensen Core Model K^{DJ} built assuming the non-existence of an inner model with a measurable cardinal. Wylie [60] proved the same for Jensen's Core Model for measures of order zero. Whereas [57] had used the older fine structure that was available at that time, Wylie's thesis used the modern arrangement of the hierarchies and the corresponding fine structure.

Several issues surface at this point: Firstly, the *indexing* of extenders plays an important role. Secondly, it was also apparent that *extender fragments* which had been introduced by Mitchell and Steel in their fine structure for $L[E]$ models ([37]) arise also naturally when considering proofs of $\Box$ for models with many measures. Thirdly, additional further condensation results would be needed to be proven both for them, and in general. Concerning this third issue: for example, whereas Skolem hulls of an initial segment of the L-hierarchy collapse to levels of L, for $L[E]$ hierarchies (even with E coding only sequences of measures) certain hulls may collapse not to an initial segment of the J^E_α hierarchy in which they were taken, but to initial segments of an ultrapower of this J^E_α hierarchy, thus leading to multi-claused condensation lemmas. (This has already been mentioned above for Jensen-Zeman [30] and is a substantial mathematical preface to their proof of Global $\Box$ in the model below $o(\kappa)=\kappa^{++}$.) Wylie's thesis contains the first kind of condensation results of a type needed to prove $\Box$ in the modern setting.

Concerning the first two issues, the indexing of extenders: several possibilities arise for how one might build extender E-hierarchies. Mitchell and Steel built in [37] an extender model $L[\vec{E}]$ with a Woodin cardinal using an indexing rule for building E-sequences which indexed with ordinal α an extender E if, when taking the ultrapower $j : M \longrightarrow N = \mathrm{Ult}(M, E)$ then $\alpha = (\nu(E)^+)^N$ where ν was the supremum of the *generators* of the extender E (or simply $(\mathrm{crit}(E)^+)^N$ if $\mathrm{crit}(E)$ was the sole generator). (This ordinal ν can be characterised as the least ordinal $\nu' \geq (\mathrm{crit}(E)^+)^N$ so that the extender of length α derived from j gives rise to the same ultrapower as E.) The *fragments* of the extender E_α which are of the form $E_\alpha \cap ([\eta]^{<\omega} \times J^{\vec{E}}_\xi)$ (where $\xi < (\kappa^+)^{J^{\vec{E}}_\alpha}$) are members of $J^{\vec{E}}_\alpha$ (see the discussion at 2.9 of [50]). Moreover the fragments just defined appear cofinally in α as ξ rises to $(\kappa^+)^{J^{\vec{E}}_\alpha}$. The extender E_α may now be coded as an *amenable* predicate which is essential for fine structure to work.

Alternative indexing schemes have been proposed. Notably, S. Friedman realised that reindexing could affect proofs of fine-structural principles and suggested the scheme whereby an index α for an extender would be $(j(\kappa))^{+N}$, with j and N as above. This suggestion has been fully worked out by Jensen in [25], and shows *inter alia* how the results of Mitchell and Steel can be performed using this indexing. Each scheme has its advantages, but for the proofs of combinatorial principles such as $\square_\kappa$ the Friedman-Jensen scheme allows certain difficulties to be postponed to later in the large cardinal hierarchy, beyond superstrong cardinals as we shall see later.

The analysis of fragments and condensation principles for the proof of $\square$ there, at that time by Wylie and S. Friedman, would be continued independently by Friedman, Jensen, Schimmerling, and Zeman.

Nothwithstanding the above analyses, in $L[E]$ models beyond measures the question of $\square_\kappa$ became increasingly complex. Schimmerling [42] proved that in the Steel Core model built below a Woodin cardinal that a) $\forall\kappa\ \square^{<\kappa}_\kappa$ held; b) that in the core model below n strong cardinals $\square^{n+1}_\kappa$ held for all κ. Below one strong cardinal Schimmerling divided a certain $S \subseteq \kappa^+$ into two parts $S = S^0 \cup S^1$ and defined a $\square_\kappa$-sequence $\langle C^i_\alpha \mid \alpha \in S^i\rangle$ for $i < 2$. However this was not a disjoint partition; hence setting $\mathcal{F}_\alpha = \{C^i_\alpha \mid \alpha \in S^i, i < 2\}$ yielded only a $\square^2_\kappa$ sequence. A modification of the latter proof by Jensen in [29] showed that $\forall\kappa\ \square_\kappa$ holds in the core model below 1 strong cardinal. Zeman [61] showed global $\square$ in the same model.

Jensen and Zeman's paper [30] proves global $\square$ in an $L[E]$ model built using Jensen's fine structure and utilising the indexing of extenders of [27]. The model is relatively small, as the assumption is that there are no inner models with a cardinal κ with Mitchell order $o(\kappa) = \kappa^{++}$. The proof establishes the existence of a "smooth category" like formulation of a set of axioms, that in turn automatically, in an entirely separate and purely combinatorial argument, can be shown to yield up global $\square$. The proof reveals some of the difficulties of extending condensation arguments to larger models. They are able to prove the necessary sequence of lemmata by adding to the requirements on the range of the maps $f : \bar{\nu} \Longrightarrow \nu$ that we specified above, that it contained certain further parameters. In short if certain canonical witness maps σ sending W_M^{ν,p_M} into M were not cofinal at the n'th stage where $\omega\rho_M^n > \nu \geq \omega\rho_M^{n+1}$ (where $\nu \in p_M$), then it is necessary to require the maps f to contain the ordinal $\sup(\mathrm{ran}(\sigma)) \cap \omega\rho_M^n$ witnessing that non-cofinality. (As there are only finitely many $\nu \in p_M$) this could be at most finitely many requirements.) With this strengthening a form of the Condensation Lemma (under the no $o(\kappa) = \kappa^{++}$ assumption) is provable and the proof can proceed. For larger models it is not clear how one can in general preserve standard parameters: one could put the witnesses themselves into the ranges of the maps, and thus preserve the standard parameters "downwards" into suitable substructures. However the smooth category approach also requires sufficient preservation in an upwards direction, so it is unclear whether it is possible to perform this in a larger model context (and is one reason we did not demonstrate this approach in our account here).

There are serious difficulties in generalising the $\Box_\kappa$ arguments to larger $L[E]$ models. The problems have to do with condensation and dealing with the extenders fragments that Schimmerling had already encountered, and which had been already seen earlier as a hurdle as described above. This was definitively solved in the last few years by work of Jensen on the former properties of condensation, and of Schimmerling and Zeman on the analysis of extender fragments. We now have a complete picture of how $\Box_\kappa$ can be proven in Jensen style $L[E]$ models and at precisely which place this first breaks down, which we now describe.

Firstly a theorem due to D. Burke [4].

2.51 Theorem. *If* $\{a \subseteq \kappa^+ \mid \mathrm{ot}(a) \in \mathrm{Card}\}$ *is stationary then* $\Box_\kappa$ *fails.*

Secondly, we make the following definition due to (but not christened by) Jensen. It was obtained by extracting the kernel of Solovay's proof of the failure of $\Box_\kappa$ from a strongly compact.

2.52 Definition. A cardinal κ is *subcompact* iff for every $B \subseteq \kappa^+$, there exists $\mu < \kappa, A \subseteq \mu^+$, and an elementary embedding $j : \langle H_{\mu^+}, \in, A\rangle \longrightarrow \langle H_{\kappa^+}, \in, B\rangle$ with $\mathrm{crit}(j) = \mu$.

It is easy to check here that μ's implicit weak inaccessibility implies the regularity of κ as $j(\mu) = \kappa$. The following is easily derived from 2.51; the argument is very similar to Solovay's, and seems to have been rediscovered by Jensen. As it is direct we give it:

2.53 Theorem (Jensen). *If* κ *is subcompact, then* $\Box_\kappa$ *fails.*

Proof. Suppose, for a contradiction, that $B \subseteq \kappa^+$ codes up a $\square_\kappa$-sequence $\langle D_\alpha \,|\, \alpha < \kappa^+\rangle$. Let μ, A, j be suitable witnesses to the definition of subcompactness of κ for this B. By elementarity we have $\langle C_\alpha \mid \alpha < \mu^+\rangle$, a $\square_\mu$ sequence, coded by A. Let $S = \{\nu < \mu^+ \mid \mathrm{cf}(\nu) < \mu\}$. Then easily $j``S$ is $<\mu$-closed; set $\gamma = \sup(j``\mu^+)$. Then $(D_\gamma)^* \cap j``S$ is unbounded in γ and $<\mu$-closed. Let $T = \{\nu < \mu^+ \mid j(\nu) \in (D_\gamma)^*\}$. Then T is unbounded in μ^+. Finally let $E = \bigcup\{C_\nu \mid \nu \in T\}$; then E is a union of cohering sets and hence is closed and unbounded in μ^+ and $E \cap \nu = C_\nu$ for all $\nu \in T$. However this would imply that for all sufficiently large $\nu \in T$, $\mathrm{ot}(C_\nu) > \mu$ which contradicts the definition of $\square_\mu$ sequence. ⊣

A modification of the argument above yields also Burke's theorem 2.51. Schimmerling and Zeman's main theorem is:

2.54 Theorem ([45]). *In any Jensen-style $L[E]$ model the following are equivalent:*

(i) $\square_\kappa$;

(ii) $\square_\kappa^{<\kappa}$;

(iii) *κ is not subcompact;*

(iv) $\{\nu < \kappa^+ \mid E_\nu \neq \varnothing\}$ *is not stationary in κ^+.*

(Here (i) implies (iii) is the Burke/Jensen result mentioned above, and (ii) implies (iii) is an easy modification of that; Schimmerling and Zeman proved the main result here that (iv) implies (i), and Jensen proved (i) implies (iv) around the time that the notion of 2.52 was formulated. Later Zeman noticed that this argument can be modified to obtain that (iii) implies (iv).)

Thus in such $L[E]$ models $\square_\kappa$ holds wherever $\square_\kappa^{<\kappa}$ does. Recall that in this style of $L[E]$ model, if $J^E_\nu \models$ "κ is the largest cardinal", then if $\lambda = \mathrm{crit}(E_\nu)$, then $E_\nu(\lambda) = \kappa$.

As $D =_{\mathrm{df}} \{\nu \,|\, J^E_\nu \models \text{“}\kappa \text{ is the largest cardinal”}\}$ is closed and unbounded in κ^+, we have, if (iv) fails, that for stationarily many ν that $\nu \in D \wedge E_\nu \neq \varnothing$. Then for some cardinal $\lambda < \kappa$ there are unboundedly many (in fact stationary many) $\nu < \kappa^+$ with $E_\nu \neq \varnothing \wedge \mathrm{crit}(E_\nu) = \lambda \wedge j_{E_\nu}(\lambda) = \kappa$. Thus λ is superstrong in $L[E]$, as κ is a cardinal of $L[E]$ and $V_\kappa^{L[E]} \subseteq \mathrm{Ult}(L[E], E_\nu)$.

The proof of $\square_\kappa$ for a non-subcompact κ involves splitting κ^+ into *disjoint* pieces S^0, S^1. On one of these the E-sequence is relatively benign and an L-like construction of a $\square_\kappa$-sequence is possible. The harder, and new, part is on the set S^1 where the work is dealing with extender fragments rather than levels of $L[E]$, and in both cases it has to be shown that there is no conflict between the sequences of the two sides, e.g. that $\alpha \in S^i \implies C_\alpha \subseteq S^i$. However the important feature remains that in either case the sequence C_ν is first order definable over the least level $J^E_{\beta(\nu)}$ of the $L[E]$ hierarchy where ν becomes singular. Finally:

2.55 Theorem (Zeman). *In any Jensen style $L[E]$ model* $\Box$(SingCard) *holds.*

The use of methods from Woodin's core model induction enabled strong conclusions to be obtained from the failure of, first $\Box_\kappa^{<\omega}$ and now with the above results, that of $\Box_\kappa$. We give some samples of the applications of Schimmerling's and Zeman's Theorem 2.54, but refer the reader to their [44] for a full survey.

2.56 Theorem. *Let κ be a singular cardinal satisfying $\mu^{\aleph_0} < \mu$ for all $\mu < \kappa$. Then if $\Box_\kappa$ fails Projective Determinacy holds.*

2.57 Theorem. *Let κ be a weakly compact cardinal and suppose $\Box_\kappa$ fails; then every set of reals in $L(\mathbb{R})$ is determined.*

The last two were first proven by Schimmerling and Steel essentially in [43] assuming the failure of $\Box_\kappa^{<\omega}$. The latter theorem concluded with PD, but was then improved by Woodin using the full Core Model Induction to get the result quoted, still from the failure of $\Box_\kappa^{<\omega}$.

2.58 Theorem. *Let κ be a measurable cardinal and suppose $\Box_\kappa$ fails; then there is a transitive proper class model of* $\mathrm{ZF} + \mathrm{AD}_\mathbb{R}$.

We lastly remark that:

2.59 Theorem (Steel [51]). *If κ is a singular strong limit cardinal and $\Box_\kappa$ fails then every set in $L(\mathbb{R})$ is determined.*

It is an open question whether the requirement on being a strong limit can be lifted here, for example at $\aleph_\omega$, or whether the result can be proven from the failure of $\Box_\kappa$ at a Jónsson cardinal κ. In these last four theorems the use of the failure of $\Box_\kappa$ is not as a simple quotation of the failure of correct cardinal successor computation in an inner model: there may be no canonical inner model at hand. In essence fine structural segments are pieced together to approximate some form of a hierarchy coded by subsets $A_\alpha \subseteq \kappa$ for $\alpha < \kappa^+$. Typically some first subset A_0 codes V_κ and then the constuction puts together mouse-like segments over A_α's. In these segments parts of a $\Box_\kappa$ sequence are pieced together (or of $\Box_\kappa^{<\omega}$ in fine structural hierarchies using the earlier work of Schimmerling). The definitions of the various $\langle C_\nu \mid \nu < \lambda_\alpha\rangle$'s are purely local (for λ_α increasing unboundedly below κ^+; as we remarked just before 2.55, the sequence C_ν is definable in some sort of similar fashion to our proof above in L: it is first-order definable over $J^E_{\beta(\nu)}$ where $\beta(\nu)$ names the first place in the $L[E]$-hierarchy where ν is definably singularised). Hence the result can be derived without knowing that there is some inner model (which might have required some further assumption concerning a larger measurable cardinal, for example) over which the Covering Lemma held.

The reader should consult [41] Sect.5 for further applications in this area.

3. Morasses

The notion of morass is a somewhat complex one. In many ways morasses seem to encapsulate the totality of the fine structure available in a model, although the $\Box$ concept has been more influential. The notion can be motivated through its original application to the Gap-2 Cardinal Transfer Theorems of Jensen (cf. 3.9 below). However they can be construed as pictures of the extremely regular behaviour of, e.g. the inner model L, in that one sees how structures, or levels of the L_α-hierarchy, say L_{κ^+}, can be approximated by directed systems of levels L_β for β of smaller cardinality then κ. A morass at a cardinal κ is characterised by a gap parameter. A morass at κ of gap 1 (a "$(\kappa, 1)$ morass") then is a system approximating the levels L_α for $\alpha \in (\kappa, \kappa^+]$ by means of levels L_β for $\beta < \kappa$ and maps between them: $f_{\beta\beta'}$. A gap-2 morass at κ is a "higher gap morass", and approximates $L_{\kappa^{++}}$ by means again of structures L_β for $\beta < \kappa$ and a more complex system of maps, which can be construed as a "morass of morasses" building up a double gap approximating process through systems of maps. There is no need to restrict to gaps of length two, and indeed for regular κ one can define (κ, γ) morasses for any $\gamma < \kappa$, once one has seen how to do it for 2.

We shall first give a definition of a gap-1 morass at κ in a manner somewhat reminiscent of a category. This is a formal definition of maps and structures which encapsulate the notion. The notation is intentionally similar to what we have done for Global $\Box$: this will bring out the similarities, and also make plausible the existence of such a structure in L.

3.1 Definition. Let $\kappa > \omega$ be regular. A *gap-1 morass at* κ is a pentuple $\langle S, S^1, \prec, \langle \mathfrak{A}_\nu \mid \nu \in S^1 \rangle, \langle \widehat{f}_{\bar\nu\nu} \mid \bar\nu \prec \nu \rangle\rangle$ satisfying the following:

M0(i) S is a set of p.r. closed pairs of ordinals $\langle \alpha, \nu\rangle$ with $\alpha < \nu < \kappa^+$ such that: $\langle \alpha, \nu\rangle, \langle \alpha', \nu'\rangle \in S \wedge \alpha' < \alpha \longrightarrow \nu' < \alpha$.

M0(ii) $S^1 =_{\mathrm{df}} \{\nu \mid \exists\alpha \langle \alpha, \nu\rangle \in S\}$; for $\nu \in S^1$ we let α_ν denote the unique α such that $\langle \alpha, \nu\rangle \in S$; $S^0 =_{\mathrm{df}} \{\alpha_\nu \mid \nu \in S^1\}$; $S_\alpha =_{\mathrm{df}} \{\nu \mid \alpha_\nu = \alpha\}$. Then: For $\alpha \in S^0, S_\alpha$ is closed in $\sup(S_\alpha)$; $\kappa = \sup(S^0 \cap \kappa) = \max(S^0)$; S_κ is unbounded in κ^+.

M0(iii) $\prec$ is a tree on S^1; $\overline{\nu} \prec \nu \longrightarrow \alpha_{\bar\nu} < \alpha_\nu$.

M0(iv) *(a)* For $\nu \in S^1$, $\mathfrak{A}_\nu$ is a transitive amenable structure, $J_\nu \subseteq |\mathfrak{A}_\nu|$; if $\tau \in S_{\alpha_\nu} \cap \nu$ then $|\mathfrak{A}_\tau| \subseteq |\mathfrak{A}_\nu|$.

(b) If h is a Σ_1-Skolem function for $\mathfrak{A}_\nu$ then $|\mathfrak{A}_\nu| = h``(\omega \times J_{\alpha_\nu}^{<\omega})$.

M0(v) $\langle \widehat{f}_{\bar\nu\nu} \mid \bar\nu \prec \nu\rangle$ is a commuting system of maps; $\widehat{f}_{\bar\nu\nu} : \mathfrak{A}_{\bar\nu} \longrightarrow_{\Sigma_1} \mathfrak{A}_\nu$; $\widehat{f}_{\bar\nu\nu}\restriction J_{\bar\nu} : \langle J_{\bar\nu}, S_{\alpha_{\bar\nu}} \cap \bar\nu\rangle \longrightarrow_Q \langle J_\nu, S_{\alpha_\nu} \cap \nu\rangle$; $\widehat{f}_{\bar\nu\nu}(\alpha_{\bar\nu}) = \alpha_\nu$; $\widehat{f}_{\bar\nu\nu}\restriction\alpha_{\bar\nu} = \mathrm{id}\restriction\alpha_{\bar\nu}$.

CP1 If $B_\nu =_{\mathrm{df}} \{\alpha_{\bar\nu} \mid \bar\nu \prec \nu\}$ and $B_\nu^+ =_{\mathrm{df}} B_\nu \cup \{\alpha_\nu\}$ then B_ν^+ is closed.

M1 If $\tau \in S_{\alpha_\nu} \cap \nu$ then B_τ is unbounded in α_ν.

M2 If $\bar\nu \prec \nu, \bar\tau \in S_{\alpha_{\bar\nu}} \cap \bar\nu$ and $\tau = \widehat{f}_{\bar\nu\nu}(\bar\tau)$, then $\bar\tau \prec \tau \wedge \widehat{f}_{\bar\tau\tau}{\restriction}J_{\bar\tau} = \widehat{f}_{\bar\nu\nu}{\restriction}J_{\bar\tau}$.

M3 B_ν unbounded in α_ν implies $|\mathfrak{A}_\nu| = \bigcup_{\bar\nu \prec \nu} \mathrm{ran}(\widehat{f}_{\bar\nu\nu})$.

CP2 If $\bar\nu \in (S_{\alpha_{\bar\nu}})^*$ then:

(a) $\bar\nu \prec \nu \wedge \sup(\widehat{f}_{\bar\nu\nu}``\bar\nu) = \lambda < \nu \longrightarrow \bar\nu \prec \lambda \wedge \widehat{f}_{\bar\nu\lambda}{\restriction}J_{\bar\nu} = \widehat{f}_{\bar\nu\nu}{\restriction}J_{\bar\nu}$;

(b) $\bar\nu \prec \nu \wedge \sup(\widehat{f}_{\bar\nu\nu}``\bar\nu) = \nu \,\wedge\, \alpha > \alpha_{\bar\nu} \longrightarrow$
$[(\forall\tau \in S_{\alpha_\nu} \cap \mathrm{ran}(\widehat{f}_{\bar\nu\nu})\ \alpha \in B_\tau) \longrightarrow \alpha \in B_\nu]$.

A morass is *universal* if $H(\kappa^+) = \bigcup_{\nu \in S^1} |\mathfrak{A}_\nu|$.

Note (i) Other forms of closure property can be used in *M0(i)* for example that of being a limit of admissibles. By *M0(i)* the definition of α_ν (in *M0(ii)*) makes sense. In *M0(v)* recall ([46] 1.18) the definition of *Q-embedding*: A *Q-formula* is one of the form $\forall u \exists v \supseteq \varphi(v)$, where φ is Σ_1 and must not contain u. Notice that this can suitably express the notion of cofinality. Then $\pi : M \longrightarrow_Q N$ if π preserves all Q-formulae.

Note (ii) If we are not interested in universality, then it is possible to give a similar "axiomatic" treatment of a gap-1 morass which does not mention L or J structures, but simple ordinal structures $\langle \nu, \in \rangle$: one omits the references to structures, thus *M3* becomes "B_ν is unbounded α_ν implies $\nu + 1 = \bigcup_{\bar\nu \prec \nu} \mathrm{ran}(f_{\bar\nu\nu})$" where now we only consider maps$f_{\bar\nu\nu} : \bar\nu + 1 \longrightarrow \nu + 1$. *M0(v)* now must be formulated in a way that expresses the similarity of the structure $\langle \bar\nu + 1, S_{\alpha_{\bar\nu}} \cap \bar\nu \rangle$ to that of $\langle \nu + 1, S_{\alpha_\nu} \cap \nu \rangle$ (see, for example, [8] for details). *M0-M3* are known as the *morass axioms* and *CP1, CP2* as the first and second *continuity properties* respectively. It is in particular the latter that gives the morass its strength. *CP2(b)* can be equivalently stated as:

CP2(b)′ $\big[\bar\nu \prec \nu \wedge \sup(\hat{f}_{\overline{\nu}\nu}``\bar\nu) = \nu \,\wedge$

$$\wedge \quad \alpha \in \bigcap_{\overline{\tau} \in S_{\alpha_{\overline{\nu}}} \cap \overline{\nu}} \{\alpha_{\tau'} \mid \overline{\tau} \prec \tau' \prec \widehat{f}_{\overline{\nu}\nu}(\overline{\tau})\}\big] \longrightarrow (\exists \nu' \prec \nu)(\alpha_{\nu'} = \alpha).$$

3.1. Construction of Gap-1 Morasses in L

Let κ be a regular cardinal. We give the construction of a universal $(\kappa, 1)$-morass in the constructible universe. We assume then $V = L$. In this we can make use of some of the notions and lemmata of the section on global $\square$.

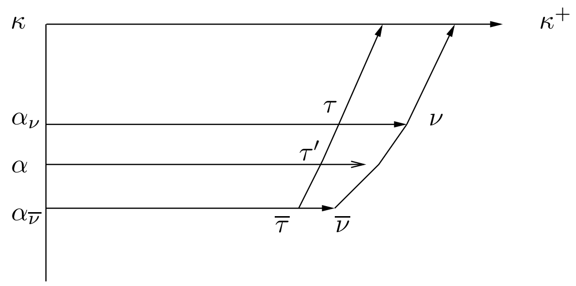

Figure I.3: Continuity Property *CP2b)*

Let $S = \{\langle\alpha,\nu\rangle \mid \alpha < \nu,\ \nu$ is a limit of admissibles, $\nu \in \mathrm{Sing} \cap \kappa^+,\ J_\nu \models$ "α is regular and is the largest cardinal"$\}$.

Note that for $\langle\alpha,\nu\rangle \in S$, as α is a cardinal in the sense of J_ν then it is *a fortiori* a limit of admissibles. We remark that our taking of ν as a limit of admissibles is overkill: much less would suffice: we just need to be able to take the transitivisation of certain simply defined hulls of smaller J_η inside J_ν. (As, for example, the intersection of On with such a hull has order type less than or equal to η, very mild recursions of length at most η can define the transitive collapse maps.) Using the p.r. closed ordinals here would be possible. $M0(i)$ is then clear (by appealing to the acceptability of the J-hierarchy!). Likewise $M0(ii)$ is true of the sets S^0, S^1, S_α defined there. As $\nu \in \mathrm{Sing}$ the concepts of the least level $\beta(\nu)$ at which ν is singularised, by some $\mathbf{\Sigma_1^{(n)}}(J_{\beta(\nu)})$ function etc., as used in the proof of global $\Box$ make sense, and we adopt them here too.

3.2 Definition. Let $\nu \in S^1$. Then we associate the following objects to ν:

a) The same n_ν, $M^k_\nu, h^k_\nu, h_\nu, \widetilde{h}_\nu$, ρ_ν from Definition 2.6.

b) $\alpha_\nu =_{\mathrm{df}}$ the largest cardinal of J_ν

c) $p_\nu =_{\mathrm{df}} p_{M_\nu} \backslash \alpha_\nu$; $q_\nu =_{\mathrm{df}} p_\nu \cap \omega\rho^{n_\nu}_{M_\nu}$.

Note that p_ν is thus the $<^*$-least parameter so that $\widetilde{h}_\nu(\alpha_\nu \cup \{p_\nu\}) = J_{\beta(\nu)}$. It is really also the same p_ν from before; it is only that we have renamed κ_ν there as α_ν. It has become customary to use α_ν for the largest cardinal in the sense of J_ν. Although "κ_ν" would have been more consistent with the global $\Box$ proof, this is only apparent: even for the $\Box$-proof, it is the ordinal α_ν that is important, rather than the κ_ν; here the important ordinal is again this α_ν. q_ν also has the same definition.

We set $\mathfrak{A}_\nu = M^{n_\nu}_\nu = \langle J_{\rho_\nu}, A^{n_\nu}_{\beta(\nu)}\rangle$. *M0(iv)* is then immediate, as h_ν is indeed a suitable Σ_1-Skolem function.

3.3 Definition. For $\nu, \overline{\nu} \in S^1$:

(i) We set $f : \overline{\nu} \Longrightarrow \nu$ if $f : J_{\overline{\nu}} \longrightarrow_{\Sigma_1} J_\nu$, and f is the restriction of $f^* : J_{\beta(\overline{\nu})} \longrightarrow_{\Sigma_1^{(n)}} J_{\beta(\nu)}$ where $n = n_\nu, \nu = f^*(\overline{\nu})$ if $\nu < \beta(\nu); \alpha_\nu \in \mathrm{ran}(f)$; $p_\nu \in \mathrm{ran}(f^*)$.

(ii) $\mathbb{F} = \{\langle \overline{\nu}, f, \nu\rangle \mid f : \overline{\nu} \Longrightarrow \nu\}$.

(iii) If $\nu < \beta$, we set $p(\nu) =_{\mathrm{df}} p_\nu \cup \{\alpha_\nu, \nu\}$; otherwise $p(\nu) =_{\mathrm{df}} p_\nu \cup \{\alpha_\nu\}$.

(iv) $f^*_{(\delta,q,\nu)}$ is the inverse of the transitive collapse of the hull $\widetilde{h}_\nu(\delta, \{p(\nu) \cup q\})$ in $J_{\beta(\nu)}$.

Now we have a new version of Lemma 2.9:

3.4 Lemma. *If $\exists\overline{\nu}(f : \overline{\nu} \Longrightarrow \nu)$ then f and f^* are uniquely determined by* $\mathrm{ran}(f) \cap \alpha_\nu$.

This justifies the following definition:

3.5 Definition. If $f : \overline{\nu} \Longrightarrow \nu$ then we set:

(i) $f^*_{\overline{\nu}\nu}$ to be that unique extension f^* given by the last lemma;

(ii) $\widehat{f}_{\overline{\nu}\nu} = f^*_{\overline{\nu}\nu}{\restriction}\mathfrak{A}_{\overline{\nu}} \longrightarrow \mathfrak{A}_\nu$.

Lemma 2.11 still applies in this context, where now we amend the notation to read "$\nu \in S^1$", or "$\bar{\nu} \in S^1$" rather than just plain "S", and drop all references to κ. If $f : \bar{\nu} \Longrightarrow \nu$ we see that $f(\alpha_{\bar{\nu}}) = \alpha_\nu$. This is clear if $\nu < \beta(\nu)$ (or equivalently $\bar{\nu} < \beta(\bar{\nu})$) since then $f : J_{\bar{\nu}} \longrightarrow J_\nu$ is in fact elementary. However even if $\beta(\nu) = \nu$, $\alpha_\nu \in \mathrm{ran}(f)$, hence $J_\nu \models$ "$f(\alpha_{\bar{\nu}})$ is a cardinal"; but clearly $f(\alpha_{\bar{\nu}})$ cannot be strictly greater than α_ν (as such are not J_ν-cardinals) nor strictly less (since $f^{-1}(\alpha_\nu)$ is a $J_{\bar{\nu}}$-cardinal $\geq \alpha_{\bar{\nu}}$).

3.6 Definition.

(i) For $f = \langle \bar{\nu}, f, \nu\rangle \in \mathbb{F}$ we call f^* *good* if

$$f^*{\restriction}J_{\bar{\nu}} : \langle J_{\bar{\nu}}, S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}\rangle \longrightarrow_Q \langle J_\nu, S_{\alpha_\nu} \cap \nu\rangle .$$

(ii) $\bar{\nu} \prec \nu$ iff there is an $f = \langle \bar{\nu}, f, \nu\rangle \in \mathbb{F}$ with f^* good, and $f^*{\restriction}\alpha_{\bar{\nu}} = \mathrm{id}{\restriction}\alpha_{\bar{\nu}}$. We shall write in this case $f : \bar{\nu} \prec \nu$. We set $\bar{\nu} \preccurlyeq \nu \longleftrightarrow \bar{\nu} \prec \nu \vee \bar{\nu} = \nu$.

One should bear in mind that again, just as for similar remarks about E_ν in the proof of Global $\Box$, "$y = S_\gamma \cap \eta$" for $\eta < \nu$ is a simple $\Sigma_0^{J_\nu}$ relation of y, γ and J_η. Thus, in particular, if $f = \langle \bar{\nu}, f, \nu\rangle \in \mathbb{F}$, then $f(S_{\alpha_{\bar{\nu}}} \cap \eta) = S_{f(\alpha_{\bar{\nu}})} \cap f(\eta) = S_{\alpha_\nu} \cap f(\eta)$. What is the Q-preservation property for? For the simple reason that if $f = \langle \bar\nu, f, \nu\rangle \in \mathbb{F}$, then without Q-preservation of $f^*{\restriction}J_{\bar\nu}$ we may have $S_{\alpha_{\bar\nu}} \cap \bar\nu$ unbounded in $\bar\nu$ (a Q-property) whilst $S_{\alpha_\nu} \cap \nu$ is bounded in ν. If we insist on Q-preservation then we shall have, if $f : \bar\nu \Longrightarrow \nu$:

- $S_{\alpha_{\bar\nu}} \cap \bar\nu$ unbounded in $\bar\nu$ iff $S_{\alpha_\nu} \cap \nu$ is unbounded in ν.

The next three properties are Σ_1:

- $\eta = \min(S_{\alpha_{\overline{\nu}}} \cap \overline{\nu}) \longrightarrow f(\eta) = \min(S_{\alpha_\nu} \cap \nu)$;

- η the successor element in $S_{\alpha_{\bar\nu}} \cap \bar\nu$ of η' implies $f(\eta)$ is the successor element of $f(\eta')$ in $S_{\alpha_\nu} \cap \nu$.

- $\eta \in (S_{\alpha_{\bar\nu}} \cap \bar\nu)^* \longrightarrow f(\eta) \in (S_{\alpha_\nu} \cap \nu)^*$.

The above can be easily checked. If we simply define gap-1 morasses without "structural properties" i.e. simply as maps between ordinals, one specifies these last three bullet points as an axiom on the S_α's (see [8] p.341).

For almost all triples $f = \langle \bar\nu, f, \nu\rangle \in \mathbb{F}$ the Q-preservation property holds anyway. Just as remarked before this definition, this is clear if $\nu < \beta(\nu)$ since then $f : J_{\bar\nu} \longrightarrow J_\nu$ is even elementary. It is still clear if $n_\nu > 1$ since then $f : J_{\bar\nu} \longrightarrow J_\nu$ is Σ_2-preserving. It is only if $n_\nu = 1$ and $\beta(\nu) = \nu$ that we may not get Q-preservation automatically.

It is now reasonably clear that $\prec$ will be a tree: this is because if $f : \bar\nu \prec \nu$ and $g : \nu' \prec \nu$ then, if $\alpha_{\bar\nu} < \alpha_{\nu'}$ we shall have $\mathrm{ran}(f^*) \subsetneq \mathrm{ran}(g^*)$ and hence $\langle \bar\nu, g^{-1}{\circ}f, \nu'\rangle \in \mathbb{F}$. It is also easy to check that the composed map $g^{-1} \circ f : J_{\bar\nu} \longrightarrow J_{\nu'}$ is Q-preserving, and this map witnesses that $\bar\nu \prec \nu'$. Equally clearly if $\alpha_{\bar\nu} = \alpha_{\nu'}$, then we must have $\bar\nu = \nu'$. Thus we have $\prec$ is a tree, and the second part of *M0(iii)* follows too.

The only remaining part of *M0* is part *(v)*. That the system of maps commutes follows from the same for the system for $\Longrightarrow$. The Q-preservation property of $f : J_{\bar\nu} \longrightarrow_Q J_\nu$ has been built into the definition of $\prec$.

For the first continuity property *CP1*, if $D \subseteq B^+_\nu$ then for each $\alpha_{\bar\nu} \in D$ we may consider the hull sets $H_{\bar\nu} =_{\mathrm{df}} \mathrm{ran}(f^*_{\bar\nu\nu})$ in $J_{\beta(\nu)}$. Let $Y = \bigcup_{\alpha_{\bar\nu}\in D} H_{\bar\nu}$, and then $\alpha =_{\mathrm{df}} \sup(D) \subseteq Y$. Then one may check that if the inverse of the transitive collapse of Y is $g^* : J_{\bar\beta} \longrightarrow_{\Sigma_1^{(n)}} J_\beta$ where $\mathrm{ran}(g^*) = Y$, then setting $g = g^*{\restriction}J_\tau$, $g^*(\tau) = \nu$ (if $\nu < \beta(\nu)$) and $\alpha_\tau = \alpha$, we have by Lemma 2.11 $\bar\beta = \beta(\tau)$, and $\langle \tau, g, \nu\rangle \in \mathbb{F}$. That g is a Q-preserving embedding between J_τ and J_ν is also easily checked: as before this only comes into question if $n = 1$ and $\nu = \beta(\nu)$. But here we can see that a union of "Q-hulls" of the form $H_{\bar\nu}$ forms a Q-elementary hull Y. Thus $g = g^*{\restriction}\tau$ is Q-preserving. Thus $\tau \preccurlyeq \nu$ and $\alpha_\tau \in B^+_\nu$.

For *M1*: let $\tau \in S_{\alpha_\nu} \cap \nu$. As $J_\nu \models$ "α_ν is the largest cardinal" we must have that $\beta(\tau) < \nu$. Now by recursion form a chain of $\Sigma_1^{(n_\tau)}$ hulls $H_\gamma = \widetilde{h}_\tau(\gamma \cup \{p(\tau)\})$ for $\gamma < \alpha_\nu$. This recursion can be effected inside J_ν as the latter is an admissible set (or is the union of such); moreover we can (i) pick out a closed and unbounded $D \subseteq \alpha_\nu$ where $\gamma \in D \longrightarrow H_\gamma \cap \alpha_\nu = \gamma$; and (ii) form the transitive collapses of such H_γ. This is because α_ν is a regular cardinal of J_ν. Then $\gamma \in D \longrightarrow \gamma = \alpha_{\bar\tau}$ for some $\alpha_{\bar\tau} \in B_\tau$.

Note that we have just shown for such τ that B_τ is unbounded in τ; but we have also shown that $B_\tau \in J_\nu$. In fact the relation "$\alpha_{\bar\tau} \in B_\tau$" is a uniform $\Sigma_0^{J_\nu}$ relation of $\bar\tau, \tau$ and $\beta(\tau)$. Consequently J_ν knows all about the morass relations for $\langle \alpha, \tau \rangle \in S$ with $\langle \alpha, \tau \rangle <_{\text{lex}} \langle \alpha_\nu, \nu \rangle$.

For *M2*: note that, with the notation of the hypothesis, $f^*_{\bar\nu\nu}(J_{\beta(\bar\tau)}) = J_{\beta(\tau)}$ and thus $f^*_{\bar\nu\nu}{\restriction}J_{\beta(\bar\tau)} : J_{\beta(\bar\tau)} \longrightarrow_{\Sigma_\omega} J_{\beta(\tau)}$. However $\text{ran}(f^*_{\bar\tau\tau})$ is determined by $\text{ran}(f^*_{\bar\tau\tau}) \cap \alpha_\nu = \alpha_{\bar\tau} = \alpha_{\bar\nu} = \text{ran}(f^*_{\bar\nu\nu}) \cap \alpha_\nu$. Hence $f^*_{\bar\nu\nu}{\restriction}J_{\bar\tau} = \widehat{f}_{\bar\nu\nu}{\restriction}J_{\bar\tau} = \widehat{f}_{\bar\tau\tau}{\restriction}J_{\bar\tau}$. As $f^*_{\bar\nu\nu}(J_{\bar\tau}) = J_\tau$ we certainly have $\widehat{f}_{\bar\nu\nu}{\restriction}J_{\bar\tau} : \langle J_{\bar\tau}, S_{\alpha_{\bar\tau}} \cap \bar\tau \rangle \longrightarrow_Q \langle J_\tau, S_{\alpha_\tau} \cap \tau \rangle$. Hence we have all the conditions for $\bar\tau \prec \tau$.

For *M3*: Suppose $x \in \mathfrak{A}_\nu = M_\nu^{n_\nu} = \langle J_{\rho_\nu}, A^{n_\nu}_{\beta(\nu)} \rangle$. Let $h = h_{\mathfrak{A}_\nu} = h_\nu$. Then $x = h_\nu(i, \vec\alpha, \alpha_\nu, q_\nu \backslash \alpha_\nu)$ for some $\vec\alpha \in [\alpha_\nu]^{<\omega}$. As B_ν is unbounded in α_ν find $\alpha' > \max(\vec\alpha)$, with $\alpha' = \alpha_\tau \in B_\nu$. Then $x = \widehat{f}_{\tau\nu}(h_\tau(i, \vec\alpha, \alpha_\tau, q_\tau \backslash \alpha_\tau))$.

Now for the distinguishing axioms of the morass: *CP2*. For (a) we use the Interpolation Lemma in the form of a direct application of Lemma 2.13 to get that there is an $f_0 : \bar\nu \Longrightarrow \lambda$ with $f{\restriction}\overline{\nu} = f_0{\restriction}\overline{\nu}$ (where $f = f^*_{\bar\nu\nu}$). However f_0 is nothing other than the required map $f^*_{\bar\nu\lambda}$. (That $f^*_{\bar\nu\lambda}{\restriction}J_{\bar\nu}$ is a Q-embedding is also immediate, as by assumption it is cofinal into λ!). Hence $\bar\nu \prec \lambda \wedge f^*_{\bar\nu\lambda}{\restriction}J_{\bar\nu} = f^*_{\bar\nu\nu}{\restriction}J_{\bar\nu}$.

For *CP2(b)*: We work in the terms of the equivalent statement *CP2(b)′*. Given the hypotheses there, let $\tau = \widehat{f}_{\bar\nu\nu}(\bar\tau)$ for $\bar\tau \in S_{\alpha_{\bar\nu}} \cap \bar\nu$. Let $X = \bigcup_{\bar\tau \in S_{\alpha_{\bar\nu}} \cap \bar\nu} X_{\bar\tau}$ where $X_{\bar\tau} = \text{ran}(\widehat{f}_{\tau'\tau}{\restriction}J_{\tau'})$ and τ' is that unique η satisfying $\bar\tau \prec \eta \prec \tau$ and with $\alpha_\eta = \alpha$. Note that $X_{\bar\tau} \cap \alpha_\nu = \alpha$ and so $X \cap \alpha_\nu = \alpha$. $X \prec_{\Sigma_0} J_\nu$. Let the transitive collapse map be $g_1^{-1} : J_\nu{\restriction}X \cong J_{\nu'}$. Define $g_0 = g_1^{-1} \circ (\widehat{f}_{\bar\nu\nu}{\restriction}J_{\bar\nu})$. Note moreover, that as $\bar\nu \in (S_{\alpha_{\bar\nu}})^*$:

$$w \in J_{\bar\nu} \Longrightarrow \exists \bar\tau \in S_{\alpha_{\bar\nu}}[w \in J_{\bar\tau} \wedge \widehat{f}_{\bar\tau\tau}(w) = \widehat{f}_{\tau'\tau}(\widehat{f}_{\bar\tau\tau'}(w)) = \widehat{f}_{\bar\nu\nu}(w)]$$

by *M2*. Thus g_0 is total on $J_{\overline{\nu}}$. By assumption $f^*_{\overline{\nu}\nu}$ is cofinal into ν. We then have that both g_0 and g_1 are cofinal. Now we may apply the Generalised Interpolation Lemma, setting g, σ and σ_1 there as $f^*_{\bar\nu\nu}$, g_0, g_1 here respectively. We obtain $\widetilde{g} : M_{\overline{\nu}} \longrightarrow_{\Sigma_1^{(n)}} J_{\beta'}$ and $g' : J_{\beta'} \longrightarrow_{\Sigma_0^{(n)}} M_\nu$ with $\widetilde{g}{\restriction}J_{\overline{\nu}} = g_0$, $g'{\restriction}J_{\nu'} = g_1$. However g_1 is cofinal into ν and $f^*_{\bar\nu\nu}$ is cofinal at the n'level (as $f_{\bar\nu\nu} : \bar\nu \Longrightarrow \nu$ cofinally, we can obtain this latter fact from Lemma 2.14); the proof of the Generalised Interpolation Lemma shows that in this case g' is cofinal at the n'th level, and so is in fact also $\Sigma_1^{(n)}$-preserving. As $g_1(\alpha) = \alpha_\nu$ and $p_\nu \backslash \alpha_\nu \in \text{ran}(f^*_{\overline{\nu}\nu})$ then $p_\nu \backslash \alpha_\nu \in \text{ran}(g')$. We have, setting $g'(p') = p_\nu \backslash \alpha_\nu$, by an application of Lemma 2.11, that $p' = p_{\nu'} \backslash \alpha$, $M_{\nu'} = J_{\beta'}$ and $g' = f^*_{\nu'\nu}$, whence $\langle \nu', g_1, \nu \rangle \in \mathbb{F}$. However then we have that $\langle \overline{\nu}, g_0, \nu' \rangle \in \mathbb{F}$ with $\widetilde{g}_0 = f^*_{\overline{\nu},\nu'}$ by the same Lemma again.

The Σ_0-preserving yet cofinality of the maps g_0 and g_1 guarantees that they are Q-preserving (note that $S_{\alpha_{\overline{\nu}}} \cap \overline{\nu}, S_{\alpha_{\nu'}} \cap \nu', S_{\alpha_\nu} \cap \nu$ are all cofinal in their respective ordinals). Thus $f^*_{\overline{\nu}\nu'}$, and $f' = f^*_{\nu'\nu}$ are good (in the sense of Definition 3.6), and $\overline{\nu} \prec \nu' \prec \nu$ with $\alpha_{\nu'} = \alpha \in B_\nu$ as required $\dashv$

3.2. Variants

Higher Gap Morasses

In the construction of the gap-1 morass at ω_1 say, we could have defined S^1 as the class of pseudo-successor cardinals, ordinals ν so that there exists $\alpha = \alpha_\nu < \nu$ so that $J_\nu \models$ "$\alpha = \omega_1$ and is the largest cardinal". A gap-n morass at ω_1 $(n < \omega)$ would ask for classes $S^i\,(1 \leq i \leq n)$ with $\nu \in S^i \longrightarrow \exists\alpha < \nu J_\nu \models$ "$\alpha = \omega_1 \wedge \mathrm{ot}\{\gamma > \omega \mid \gamma \in \mathrm{Card}\} = i$". Here we are allowing any $\nu < \omega_{n+1}$. In the gap-2 case, we would think of a system that approximates to ω_3 via a system of maps built out of countable objects approximating objects of size ω_1 reaching up to ω_2 much as a gap-1 morass does, but then additional maps piece these together in turn to obtain objects of size ω_2 that will reach up to ω_3. An enhanced set of axioms extending those of the Definition 3.1 regulate this overall structure. Additional axioms (the "logical axioms") are also needed to control continuity properties and ensure that maps correctly transfer pieces of the morass from one structure to another. These higher gap morasses were introduced by Jensen ([26]).

There is no particular need to consider only finite gaps: countable length gaps (for morasses at ω_1); and generally gap-η morasses at κ can be constructed in L for $\eta < \kappa$, $\kappa \in \mathrm{Reg}$. Venturing beyond that, Irrgang has proposed definitions for gap-β morasses at ω_1 for $\beta > \omega_1$ and demonstrated their construction.

Hyperfine structural constructions exist of gap-1 morasses (Friedman, Koepke, and Piwinger [20]), and for the gap-2 case (Friedman, Piwinger [21]); the latter is ostensibly different from the Jensen morass as they come with enhanced preservation properties. This "perfect preservation" comes about from the extremely fine gradations of locations, that enables even more stages in the hierarchy than for the J-hierarchy – thus developing a structure not *prima facie* available for the usual J-hierarchy.

Coarse Morasses

On occasions the full structure of the morass is simply not used and a coarse morass suffices. As its name implies, it is obtained by dropping some clauses from that of a full morass: namely those of the second continuity property *CP2* and *M0(v)*. As we don't need such fine elementarity between the structures we can simplify *M0(iv)* as well. (Indeed as we shall see, this principle requires no fine structure for its construction at all.)

3.7 Definition. Let $\kappa > \omega$ be regular. A *coarse gap-1 morass* at κ is a quadruple $\langle S, S^1, \prec, \langle \widehat{f}_{\bar{\nu}\nu} \mid \bar{\nu} \prec \nu \rangle \rangle$ satisfying *M0(i)-(iii), M0(iv)(a), CP1, M1-3* of Definition 3.1, together with

M0(v)' $\langle \widehat{f}_{\bar{\nu}\nu} \mid \bar{\nu} \prec \nu \rangle$ is a commuting system of maps; $\widehat{f}_{\bar{\nu}\nu} : \mathfrak{A}_{\bar{\nu}} \longrightarrow_{\Sigma_\omega} \mathfrak{A}_\nu$; $\widehat{f}_{\bar{\nu}\nu}(\alpha_{\bar{\nu}}) = \alpha_\nu$; $\widehat{f}_{\bar{\nu}\nu} \restriction \alpha_{\bar{\nu}} = \mathrm{id} \restriction \alpha_{\bar{\nu}}$.

In L coarse morasses are easily constructed: take the class S of pairs $\langle \alpha, \nu \rangle$ such that:

• Both J_α, J_ν are limits of ZF^- models, and $J_\nu \models$ "$\alpha = \alpha_\nu$ is the largest cardinal".

If $\langle \alpha, \nu \rangle \in S$ set $\mathfrak{A}_\nu =_{\mathrm{df}} J_\nu$. We define ν^* to be the least $\nu' \geq \nu$ so that (i) $J_{\nu'} \models \mathrm{ZF}^-$; (ii) for some $p \in J_{\nu'}$, every $x \in J_{\nu'}$ is $J_{\nu'}$-definable from parameters in $\alpha_\nu \cup \{p\}$. An elementary argument shows that such a ν^* must exist, and we set p_ν to be the $<_L$-least p satisfying (ii). Then set $q_\nu = p_\nu$ if $\nu^* = \nu$, or to equal $\langle p_\nu, \nu \rangle$ otherwise. The standard kinds of argument we have been using show quickly that if $f^* : J_{\nu'} \longrightarrow_{\Sigma_\omega} J_{\nu^*}$ with $q_\nu \in \mathrm{ran}(f^*)$, then $f^*(\bar{\nu}) = \nu$ implies $\nu' = \bar{\nu}^*$; we may define $\bar{\nu} \prec \nu$ if there exists an $f^* : J_{\bar{\nu}^*} \longrightarrow_{\Sigma_\omega} J_{\nu^*}$ such that $f^* \restriction \alpha_{\bar{\nu}} = \mathrm{id} \restriction \alpha_{\bar{\nu}}$, $f^*(\alpha_{\bar{\nu}}) > \alpha_{\bar{\nu}}$ and $q_\nu \in \mathrm{ran}(f^*)$. We set $\widehat{f}_{\bar{\nu}\nu} =_{\mathrm{df}} f^* \restriction J_{\bar{\nu}}$. It can be verified without much new work that if we now fix a regular cardinal κ and restrict our pairs $\langle \alpha, \nu \rangle$ so that $\nu < \kappa^+$ with $J_\nu \models$ "$\alpha_\nu \in \mathrm{Reg}$", then (setting S^1 to be the obvious class of such ν) our relation $\prec \restriction \kappa^+ \times \kappa^+$, together with all possible maps $\widehat{f}_{\bar{\nu}\nu}$, satisfies the coarse morass definitions. For more detail the reader may consult [11], where it is further shown how to extend the gap-1 coarse notion easily to a generalised gap *global coarse morass* of arbitrarily large gaps.

Morasses with Extra Structure.

Various extra structure has been imposed on the morass concept in order to get strengthened results. This has occasioned so-called *morasses with linear limits* [12], or *morasses with built in* $\diamondsuit$, or the *morasses with built in* $\square$ of Friedman [17].

Simplified Morasses.

A substantial simplification of the morass notion is that of the *simplified morass* of D. Velleman [53]. Working to try and obtain a Martin's axiom-like postulate equivalent to the existence of gap-1 morasses, he derived a very simple and short list of axioms concerning sets of maps between ordinals. He showed that that given a morass one could show (by forcing) the existence of a structure satisfying this axiom list: the simplified morass. This work derived from earlier work of Shelah and Stanley who were trying also to obtain similar forcing axioms. The advantage here is indeed that of simplicity: it delivers an easily comprehended structure ready for possible application.

Velleman left open the question of whether such morasses could be constructed directly in L. He later went on to develop higher gap simplified morasses [54] and morasses with linear limits. That simplified morasses existed in L was proven first by Donder [12] in the gap-1 case, and by Jensen (unpublished), building on Donder's work, for the higher gaps. For a construction of gap-1 and -2 morasses from simplified morasses see [38]. A Silver Machine construction of a gap-1 morass was given in [40].

Applications

Jensen's original applications were to cardinal transfer theorems.

3.8 Definition. $(\kappa,\lambda) \longrightarrow (\theta,\eta)$ holds if for every structure whose universe has size κ with a distinguished unary predicate with extension in the structure of size λ, also has a model of size θ whose predicate has size η.

3.9 Theorem (Jensen)(The Gap-n+1 Cardinal Transfer Theorem). *If there is a gap-n morass at ω_1 then $(\kappa^{+(n+1)},\kappa) \longrightarrow (\omega_{n+1},\omega)$.*

Jensen also derived the following form of *Prikry's principle* from gap-1 morasses, symbolically this is:

$$\binom{\omega_2}{\omega_1} \not\longrightarrow \begin{bmatrix}\aleph_0\\ \aleph_1\end{bmatrix}_{\aleph_1}$$

Interpreted this says that there is a partition of $\omega_2\times\omega_1$ into $\aleph_1$ many sets $\{I_\gamma \mid \gamma<\omega_1\}$ such that $\forall A\subseteq[\omega_2]^{\aleph_0}\forall B\subseteq[\omega_1]^{\aleph_1}\forall\gamma<\omega_1(A\times B)\cap I_\gamma\neq\varnothing$. (See [39] for more on this and some further generalisations.)

We state as a further example here, versions of the non-existence of free subsets from gap-2 morasses. The notion of "free" here is as a kind of maximal independent set. We typically ask that for every means of associating ordinals with tuples we can find some large set which is free of associations. Let $\mathrm{Fr}(\omega_3,4,\omega_1)$ denote: "For every $F:[\omega_3]^4\longrightarrow\omega_3$ such that for every quadruple a from ω_3 satisfies $F(a)\notin a$, there exists X a subset of ω_3 of cardinality ω_1 with $X\cap F``[X]^4=\varnothing$".

3.10 Theorem (Jensen, unpublished). *Suppose* CH + *"there exists a gap-2 morass". Then* $\neg\,\mathrm{Fr}(\omega_3,4,\omega_1)$.

Set-theoretical applications of morasses are not legion. The reader may consult [39] and [31] as a survey of early principles derivable from gap-1 morasses. A Morass with built in $\square$ forms a building block of the Strong Coding forcing constructed in [16]. For applications of gap-1 and global coarse morasses to derive in L one and two cardinal versions of $\diamondsuit$ sequences, Kurepa trees without Aronszajn subtrees, and some negative partition relations, see Donder [11].

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