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Dimension of Fat Sierpiński Gaskets

Abstract

In this paper we continue the work started by Broomhead, Montaldi and Sidorov investigating the Hausdorff dimension of fat Sierpiński gaskets. We obtain generic results where the contraction rate λ is in a certain region.

1 Introduction

Let $F = \{S_1, \dots, S_k\}$ be a family of contractions on $\mathbb{R}^d$. It was shown in [5] that there exists a unique non-empty compact set $\Lambda(\lambda)$, called the attractor of F, such that,

$$\Lambda(F) = \cup_{i=1}^{k} S_i(\Lambda(F)).$$

In the case where the contractions are similarities and a technical condition called the open set condition (OSC) is satisfied it is a straight forward problem to calculate the Hausdorff dimension of $\Lambda(F)$ (see [3]). Not satisfying the OSC essentially means that the images $s_i(\Lambda(F))$ overlap in a non trivial manner. In this case calculating the Hausdorff dimension of the attractor of the IFS becomes a much more difficult question. We study a specific case in $\mathbb{R}^2$.

The fat Sierpiński gasket was introduced by Simon and Solomyak in [10]. It is defined to be the attractor, $\Lambda(\lambda) \subset \mathbb{R}^2$ of the IFS, $F = \{T_0, T_1, T_2\}$ where,

$$\begin{aligned} T_0(x) &= \lambda x \\ T_1(x) &= \lambda x + (1,0) \\ T_2(x) &= \lambda x + \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \end{aligned}$$

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[†] The pictures in this paper were drawn using Matlab

for $\lambda > \frac{1}{2}$. In proposition 3.3 of [10] they show that there exists a dense subset, $A \subset [\frac{1}{2}, \frac{1}{\sqrt{3}}]$, such that for all $\lambda \in A$, $\dim_{\mathcal{H}} \Lambda(\lambda) < -\frac{\log 3}{\log \lambda}$.

A systematic investigation of the Hausdorff dimension of $\Lambda(\lambda)$ was started by Broomhead, Montaldi and Sidorov in [2]. They were able to compute the exact Hausdorff dimension of $\Lambda(\lambda)$ when λ is in a special class of algebraic numbers they call the multinacci numbers. These are the positive solutions, ω_n, to the equations $\sum_{k=1}^{n} \lambda^k = 1$. In particular ω_2 is equal to the reciprocal of the golden ratio. They obtain the following result,

Theorem 1 (Broomhead, Montaldi, Sidorov).

$$\dim_{\mathcal{H}}(\Lambda(\omega_n)) = \frac{\log \tau_n}{\log \omega_n},$$

where τ_m is the smallest positive root of the polynomial $3z^{n+1} - 3z + 1$.

It should be noted that $\frac{\log \tau_n}{\log \omega_n} < -\frac{\log 3}{\log \omega_n}$.

In this paper we continue the investigation into the Hausdorff dimension of $\Lambda(\lambda)$. The following is our main result.

Theorem 2. *1. For almost all $\lambda \in [\frac{1}{2}, \frac{\sqrt[3]{4}}{3} \approx 0.529]$,*

$$\dim_{\mathcal{H}} \Lambda(\lambda) = -\frac{\log 3}{\log \lambda}.$$

2. For almost all $\lambda \geq 0.5853$,

$$\dim_{\mathcal{H}} \Lambda(\lambda) = 2.$$

Our methods only enable us to show that $\dim_{\mathcal{H}} \Lambda(\lambda) = 2$ for almost all $\lambda \leq 0.649$. However it is clear that for all $\lambda \geq \frac{2}{3}$, $\dim_{\mathcal{H}} \Lambda(\lambda) = 2$ and in [2] it is shown that for all $\lambda \geq 0.648$, $\Lambda(\lambda)$ has non-empty interior and hence Hausdorff dimension 2. It should be noted that the results in [10] and [2] mean that the equality in Theorem 2 certainly does not hold for all λ. It would be interesting to know whether the region of λ for which Theorem 2 is true can be extended to a larger region. However the method used in this paper only provides almost sure lower bounds for $\lambda \in (0.529, 0.5853]$ which are strictly less than $-\frac{\log 3}{\log \lambda}$. Theorem 2 has the following topological analogue.

Corollary 1. *1. There exists a residual set $A \subset [\frac{1}{2}, \frac{\sqrt[3]{4}}{2}]$ such that for any $\lambda \in A$,*

$$\dim_{\mathcal{H}} \Lambda(\lambda) = -\frac{\log 3}{\log \lambda}.$$

2. There exists a residual set $B \subset [0.5853, 1]$ such that for any $\lambda \in B$

$$\dim_{\mathcal{H}} \Lambda(\lambda) = 2.$$

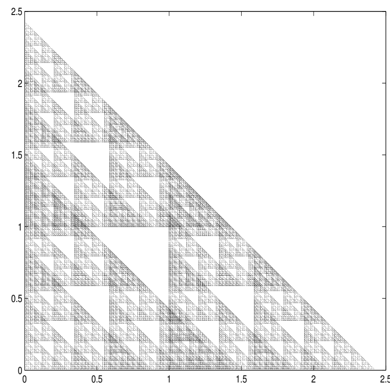

Figure 1: $\Lambda(\lambda)$ for $\lambda = 0.59$. Theorem 2 states that for almost all $\lambda > 0.5853$, $\dim_{\mathcal{H}} \Lambda(\lambda) = 2$.

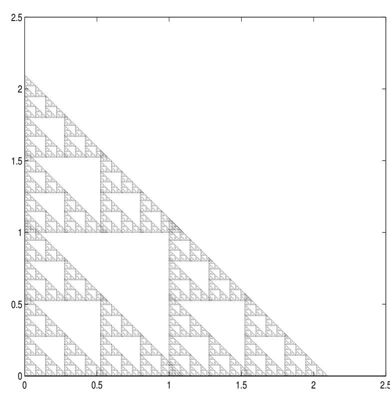

Figure 2: $\Lambda(\lambda)$ for $\lambda = 0.521$. Theorem 2 shows that for almost all $\lambda \in [0.5, 0.529]$ $\dim_{\mathcal{H}} \Lambda(\lambda) = -\frac{\log 3}{\log \lambda}$.

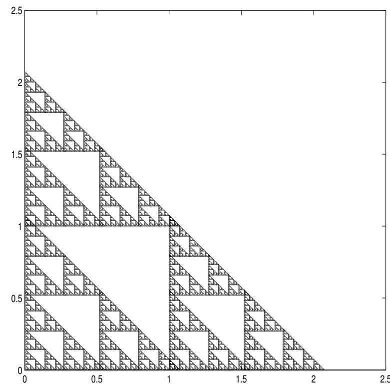

Figure 3: $\Lambda(\lambda)$ for $\lambda = \omega_4 \approx 0.519$. It is shown in [2] that $\dim_{\mathcal{H}} \Lambda(\lambda) = \frac{\log \tau_4}{\log \omega_4} \approx 1.654 < -\frac{\log 3}{\log \omega_4}$. Theorem 2 shows that this is an exceptional value.

Hence the results found in [2] and [10] in the above region were exceptional cases both in a topological and measure theoretic sense.

It is notationally more convenient to look at a slightly different IFS. This is defined by the similarities,

$$\begin{aligned} T_0(x) &= \lambda x \\ T_1(x) &= \lambda x + (1,0) \\ T_2(x) &= \lambda x + (0,1). \end{aligned}$$

However the attractor of this IFS can be obtained by an affine transformation applied to the set $\Lambda(\lambda)$ and hence has the same Hausdorff dimension. There has been a lot of study of overlapping IFS's in one dimension ([12],[9],[8],[11]). Most of this work has used the idea of transversality introduced in [9] to obtain generic results. Typically these results compute the Hausdorff dimension of the attractor for a set of full measure. Various work has been done on lower semi-continuity of the dimension overlapping IFS. This includes unpublished work by Pollicott and Simon-Solomyak as well as the published work by Jonker and Veerman [6]. Using this work it is often possible to compute the Hausdorff dimension for a residual set (a subset which contains a dense countable intersection of open sets). We examine cross sections to enable us to use the method of transversality which has been so effective in the one-dimensional setting.

2 Definitions and Technical Lemmas

For a set $F \subseteq \mathbb{R}^n$ the s-dimensional Hausdorff dimension is defined by

$$H^s(F) = \lim_{\epsilon \to 0} \inf \left\{ \sum |u_i|^s \mid \{u_i\}_i \text{ is a finite or countable } \epsilon\text{-cover of } F \right\}.$$

The Hausdorff dimension of F is then defined as,

$$\dim_{\mathcal{H}} F = \inf\{s : H^s(F) = 0\} = \sup\{s : H^s(F) = \infty\}.$$

For a probability measure μ on $\mathbb{R}^n$ the Hausdorff dimension is defined by,

$$\dim_{\mathcal{H}} \mu = \inf\{\dim_{\mathcal{H}} F : \mu(F) = 1 \text{ and } F \text{ is a Borel set}\}.$$

The mass distribution principle can be used to show the following equality concerting the dimension of a measure,

$$\dim_{\mathcal{H}} \mu = \text{ess-sup} \left\{ \frac{\log \mu(B(x,r))}{\log r} : x \in \mathbb{R}^n \right\}. \tag{1}$$

Here ess-sup means the essential supremum.

We now prove a slight variation of the potential theoretic method for calculating lower bounds of Hausdorff dimension, [3]. For more details and links to generalised dimension see [4].

Lemma 1. *Let $A \subseteq \mathbb{R}$ be a Borel set and $\alpha, s \in (0,1]$. If there exists a measure μ on A such that,*

$$\int \left(\int \frac{d\mu(x)}{|x-y|^s} \right)^\alpha d\mu(y) < \infty \tag{2}$$

then $\dim_{\mathcal{H}} A \geq s$ and $\dim_{\mathcal{H}} \mu \geq s$.

Proof. Let $\phi_\mu(y) = \left(\int \frac{\mathrm{d}\mu(x)}{|x-y|^s} \right)$. If the inequality (2) holds for a measure μ on a set A then it follows that $(\phi_\mu(y))^\alpha$ is integrable with respect to μ. This means that there exists M such that,

$$A_M = \{y : (\phi_\mu(y))^\alpha \leq M\}$$

satisfies $\mu(A_M) > 0$. Thus we can define a measure ν simply by the restriction of μ to A_M. Hence for any $x \in A$,

$$M^{\frac{1}{\alpha}} \geq \int_A \frac{\mathrm{d}\nu(x)}{|x-y|^s} \geq \int_{B(x,r)} \frac{\mathrm{d}\nu(y)}{|x-y|^s} \geq \frac{1}{r^s} \nu(B(x,r)).$$

Thus for any $x \in A$, $\nu(B(x,r)) \leq M^{\frac{1}{\alpha}} r^s$ and by the mass distribution principle $\dim_{\mathcal{H}} A \geq s$ and $\dim_{\mathcal{H}} \mu \geq s$. $\square$

We also need a lemma which relies on the idea of transversality of a power series. This idea was first used in [9] and has since been the main tool in investigating IFS with overlaps. A power series g is said to satisfy the ϵ-transversality condition if g crosses any line within ϵ of the origin with slope at most $-\epsilon$. Consequences of transversality include the absolute continuity of Bernoulli convolutions ([12],[8]) and almost sure results for the dimension of several fractal families ([9],[11]). Consider power series of the form,

$$g(x) = 1 + \sum_{k=1}^{\infty} g_k x^k, \text{ with } g_k \in \{-1, 0, 1\}. \tag{3}$$

Let

$$b(1) = \inf\{\lambda > 0 : \exists g(x) \text{ of the form (3) such that } g(\lambda) = g'(\lambda) = 0\}.$$

Thus for any $0 < a < c < b(1)$ and any g of the form (3) there exists $\epsilon > 0$ such that for any λ where $|g(\lambda)| < \epsilon$, $|g'(\lambda)| \geq \epsilon$. Thus any power series of the form (3) where λ takes values less than c for some $c < b(1)$ satisfies ϵ-transversality for some ϵ. Peres and Solomyak have computed values for $b(1)$ (Lemma 5.2 in [8]) . They obtain,

$$b(1) \approx 0.649.$$

This allows us to prove the following Lemma which is almost identical to Lemma 2 in [9].

Lemma 2. *For any interval $I = [a, c]$ where $0 < a < c < b(1)$, $s < 1$ and any $\{a_k\}_{k\in\mathbb{N}}$ where $a_0 \neq 0$ and $a_k \in \{0, \pm 1\}$ there exists $K(s)$ such that,*

$$\int_I \frac{d\lambda}{|a_0 + \sum_{n=1}^{\infty} a_n \lambda^n|^s} \leq K(s).$$

Proof. From above we know there exists $\epsilon > 0$ such that if $|g(\lambda)| \leq \epsilon$ then $|g'(\lambda)| \geq \epsilon$ for any $\lambda \in [a, c]$. This allows exactly the same method of proof as used to proof Lemma 2 in [9]. □

The tool which allows us to use these one-dimensional methods to obtain a result about a subset of $\mathbb{R}^2$ is a generalisation of the Marstrand slicing theorem, [7]. It also appears in [3] as Corollary 7.12 and it is stated and proved as Theorem 4.1 in Chapter 3 of [1].

Lemma 3. *Let F be any subset of $\mathbb{R}^2$, and let E be a subset of the y-axis. Let $L_y = \{(x, z) \in \mathbb{R}^2 : z = y\}$. If $\dim_{\mathcal{H}}(F \cap L_x) \geq t$ for all $y \in E$, then $\dim_{\mathcal{H}} F \geq t + \dim_{\mathcal{H}} E$.*

3 Biased Bernoulli convolutions

Let $\lambda \in [0.5, 0.649\ldots]$ and $\underline{p} = (p_0, p_1)$ be a probability vector. We let,

$$\begin{aligned} T_0(x) &= \lambda x \\ T_1(x) &= \lambda x + 1. \end{aligned}$$

Let $\nu = \nu_\lambda^{p_0,p_1}$ be the self-similar measure such that for all $J \subset \left[0, \frac{1}{1-\lambda}\right]$,

$$\nu(J) = p_0 \nu(T_0^{-1}(J)) + p_1 \nu(T_1^{-1}(J)).$$

We will also let $\mu = \mu_{p_0,p_1}$ be (p_0,p_1)-Bernoulli measure defined on the sequence space, $\{0,1\}^{\mathbb{N}}$. We let $\Pi_\lambda : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ be defined by,

$$\Pi_\lambda(\underline{i}) = \sum_{n=0}^{\infty} i_n \lambda^n.$$

This gives $\nu_\lambda^{(p_0,p_1)} = \mu^{(p_0,p_1)} \circ \Pi_\lambda^{-1}$. We will also use the following notation: $|\underline{i} \wedge \underline{j}| = \min\{k : i_k \neq j_k\}$, $W_{\omega,k} = \{\tau \in \Omega : \tau_j = \omega_j : j \leq k\}$, W_k consists of all kth level cylinders,

$$[i, 0, i_1, \ldots, i_{k-1}] = \{\underline{j} : i_r = j_r \text{ for } 0 \leq r \leq k-1\}$$

and $k_r(\underline{i}) = \text{card}\{0 \leq j \leq k-1 : x_j = r\}$.

Proposition 1. *Fix (p_0,p_1). For almost all $\lambda \in [0.5, 0.649\ldots]$,*

$$\dim_{\mathcal{H}} \nu_\lambda^{(p_0,p_1)} = \min\left(\frac{p_0 \log p_0 + p_1 \log p_1}{\log \lambda}, 1\right).$$

This result could be deduced as a Corollary to Theorem 7.2 in [11]. However in the present simpler setting it is possible to construct a more elementary proof which is based on methods used in [8].

Proof of Proposition 1

The proof of the upper bound is standard. Note that by the strong law of large numbers,

$$\lim_{n\to\infty} \frac{1}{n} \log \mu([i_0, \ldots, i_{n-1}]) \to p_0 \log p_0 + p_1 \log p_1 \text{ for } \mu\text{-almost all } \underline{i}.$$

Thus for all $\epsilon > 0$ there exists N such that for all $n \geq N$

$$\nu_\lambda(B(\Pi_\lambda \underline{i}, \lambda^n))) \geq n(p_0 \log p_0 + p_1 \log p_1 - \epsilon)$$

for μ almost every $\underline{i}$. However because $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$

$$\frac{log(\nu_\lambda(B(x, \lambda^n)))}{\log \lambda^n} \leq \frac{p_0 \log p_0 + p_1 \log p_1 - \epsilon}{\log \lambda}$$

for ν_λ almost all x. Hence by (1) the proof of the upper bound is complete.

For the lower bound the following lemma is needed. It involves the use of an exponent $\alpha \in (0,1]$. The idea to use this exponent came from [8].

Lemma 4. *Fix (p_0, p_1). For all $\alpha \in (0,1]$ we have that for almost all $\lambda \in [0.5, b(1)]$*

$$\dim_{\mathcal{H}} \nu_\lambda^{(p_0,p_1)} \geq \min\left(\frac{\log((p_0^{\alpha+1} + p_1^{\alpha+1})^{\frac{1}{\alpha}})}{\log\lambda}, 1\right).$$

Proof. Fix (p_0, p_1) and let $\epsilon > 0$.
For simplicity denote $d(\alpha,\epsilon) = (p_0^{\alpha+1} + p_1^{\alpha+1} + \epsilon)^{\frac{1}{\alpha}}$. We let $S_\epsilon(\lambda) = \min\left(\frac{\log(d(\alpha,\epsilon))}{\log\lambda}, 1-\epsilon\right)$.
We use Lemma 1 together with Fubini's theorem and Lemma 2.

$$I = \int_{0.5}^{b(1)} \int \left(\int \frac{\mathrm{d}\nu_\lambda(x)}{|x-y|^{S_\epsilon(\lambda)}}\right)^\alpha \mathrm{d}\nu_\lambda(y)\mathrm{d}\lambda = \int_{0.5}^{b(1)} \int \left(\int \frac{\mathrm{d}\mu(\underline{i})}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{S_\epsilon(y)}}\right)^\alpha \mathrm{d}\mu(\underline{j})\mathrm{d}\lambda$$

Apply Fubini's theorem and Hölder's inequality $\int f^\alpha \leq C(\int f)^\alpha$ for $\alpha \in (0,1]$.) to give,

$$\begin{aligned}
I &\leq C \int \left(\int_{0.5}^{b(1)} \int \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\lambda}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda)}}\right)^\alpha \mathrm{d}\mu(\underline{j}) \\
&\leq C_1 \int \left(\int_{0.5}^{b(1)} \int \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\lambda}{|\sum_{n=0}^\infty (i_n - j_n)\lambda^n|^{s_\epsilon(\lambda)}}\right)^\alpha \mathrm{d}\mu(\underline{j}) \\
&\leq C_1 \int \left(\int_{0.5}^{b(1)} \int \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\lambda}{\left(\lambda^{|\underline{i}\wedge\underline{j}|}\, |a_0 + \sum_{n=1}^\infty a_n\lambda^n)\right|^{s_\epsilon(\lambda)}}\right)^\alpha \mathrm{d}\mu(\underline{j})
\end{aligned}$$

where $a_n \in \{-1,0,1\}$ for $n \geq 1$ and $a_0 \in \{-1,1\}$. We now use Lemma 2 to continue,

$$\begin{aligned}
I &\leq C_1 \int \left(\int_{0.5}^{b(1)} \int \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\lambda}{\left(d(\alpha,\epsilon)^{|\underline{i}\wedge\underline{j}|}\, |a_0 + \sum_{n=1}^\infty a_n\lambda^n|\right)^{s_\epsilon(\lambda)}}\right)^\alpha \mathrm{d}\mu(\underline{j}) \\
&\leq C_1 \int \left(\int_{0.5}^{b(1)} \frac{\mathrm{d}\lambda}{|a_0 + \sum_{n=1}^\infty a_n\lambda^n|^{s_\epsilon(\lambda)}} \int \frac{\mathrm{d}\mu(\underline{i})}{d(\alpha,\epsilon)^{|\underline{i}\wedge\underline{j}|}}\right)^\alpha \mathrm{d}\mu(\underline{j}) \\
&\leq C_2 \int \left(\int \frac{\mathrm{d}\mu(\underline{i})}{d(\alpha,\epsilon)^{|\underline{i}\wedge\underline{j}|}}\right)^\alpha \mathrm{d}\mu(\underline{j}) \\
&\leq C_2 \int \left(\sum_{k=0}^\infty \frac{\mu(W_{\omega,k})}{d(\alpha,\epsilon)^k}\right)^\alpha \mathrm{d}\mu(\omega)
\end{aligned}$$

We proceed by using the inequality $(\sum_i b_i)^\alpha \leq \sum_i b_i^\alpha$ for $b_i > 0$ and $\alpha \in (0,1]$

$$\begin{aligned} I &\leq C_2 \sum_{k=0}^{\infty} \sum_{w \in W_k} \frac{\mu(W)^{\alpha+1}}{d(\alpha,\epsilon)^{\alpha k}} \\ &\leq C_2 \sum_{k=0}^{\infty} d(\alpha,\epsilon)^{-\alpha k} (p_0^{\alpha+1} + p_1^{\alpha+1})^k. \end{aligned}$$

Thus because $d(\alpha,\epsilon)^\alpha > p_0^{\alpha+1} + p_1^{\alpha+1}$ we have $I < \infty$. Hence by Lemma 1 $\dim_{\mathcal{H}} \nu_\lambda \geq \min\left(\frac{d(\alpha,\epsilon)}{\log \lambda}, 1-\epsilon\right)$ for almost all λ. To complete the proof we let $\epsilon = \frac{1}{n}$ for $n \in \mathbb{N}$ and let $n \to \infty$. $\square$

To complete the proof of Proposition 1 we let $\alpha_n = \frac{1}{n}$ for $n \in \mathbb{N}$ and observe that,

$$\lim_{n\to\infty} \frac{\log(p_0^{\alpha_n+1} + p_1^{\alpha_n+1})}{\alpha_n \log \lambda} = \frac{p_0 \log p_0 + p_1 \log p_1}{\log \lambda}.$$

4 Cross sections of fat gaskets

Consider a sequence $\{i_n\} \in \{0,1,2\}^{\mathbb{N}}$ we can then represent each point in $\Lambda(\lambda)$ using the expansion,

$$\sum_{n=0}^{\infty} a_{i_n} \lambda^n,$$

where $a_0 = (0,0)$, $a_1 = (1,0)$ and $a_2 = (0,1)$. It should be noted that for $\lambda > \frac{1}{2}$ this expansion is not unique. Consider a sequence $\underline{x} \in \{0,1\}^{\mathbb{N}}$. Intuitively we think of the case when $x_n = 0$ as corresponding to the bottom two triangles in the gasket and x_n=1 corresponding to the top triangle. We then define a complementary sequence $\underline{j} \in \{0,1\}^N$ such that $j_n = 0$ whenever $x_n = 1$. The idea of this sequence is to determine a horizontal point on the gasket corresponding to the sequence $\underline{x}$. Thus whenever $x_n = 0$ there are two choices either 0 or 1 corresponding to the bottom two triangles in the gasket. However when $x_n = 1$ there is just the one choice and j_n must equal 0. This means if we define $\underline{i} \in \{0,1,2\}^{\mathbb{N}}$ such that

$$i_n = \begin{cases} 0 \text{ if } x_n = 0, j_n = 0 \\ 1 \text{ if } x_n = 0, j_n = 1 \\ 2 \text{ if } x_n = 1, j_n = 0 \end{cases}$$

then

$$\left(\sum_{n=0}^{\infty} j_n \lambda^n, \sum_{n=0}^{\infty} x_n \lambda^n\right) = \left(\sum_{n=0}^{\infty} a_{i_n} \lambda^n\right) \in \Lambda(\lambda).$$

Thus if we let,

$$L_{\Pi_\lambda(\underline{x})}(\Lambda(\lambda)) = \{z \in \mathbb{R} : (z, \Pi_\lambda(\underline{x})) \in \Lambda(\lambda)\}$$

then for any sequence $\underline{i} \in \{0,1\}^N$ such that $i_n = 0$ if $x_n = 1$ we have that $\Pi_\lambda(\underline{i}) \in L_{\Pi_\lambda(\underline{x})}(\Lambda(\lambda))$.

We look at the dimension of the set $L_{\Pi_\lambda(\underline{x})}(\Lambda(\lambda))$. We will fix (p_0, p_1). Let $\mu = \mu_{p_0,p_1}$ be (p_0, p_1)-Bernoulli measure on $\{0,1\}^{\mathbb{N}}$. We can define another measure $\tilde{\mu}_{\underline{x}}$ on $\{0,1\}^N$ such that,

$$\tilde{\mu}_{\underline{x}}(\{\underline{i} : i_n = 0\}) = \begin{cases} 1 \text{ if } x_n = 1 \\ \frac{1}{2} \text{ if } x_n = 0 \end{cases}.$$

This means for kth level cylinders,

$$\tilde{\mu}_{\underline{x}}([i_0, \ldots, i_{k-1}]) = \begin{cases} 0 \text{ if } \exists j \text{ such that } i_j = x_j = 1 \\ 2^{-k_0}(\underline{x}) \text{ if for all } x_j = 1 \text{ we have } i_j = 0 \end{cases}.$$

Intuitively this means whenever $x_k = 0$ this corresponds to the bottom two triangles in the gasket and we have a choice of the two triangles but whenever $x_k = 1$ we are in the top triangle in the gasket so there is only one choice.

Let $\tilde{\nu}_{\lambda,\underline{x}} = \tilde{\mu}_{\underline{x}} \circ \Pi_\lambda^{-1}$ and note that it is supported on a subset of $L_{\Pi_\lambda(\underline{x})}(\Lambda(\lambda))$.

Lemma 5. *For almost all $\lambda \in [0.5, 0.649\ldots]$, and for ν_λ almost all $y \in \mathbb{R}$*

$$\dim_{\mathcal{H}} L_y(\Lambda(\lambda)) \geq \min\left(-\frac{p_0 \log 2}{\log \lambda}, 1\right).$$

Proof. We shall show that for all $\alpha \in (0,1]$,

$$\dim_{\mathcal{H}} L_{\Pi_\lambda(\underline{x})}(\Lambda(\lambda)) \geq -\frac{\log(1 - p_0(1 - 2^{-\alpha}))}{\alpha \log \lambda},$$

for almost all λ and μ almost all $\underline{x} \in \{0,1\}^N$. The result then follows because if we let $\alpha_n = \frac{1}{n}$ for $n \in \mathbb{N}$ then,

$$\lim_{n\to\infty} \frac{\log(1 - p_0(1 - 2^{-\alpha_n}))}{\alpha_n \log \lambda} = -\frac{p_0 \log 2}{\log \lambda}$$

and if $\dim_{\mathcal{H}} L_{\Pi_\lambda \underline{x}}(\Lambda(\lambda)) \geq s$ for μ-almost all $\underline{x} \in \{0,1\}^{\mathbb{N}}$ then $\dim_{\mathcal{H}} L_y(\Lambda(\lambda)) \geq s$ for ν_λ almost all $y \in \mathbb{R}$. Let $\epsilon > 0$ and for simplicity let $d(\alpha, \epsilon) = (1 - p_0(1 - 2^{-\alpha}) + \epsilon)^{\frac{1}{\alpha}}$ and $s_\epsilon(\lambda) = \min\left(-\frac{\log d(\alpha,\epsilon)}{\log \lambda}, 1 - \epsilon\right)$. We use the measure $\tilde{\nu}_{\lambda,\underline{x}}$. Using the potential theoretic method for calculating Hausdorff dimension it suffices to show that,

$$I = \int_{0.5}^{b(1)} \int \left(\int\int \frac{\mathrm{d}\tilde{\nu}_{\lambda,\underline{x}}(y)\mathrm{d}\tilde{\nu}_{\lambda,\underline{x}}(z)}{|z-y|^{s_\epsilon(\lambda)}}\right)^\alpha \mathrm{d}\mu(\underline{x})\mathrm{d}\lambda < \infty.$$

We start by lifting to the sequence space, using Fubini's theorem and Hölder's inequality, $\int f^\alpha \le C\left(\int f\right)^\alpha$ for $\alpha \in (0,1]$.

$$\begin{aligned} I &= \int_{0.5}^{b(1)} \int \left(\int\int \frac{\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{i})\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{j})}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda,\alpha)}} \right)^\alpha \mathrm{d}\mu(\underline{x})\mathrm{d}\lambda \\ &\le C \int \left(\int_{0.5}^{b(1)} \int\int \frac{\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{i})\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{j})\mathrm{d}\lambda}{|\sum_{n=0}^\infty (i_n - j_n)\lambda^n|^{s_\epsilon(\lambda,\alpha)}} \right)^\alpha \mathrm{d}\mu(\underline{x}) \\ &\le C_1 \int \left(\int_{0.5}^{b(1)} \int\int \frac{\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{i})\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{j})\mathrm{d}\lambda}{|a_0 + \sum_{n=1}^\infty a_n\lambda^n|^{s_\epsilon(\lambda,\alpha)} \lambda^{|\underline{i}\wedge\underline{j}|s_\epsilon(\lambda,\alpha)}} \right)^\alpha \mathrm{d}\mu(\underline{x}) \\ &\le C_1 \int \left(\left(\int_{0.5}^{b(1)} \frac{\mathrm{d}\lambda}{|a_0 + \sum_{n=1}^\infty a_n\lambda^n|^{s_\epsilon(\lambda,\alpha)}} \right) \left(\int\int \frac{\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{i})\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{j})}{d(\alpha,\epsilon)^{|\underline{i}\wedge\underline{j}|}} \right) \right)^\alpha \mathrm{d}\mu(\underline{x}), \end{aligned}$$

where $a_0 \in \{-1,1\}$ and $a_n \in \{-1,0,1\}$ for $n \ge 1$. This means we can apply Lemma 2. Hence

$$\begin{aligned} I &\le C_2 \int \left(\int\int \frac{\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{i})\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{j})}{d(\alpha,\epsilon)^{|\underline{i}\wedge\underline{j}|}} \right)^\alpha \mathrm{d}\mu(\underline{x}) \\ &\le C_2 \int \left(\sum_{k=0}^\infty \frac{2^{-k_0(\underline{x})}}{d(\alpha,\epsilon)^k} \right)^\alpha \mathrm{d}\mu(\underline{x}) \end{aligned}$$

As in the proof of Lemma 4 we use the inequality $\left(\sum_i b_i\right)^\alpha \le \sum_i b_i^\alpha$ for $b_i > 0$. We get,

$$\begin{aligned} I &\le C_2 \sum_{k=0}^\infty d(\alpha,\epsilon)^{-\alpha k} \int 2^{-k_0(\underline{x})\alpha} \mathrm{d}\mu(\underline{x}) \\ &\le C_2 \sum_{k=0}^\infty d(\alpha,\epsilon)^{-\alpha k} \sum_{[i_0,\ldots,i_{k-1}]\in W_k} 2^{-k_0([i_0,\ldots,i_{k-1}])\alpha} \mu(W_k) \\ &\le C_1 \sum_{k=0}^\infty \frac{(p_0 2^{-\alpha} + p_1)^k}{(d^{\alpha,\epsilon})^{\alpha k}}. \end{aligned}$$

We can now see that $I < \infty$ because $p_0 2^{-\alpha} + p_1 = 1 - p_0(1 - 2^{-\alpha}) < d(\alpha,\epsilon)$. To finish the proof let $\epsilon = \frac{1}{n}$ for $n \in \mathbb{N}$ and let $n \to \infty$. $\square$

5 Proof of Theorem 2

It is a standard result that $\dim_{\mathcal{H}} \Lambda(\lambda) \le -\frac{\log 3}{\log \lambda}$ for all λ, see, for example, [3]. Let $\underline{p} = (\frac{2}{3}, \frac{1}{3})$, let $\mu_{\underline{p}}$ be the standard $\underline{p}$-Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ and let $\nu_\lambda = \mu_{\underline{p}} \circ \Pi_\lambda^{-1}$. We know from

Proposition 1 that for almost all $\lambda \in [\frac{1}{2}, \frac{\sqrt[3]{4}}{3}]$,

$$\dim_{\mathcal{H}} \nu_\lambda = \frac{\frac{1}{3}\log\left(\frac{1}{3}\right) + \frac{2}{3}\log\left(\frac{2}{3}\right)}{\log \lambda}$$

and by Lemma 5 that for almost all $\lambda \in [\frac{1}{2}, \frac{\sqrt[3]{4}}{3}]$ and ν_λ almost all $y \in \mathbb{R}$

$$\dim_{\mathcal{H}} L_y(\Lambda(\lambda)) \geq -\frac{\frac{2}{3}\log 2}{\log \lambda}.$$

Thus using Lemma 3 we have that,

$$\dim_{\mathcal{H}} \Lambda(\lambda) \geq \frac{\frac{1}{3}\log\left(\frac{1}{3}\right) + \frac{2}{3}\log\left(\frac{2}{3}\right)}{\log \lambda} - \frac{\frac{2}{3}\log 2}{\log \lambda} = -\frac{\log 3}{\log \lambda}$$

for almost all $\lambda \in [\frac{1}{2}, \frac{\sqrt[3]{4}}{3}]$.

To proof part 2 of Theorem 2 we need to take an alternative choice of probability vector. For example if we choose $\underline{p} = (0.7729, 0.2271)$ then

$$\frac{0.7729\log 0.7729 + 0.2271\log 0.2271}{\log 0.5853} \geq 1 \text{ and } -\frac{0.7729\log 2}{\log 0.5853} \geq 1.$$

Thus by letting $\nu_\lambda = \mu_{\underline{p}} \circ \Pi_\lambda^{-1}$ and applying Proposition 1, Lemma 5 and Lemma 3 we have that,

$$\dim_{\mathcal{H}} \Lambda(\lambda) = 2$$

for almost all $\lambda \in [0.5853, b(1)]$. It is shown in [2] that $\Lambda(\lambda)$ has non-empty interior for all $\lambda \geq 0.648\ldots < b(1)$. Thus $\dim_{\mathcal{H}} \Lambda(\lambda) = 2$ for almost all $\lambda \geq 0.5853$.

6 Proof of Corollary 1

We shall only prove part 1. of Corollary 1 because the proof of part 2 can be done using exactly the same method. The method is similar to the proof of Theorem 2.3 in [10]. From Theorem 2 we know there exists a dense set of $\lambda \in [\frac{1}{2}, \frac{\sqrt[3]{4}}{3}]$ such that $\dim_{\mathcal{H}} \Lambda(\lambda) = -\frac{\log 3}{\log \lambda}$. Let F be the set of all IFS's, $\{S_i\}_{i=0}^2$ in $\mathbb{R}^2$ such that $S_i(x) = \lambda x + b_i$ for $b_i \in \mathbb{R}^2$. We define a topology on F by the natural bijection from F to $[\frac{1}{2}, \frac{\sqrt[3]{4}}{3}] \times \mathbb{R}^6$.

From Theorem B in [6] we know that the function $\alpha(F) = \dim_{\mathcal{H}}(\Lambda(F))$ is lower semi-continuous. However since for a fixed λ, $\dim_{\mathcal{H}}(\Lambda(F))$ is constant. The function, $\alpha'(\lambda) = \dim_{\mathcal{H}}(\Lambda(\lambda))$ is also lower semi-continuous. If we let $\beta(\lambda) = -\frac{\log 3}{\log \lambda}$ then we have that β is continuous and $\alpha'(\lambda) \leq \beta(\lambda)$. We now show that

$$\left\{\lambda \in \left[\frac{1}{2}, \frac{\sqrt[3]{4}}{3}\right] : \alpha'(\lambda) = \beta(\lambda)\right\} = \left\{\lambda \in \left[\frac{1}{2}, \frac{\sqrt[3]{4}}{3}\right] : \alpha' \text{ is continuous at } \lambda\right\}. \qquad (4)$$

Recall that a subset $F \subset \left[\frac{1}{2}, \frac{\sqrt[3]{4}}{3}\right]$ is said to be residual if it contains a dense G_δ set. Firstly consider $\lambda \in \left[\frac{1}{2}, \frac{\sqrt[3]{4}}{3}\right]$ such that $\alpha' = \beta(\lambda)$. We know that β is continuous and $\alpha' < \beta$ is lower semi continuous thus α' is continuous at λ. On the other hand α' cannot be continuous at λ if $\alpha' \neq \beta(\lambda)$ because $\alpha'(\lambda) = \beta(\lambda)$ for a.e. $\lambda \in \left[\frac{1}{2}, \frac{\sqrt[3]{4}}{3}\right]$. This completes the proof of (4).

The set of continuity points for any function is a G_δ set. Hence the set of points where $\alpha' = \beta(\lambda)$ contains a dense G_δ set.

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