# HAUSDORFF DIMENSION FOR RANDOMLY PERTURBED SELF AFFINE ATTRACTORS

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Abstract. In this paper we shall consider a self-affine iterated function system in $\mathbb{R}^d$, $d \geq 2$, where we allow a small random translation at each application of the contractions. We compute the dimension of a typical attractor of the resulting random iterated function system, complementing a famous deterministic result of Falconer, which necessarily requires restrictions on the norms of the contraction. However, our result has the advantage that we do not need to impose any additional assumptions on the norms. This is of benefit in practical applications, where such perturbations would correspond to the effect of random noise. We also give analogous results for the dimension of ergodic measures (in terms of their Lyapunov dimension). Finally, we apply our method to a problem originating in the theory of fractal image compression.

## 1. Introduction

In this article we consider families of self-affine Iterated Function Systems (IFS) defined on $\mathbb{R}^d$. More precisely, we consider contractions

$$\mathcal{F} := \{f_i(\mathbf{x}) = A_i \cdot \mathbf{x} + \mathbf{t}_i\}_{i=1}^m, \tag{1}$$

for $\mathbf{x} \in \mathbb{R}^d$, where the $A_i$ are $d \times d$ non-singular matrices satisfying

$$0 < \|A_i\| < \Theta < 1, \ \forall 1 \leq i \leq m, \tag{2}$$

and the vectors $\mathbf{t}_i$ are in $\mathbb{R}^d$. The following definition is standard.

**Definition 1** (The attractor $\Lambda$ of $\mathcal{F}$)**.** *Let $B$ be a ball in $\mathbb{R}^d$ centered at the origin with radius larger than* $\max_{1\leq i\leq m} \|\mathbf{t}_i\|/(1-\Theta)$*. Then the*

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*attractor of $\mathcal{F}$ is defined by:*

$$\Lambda := \bigcap_{n=1}^{\infty} \bigcup_{i_0 \dots i_{n-1}} f_{i_0 \dots i_{n-1}}(B). \tag{3}$$

*where we denote $f_{i_0 \dots i_{n-1}} = f_{i_0} \circ \dots \circ f_{i_{n-1}}$.*

It is easy to see that this definition does not depend on the choice of $B$ and that $\Lambda$ is the unique non-empty compact set for which

$$\Lambda = \bigcup_{i=1}^{m} f_i(\Lambda).$$

Since we use the translations as parameters it is sometimes convenient to explicitly show the dependence by writing $\mathcal{F}^{\mathbf{t}}$ and $f_i^{\mathbf{t}}$, instead of $\mathcal{F}$ and $f_i$, respectively. We let $\dim_{\mathrm{H}} \Lambda^{\mathbf{t}}$ and $\dim_{\mathrm{B}} \Lambda^{\mathbf{t}}$ denote the Hausdorff dimension and Box dimension of the attractor, respectively. (We refer the reader to [5] for the definitions.)

The following result is due to Falconer [4] and Solomyak [18].

**Theorem 1** (Falconer, Solomyak)**.** *If*

$$\|A_i\| < \frac{1}{2}, \ \forall 1 \le i \le m, \tag{4}$$

*then for $\mathcal{L}\mathrm{eb}_{m \cdot d}$-almost all vectors $\mathbf{t} := (\mathbf{t}_1, \dots, \mathbf{t}_m) \in \mathbb{R}^{m \cdot d}$ the dimension of the attractor $\Lambda^{\mathbf{t}} = \cup_{i=1}^{m} f_i^{\mathbf{t}}(\Lambda^{\mathbf{t}})$ is*

$$\dim_{\mathrm{H}} \Lambda^{\mathbf{t}} = \dim_{\mathrm{B}} \Lambda^{\mathbf{t}} = \min \{d, s(A_1, \dots, A_m)\}, \tag{5}$$

*where $s(A_1, \dots, A_m)$ is the singularity dimension (see* [4, Proposition 4.1] *and Definition 2 below.)*

Theorem 1 was originally proved by Falconer in 1988 under the stronger hypothesis that $\|A_i\| < \frac{1}{3}$, for all $1 \le i \le m$. Ten years later Solomyak weakened the hypotheses to their present form. Previously, Edgar [2] had already observed that the bound in (4) is optimal. Moreover, it was shown by Simon and Solomyak in [17] that the bound $\frac{1}{2}$ in (4) cannot be improved even in the special case that all the maps $f_i$ are similarities. Finally, the "almost all" hypothesis is necessary, in light of the construction by Bedford and McMullen of sporadic examples where the equality (5) fails even when the norms are smaller than $\frac{1}{2}$ [13].

If the matrices $A_i$ are orthogonal then the singularity dimension is simply the usual similarity dimension, while in the general case it is defined in terms of the singular values of repeated products of the matrices $A_1, \ldots, A_m$. More precisely, let $T$ be a non-singular linear mapping from $\mathbb{R}^d$ to $\mathbb{R}^d$. The singular values

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d$$

of $T$ are the positive square roots of the eigenvalues of the positive definite symmetric matrix $T^*T$. The singular value function $\phi^s(T)$ is defined for $s > 0$ by

$$\phi^s(T) := \begin{cases} \alpha_1 \cdots \alpha_{k-1} \alpha_k^{s-(k-1)}, & \text{if } k-1 < s \leq k \leq d; \\ (\alpha_1 \cdots \alpha_d)^{s/d}, & \text{if } s \geq d. \end{cases}$$

We can now present the definition of singularity dimension.

**Definition 2** (Singularity dimension)**.** *For a contracting self-affine IFS defined by (1) the singularity dimension is*

$$(6) \qquad s(A_1, \ldots, A_n) := \inf \left\{ t > 0 : \sum_{n=0}^{\infty} \sum_{i_0 \ldots i_n} \phi^t(A_{i_0} \cdots A_{i_n}) < \infty \right\}.$$

The following conjecture is widely believed to hold.

**Conjecture 1.** *For a typical (in an appropriate sense) self-affine IFS the Hausdorff dimension of the limit set is equal to* $\min\{d, s(A_1, \ldots, A_m)\}$.

In this general direction, we will prove a statement for a random perturbation of a given attractor $\Lambda$. Of particular importance is the fact that our Theorem will make no assumptions regarding the matrix norms. More precisely, we assume that with each application of the functions from the given IFS we make a random additive error $Y$. We assume that these errors have distribution $\eta$, where $\eta$ is an absolutely continuous distribution with bounded density supported on a disk $D$ which is centered at the origin and can be chosen to be arbitrarily small. We saw in (3) that $\Lambda$ is defined by all possible compositions $f_{i_0 \cdots i_{n-1}}$. For each such function we assume that the perturbations are independent. More precisely, let $\mathcal{T}$ be the $m$-adic tree with $m^n$ nodes on the $n$-th level. Each of these $n$-th level nodes corresponds to a word $\mathbf{i}_n := (i_0, \ldots, i_{n-1}) \in \{1, \ldots, m\}^n$. To obtain a random perturbation of the attractor $\Lambda$ we consider the random perturbations of the $n$-th level node maps $f_{\mathbf{i}_n} : \mathbb{R}^d \to \mathbb{R}^d$ of the form

$$f_{\mathbf{i}_n}^{\mathbf{y}_{\mathbf{i}_n}} := (f_{i_0} + y_{i_0}) \circ (f_{i_1} + y_{i_0 i_1}) \circ \cdots \circ (f_{i_{n-1}} + y_{i_0 \ldots i_{n-1}}),$$

where $\mathbf{i}_n = (i_0, \ldots, i_{n-1}) \in \mathcal{T}$, and the elements of

$$\mathbf{y}_{\mathbf{i}_n} := \left( y_{i_0}, y_{i_0 i_1}, \ldots, y_{i_0 \ldots i_{n-1}} \right) \in \underbrace{D \times \cdots \times D}_{n}$$

are i.i.d. with distribution $\eta$.

It is notationally convenient to label the perturbations by the natural numbers. Let $\varphi : \mathcal{T} \to \mathbb{N}$ be the natural labeling of the nodes given by setting $\varphi(k)$ to be the $k$-th element of the following infinite sequence:

$$\{1, 2, \dots, m, (1,1), (1,2), \dots, (m,m), (1,1,1), \dots, (m,m,m), \dots\}.$$

The sequence of all random errors

$$\mathbf{y} := \{y_k\}_{k=1}^{\infty} \in D^{\infty} := D \times D \times \cdots \tag{7}$$

in the construction of the random perturbation of $\Lambda$ is defined by

$$y_k := y_{\mathbf{i}_n} \text{ if } k = \varphi(\mathbf{i}_n).$$

Given $\mathbf{y} \in D^{\infty}$ we denote the associated attractor by $\Lambda^{\mathbf{y}}$. More precisely,

$$\Lambda^{\mathbf{y}} := \bigcap_{n=0}^{\infty} \bigcup_{\mathbf{i}_n} f_{\mathbf{i}_n}^{\mathbf{y}_{\mathbf{i}_n}}(B),$$

where $B$ is a sufficiently large ball in $\mathbb{R}^d$, centered at the origin. Let $\Pi^{\mathbf{y}} : \Sigma \to \mathbb{R}^d$ be the natural projection from the symbolic space

$$\Sigma = \{1, \dots, m\}^{\mathbb{Z}^+}$$

to the attractor $\Lambda^{\mathbf{y}}$ given by

$$\Pi^{\mathbf{y}}(\mathbf{i}) := \lim_{n\to\infty} f_{\mathbf{i}_n}^{\mathbf{y}_{\mathbf{i}_n}}(0) \tag{8}$$

where $\mathbf{i} \in \Sigma$ and $\mathbf{i}_n$ denotes the truncation to the first $n$ terms. More explicitly,

$$\begin{aligned} \Pi^{\mathbf{y}}(\mathbf{i}) := \lim_{n\to\infty} \Big(\mathbf{t}_{i_0} &+ \sum_{k=1}^{n} A_{i_0} \cdots A_{i_{k-1}} \cdot \mathbf{t}_{i_k} \\ &+ y_{i_0} + \sum_{k=1}^{n} A_{i_0} \cdots A_{i_{k-1}} \cdot y_{i_0 \dots i_k}\Big). \end{aligned} \tag{9}$$

On $D^{\infty}$ we define the infinite product measure by

$$\mathbb{P} := \eta \times \cdots \times \eta \times \cdots. \tag{10}$$

**Example 1.** *Fat Sierpiński gaskets were one of the examples used in* [17] *to show that Theorem 1 does not extend to contractions bigger than* $\frac{1}{2}$. *These were defined in terms of functions on* $\mathbb{R}^2$, *of the form*

$$f_i(x) = \lambda x + a_i \textit{ for } i = 0, 1, 2,$$

*where* $a_0, a_1, a_2$ *are not collinear and* $\lambda > \frac{1}{2}$. *For certain algebraic* $\lambda$ *the dimension is strictly less than* $\min\{-\frac{\log 3}{\log \lambda}, 2\}$, *the value one might expect from Theorem 1. One such example is* $\lambda = \frac{\sqrt{5}-1}{2}$ *where the*

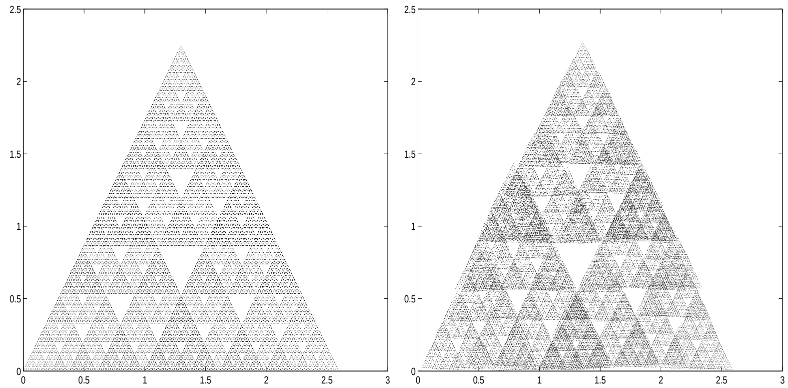


FIGURE 1. The golden gasket with and without random errors in the translation

*expected dimension from Theorem 1 is* 2 *but was computed in* [1] *as* $1.93\ldots$. *However, with the addition of random translation errors to the iterated function scheme, Theorem 2 gives that almost surely the dimension of the attractor is two and, furthermore, the Lebesgue measure is positive. The reason for the drop in dimension in the deterministic case is that several of the third level iterations are the same (for example* $f_0 \circ f_1 \circ f_1 = f_1 \circ f_0 \circ f_0$*). In particular, this means a "recoding" can give a more efficient cover. However, in the random case these third level iterations are typically no longer identical due to the random errors (see figure 1). Moreover, since no form of recoding would be possible as the random errors are different for each* $n$*th level iterate there is no longer any obvious reason to anticipate a drop in dimension.*

The following theorem gives a rigorous formulation of the intuitive notion that random attractors should have the expected dimension.

**Theorem 2** (Main Theorem)**.** *Consider contracting self-affine IFS of the form (1). For* $\mathbb{P}$*-almost all* $\mathbf{y} \in D^{\infty}$ *we have:*

(1) *If* $s(A_1, \ldots, A_m) \le d$ *then* $\dim_{\mathrm{H}} \Lambda^{\mathbf{y}} = s(A_1, \ldots, A_m)$*;*
(2) *If* $s(A_1, \ldots, A_m) > d$ *then* $\mathcal{L}\mathrm{eb}_d(\Lambda^{\mathbf{y}}) > 0$.

There is a closely related result by Peres, Simon and Solomyak in [15] for self-similar IFS with random multiplicative errors. Moreover, there the authors consider IFS which contract on average and where the $n$th level maps are assumed to have the same error.

Our method of proof of this Theorem involves an investigation of the dimension of certain measures defined on the attractor (Theorem 3). These are the images, under the natural projection, of ergodic, shift invariant Borel measures on $\Sigma$. For the rest of the paper when we refer to an ergodic measure on $\Sigma$, or a compact subset of $\Sigma$, we shall always assume it is both Borel and shift invariant. We use this method since $s(A_1, \dots, A_m)$ can be expressed in terms of the Lyapunov exponents of an ergodic measure introduced by Käenmäki in [8]. More precisely, let $\nu$ be an ergodic measure on $\Sigma$ and let $A : \Sigma \to \mathbf{M}$, where $\mathbf{M}$ denotes the set of $d \times d$ matrices with real entries be defined by

$$A(\mathbf{i}) := A_{i_0}.$$

Then for the stationary process given by the measure $\nu$ and

$$\left\{P_n(A, \mathbf{i}) := A^*_{i_{n-1}} \cdots A^*_{i_0}\right\}_{n=1}^{\infty} \tag{11}$$

we denote the Lyapunov exponents [10, Theorem 5.7] by

$$\lambda_1(\nu) \geq \lambda_2(\nu) \geq \cdots \geq \lambda_d(\nu). \tag{12}$$

Indeed, the main reason for introducing the transposed matrices $A^*_i$ is to reverse the order of the product of such matrices, and so be able to directly apply known results on Lyapunov exponents.

This leads naturally to the following definition.

**Definition 3** (Definition of the Lyapunov dimension $D(\nu)$)**.**

**(i):** *If*

$$k := k(\nu) = \max\left\{i : 0 < h_\nu + \lambda_1(\nu) + \cdots + \lambda_i(\nu)\right\} \leq d, \tag{13}$$

*then we define*

$$D(\nu) := k + \frac{h_\nu + \lambda_1(\nu) + \cdots + \lambda_k(n)}{-\lambda_{k+1}(\nu)};$$

**(ii):** *If* $h_\nu + \lambda_1(\nu) + \cdots + \lambda_d(\nu) > 0$ *then we define*

$$D(\nu) := d \cdot \frac{h_\nu}{-(\lambda_1(\nu) + \cdots + \lambda_d(\nu))}$$

*where* $h_\nu$ *is the entropy of the measure* $\nu$*.*

Note that in both cases above the Lyapunov dimension is simply the generalization of the similarity dimension (which can be higher than the dimension of the space) if the system is self-similar.

The following theorem gives a characterization of properties of the image of an ergodic measure in terms of the Lyapunov dimension.

**Theorem 3.** *Consider a contracting self-affine IFS of the form (1) and an ergodic measure* $\nu$ *on* $\Sigma$*. For* $\mathbb{P}$*-almost all* $\mathbf{y} \in D^\infty$ *the following hold:*

**(a):**

$$\dim_{\mathrm{H}} \Pi^{\mathbf{y}}_{*}(\nu) = \min\{d, D(\nu)\}$$

**(b):** *If $D(\nu) > d$ then*

$$\Pi^{\mathbf{y}}_{*}(\nu) \ll \mathcal{L}\mathrm{eb}_d.$$

We refer the reader to [6] for the definition of the Hausdorff dimension of a measure. Theorem 2 is an immediate consequence of Theorem 3 and the following proposition. Let $\mathcal{E}(\Sigma)$ denote the ergodic probability measures on $\Sigma$.

**Proposition 1.** *There exists an ergodic probability measure $\mu$ on $\Sigma$ such that*

$$s(A_1, \dots, A_m) = D(\mu) = \sup_{\nu \in \mathcal{E}(\Sigma)} D(\nu).$$

The measure $\mu$ in Proposition 1 is the same as that constructed by Käenmäki in [8].

We briefly explain the scheme of the proofs of Theorem 2 and Theorem 3. The key point is the introduction of a new *self-affine transversality condition* (26) for certain families of self-affine IFS, which was motivated by Solomyak's general projection scheme in [18]. We show that if this condition holds then for a typical parameter value the Hausdorff dimension of the attractor is the minimum of $d$ and the singularity dimension (Theorem 6). Furthermore, the Hausdorff dimension of the image $\nu = \Pi^{\mathbf{y}}(\mu)$ of an ergodic measure $\mu$ on $\Sigma$ is the Lyapunov dimension if the Lyapunov dimension $D(\nu)$ is smaller than $d$. Alternatively, if $D(\nu)$ is larger than $d$ then the image measure is absolute continuous (Theorem 7). As another application of this method we can solve a randomized version of a long standing open problem in fractal image compression [7], [9] (see Section 6).

The techniques we use in this paper also apply to the deterministic case, giving the following analogue of Theorem 1. We define a natural projection $\pi^{\mathbf{t}} : \Sigma \to \mathbb{R}^d$ by

$$\pi^{\mathbf{t}}(\mathbf{i}) := \lim_{n\to\infty} \left( \mathbf{t}_{i_0} + \sum_{k=1}^{n} A_{i_0} \cdots A_{i_{k-1}} \cdot \mathbf{t}_{i_k} \right).$$

**Theorem 4.** *Consider a contracting self-affine IFS of the form (1) and an ergodic measure $\nu$ on $\Sigma$. As in Theorem 1 we assume that*

$$\|A_i\| < \frac{1}{2},\ \forall 1 \le i \le m.$$

*Then for $\mathcal{L}\mathrm{eb}_{md}$ a.e. $\mathbf{t} \in \mathbb{R}^{md}$ we have:*

**(a):** $\dim_{\mathrm{H}} \pi^{\mathbf{t}}_{*}(\nu) = \min\{D(\nu), d\}$*;*

**(b):** *If* $D(\nu) > d$ *then* $\pi^{\mathbf{t}}_{*}(\nu) \ll \mathcal{L}\mathrm{eb}_d$*;*
**(c):** *If* $s(A_1, \ldots, A_m) > d$ *then* $\mathcal{L}\mathrm{eb}_d(\Lambda^{\mathbf{t}}) > 0$*, where* $\Lambda^{\mathbf{t}}$ *is the attractor. That is* $\Lambda^{\mathbf{t}} = \pi^{\mathbf{t}}(\Sigma)$*.*

In section 2 we describe the main properties of the singular value function and singularity dimension. In section 3 we introduce a variant on thermodynamic formalism, which is important in relating singularity dimension and Lyapunov dimension. The self-affine transversality condition is introduced in section 4 and used to get lower bounds on the dimensions, and to verify absolute continuity, in a general setting. Section 5 contains proofs of Theorems 2-7. Finally, in section 6 we apply our method to a problem in fractal image compression.

## 2. Main properties of the singular value function

Here we summarize those properties of the singular value function and singularity dimension which will be used in our proofs.

One can easily see that for every $s, h > 0$ we have

$$\alpha_1^h \geq \frac{\phi^{s+h}(T)}{\phi^s(T)} \geq \alpha_d^h. \tag{14}$$

The singular value function is submultiplicative [4]: For each $s \geq 0$,

$$\phi^s(T_1 T_2) \leq \phi^s(T_1)\phi^s(T_2).$$

We denote by $\Sigma^*$ the set of finite words from the alphabet $\{1, \ldots, m\}$. Fix a contractive IFS $\mathcal{F}$, of the form (1). We choose $0 < a_{\mathcal{F}} < b_{\mathcal{F}} < 1$ such that for each $i = 1, \ldots, m$,

$$1 > b_{\mathcal{F}} \geq \alpha_1(A_i) \geq \cdots \geq \alpha_d(A_i) \geq a_{\mathcal{F}} > 0.$$

We then obtain

$$b_{\mathcal{F}}^{|\mathbf{i}|} \geq \alpha_1(A_{\mathbf{i}}) \geq \cdots \geq \alpha_d(A_{\mathbf{i}}) \geq a_{\mathcal{F}}^{|\mathbf{i}|} \tag{15}$$

where $|\mathbf{i}|$ denotes the length of $\mathbf{i}$. Furthermore, it follows from the definition that for all $\mathbf{i} \in \Sigma^*$ and $s \geq 0$ we have

$$b_{\mathcal{F}}^{s|\mathbf{i}|} \geq \phi^s(A_{\mathbf{i}}) \geq a_{\mathcal{F}}^{s|\mathbf{i}|}. \tag{16}$$

For an arbitrary $\mathbf{i} \in \Sigma^*$ with $|\mathbf{i}| \geq n$, or $\mathbf{i} \in \Sigma$, we define

$$\psi_n^s(\mathbf{i}) := \log \phi^s(A_{i_0} \cdots A_{i_{n-1}}).$$

It follows from (16) that

$$ns \log b_{\mathcal{F}} \geq \psi_n^s(\mathbf{i}) \geq ns \log a_{\mathcal{F}}.$$

Using

$$\psi_{n+k}(\mathbf{i}) \leq \psi_n(\mathbf{i}) + \psi_k(\sigma^n \mathbf{i}),$$

it follows from the sub-additive ergodic theorem that for every ergodic measure $\nu$ on $\Sigma$ and for $\nu$ almost all $\mathbf{i} \in \Sigma$ we have that

$$\lim_{n\to\infty} \frac{1}{n}\psi_n^s(\mathbf{i}) = \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma} \psi_n^s(\mathbf{i}) d\nu(\mathbf{i}). \tag{17}$$

Finally, we need the following simple result to show that the singular values are the same if we reverse the order of the product of the matrices.

**Lemma 1.** *For every $\mathbf{i} \in \Sigma^*$ and for every $1 \leq k \leq d$ we have*

$$\alpha_k(A_{\mathbf{i}}) = \alpha_k(P_n(A, \mathbf{i})),$$

*where $P_n(A, \mathbf{i})$ was defined in (11).*

*Proof.* The proof immediately follows from the standard fact that for a $d \times d$ matrix $B$ the eigenvalues of the matrices $B^*B$ and $BB^*$ are the same (with the same multiplicity). □

2.1. **A Corollary of a proof of Oseledeč's Theorem.** Using the proof [10, p. 43-47] of Oseledeč's Theorem we can deduce the following result.

**Lemma 2.** *Consider a contractive IFS $\mathcal{F}$ defined by (1). Let $\nu$ be an arbitrary ergodic measure on $\Sigma$. We define $P_n(A, \mathbf{i})$ and $\lambda_i(\nu)$ for $i = 1, \ldots, d$ as in (11) and (12). Then for $\nu$-almost all $\mathbf{i} \in \Sigma$ and $1 \leq k \leq d$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \alpha_k \left(P_n(A, \mathbf{i})\right) = \lambda_k(\nu), \tag{18}$$

*where $\alpha_k \left(P_n(A, \mathbf{i})\right)$ is the $k$-th largest singular value of $P_n(A, \mathbf{i})$.*

This is an immediate consequence of the proof [10, Theorem 5.7].

2.2. **The singularity dimension and finite measures.** Consider the $d \times d$ non-singular matrices $A_1, \ldots, A_m$ satisfying (2) and $\Omega \subset \Sigma$ a $\sigma$-invariant compact set. For $\Omega = \Sigma$ the singularity dimension was defined by Falconer [4, p. 344]. For the convenience of the reader we summarize the most important facts about the singularity dimension $s_\Omega(A_1, \ldots, A_m)$ for $\Omega \subset \Sigma$ compact $\sigma$-invariant analogous to those in [4, p.344.].

For every $\ell > 0$ and $s > 0$ we define a set function $\mathcal{N}_\ell^s$ on the subsets of $\Omega$ by

$$\mathcal{N}_\ell^s(A) := \inf \left\{ \sum_k \phi^s(A_{\omega_k}) : \ \omega_k \in \Sigma^*, A \subset \cup_k \omega_k \text{ and } |\omega_k| \geq \ell \right\}.$$

Then it follows from [11, Theorem 4.2] that the measure

$$\mathcal{N}^s(A) := \lim_{\ell\to\infty} \mathcal{N}^s_\ell(A) = \sup_\ell \mathcal{N}^s_\ell(A)$$

is a Borel regular measure. It follows from (14) and (15) that if $t < s$ and $\mathcal{N}^t(\Omega) < \infty$ then $\mathcal{N}^s(\Omega) = 0$. This implies that we can define $s_\Omega(A_1, \ldots, A_m)$ by

$$s_\Omega(A_1, \ldots, A_m) = \inf\{s : \mathcal{N}^s(\Omega) = 0\} = \sup\{s : \mathcal{N}^s(\Omega) = \infty\}. \tag{19}$$

It follows from [4, p.344] that if $\Omega = \Sigma$ then $s_\Omega(A_1, \ldots, A_m)$ is equal to $s(A_1, \ldots, A_m)$ defined in (9).

The following lemma allows us to find a finite measure whose properties are closely related to the singularity dimension.

**Lemma 3.** *If $\mathcal{N}^s(\Omega) = \infty$ for some $s$ then there exists a finite measure $\mu$ supported on $\Omega$ and a constant $c_0$ such that*

$$\mu(\omega) \le c_0 \cdot \phi^s(A_\omega) \text{ for every } \omega \in \Sigma^*$$

*where $\mu(\omega)$ represents the measure of the corresponding cylinder set.*

*Proof.* Fix an $s$ such that $\mathcal{N}^s(\Omega) = \infty$. It follows from [16, Theorem 54] that there exists a compact set $\Omega' \subset \Omega$ such that $0 < \mathcal{N}^s(\Omega') < \infty$. We define a metric $\rho$ on $\Omega'$ as follows:

$$\rho(\mathbf{i}, \mathbf{j}) := \begin{cases} \phi^s(A_{\mathbf{i}\wedge\mathbf{j}}), & \text{if } \mathbf{i} \ne \mathbf{j}; \\ 0, & \text{if } \mathbf{i} = \mathbf{j}. \end{cases}$$

We can see that $\rho$ is a metric since for every $\omega \in \Sigma^*$ we have

$$\phi^s(A_{\mathbf{i}\wedge\mathbf{j}}) \le \max\{\phi^s(A_{\mathbf{i}\wedge\omega}), \phi^s(A_{\omega\wedge\mathbf{j}})\}.$$

Then it follows from [16, Theorem 53] that the measure $\mathcal{N}^s$ is the 1-dimensional Hausdorff measure on the compact set $\Omega'$. Now the assertion of the Lemma immediately follows from [11, Theorem 8.17]. □

## 3. Thermodynamical formalism

Following [8], we define the *energy* of an ergodic measure $\nu$ by

$$E_\nu(s) := \lim_{n\to\infty} \frac{1}{n} \int \psi^s_n(\mathbf{i}) d\nu(\mathbf{i}). \tag{20}$$

The following lemma relates the energy to the Lyapunov exponents.

**Lemma 4.** *Assume the same hypotheses as in Lemma 2. For $k < s \le k+1$, we can write*

$$E_\nu(s) = \begin{cases} \lambda_1(\nu) + \cdots + \lambda_k(\nu) + [s-k]\,\lambda_{k+1}(\nu), & \text{if } s < d; \\ \frac{s}{d}[\lambda_1(\nu) + \cdots + \lambda_d(\nu)], & \text{if } s \ge d. \end{cases} \tag{21}$$

*From this we obtain that*

$$h_\nu + E_\nu(D(\nu)) = 0. \tag{22}$$

*Proof.* It follows from (17) that for $\nu$-almost all $\mathbf{i}$ we have $E_\nu(s) = \lim\limits_{n\to\infty} \frac{1}{n}\psi_n^s(\mathbf{i})$. It follows from Lemma 2 that for $\nu$-almost all $\mathbf{i}$ we have $\lim\limits_{n\to\infty} \frac{1}{n}\psi_n^s(\mathbf{i})$ is equal to the right hand side of (21). □

We can define the *pressure function* by

$$P(s) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\mathbf{i}|=n} \phi^s(A_{\mathbf{i}}).$$

In particular, we see that $P(0) = \log m > 0$. Comparing (14) and (15) we see that

$$\log b_{\mathcal{F}} \geq \frac{\frac{1}{n}\cdot \log \sum\limits_{|\mathbf{i}|=k} \phi^{s+h}(A_{\mathbf{i}}) - \frac{1}{n}\cdot \log \sum\limits_{|\mathbf{i}|=k} \phi^{s}(A_{\mathbf{i}})}{h} \geq \log a_{\mathcal{F}}$$

and from this we see that the function $P(s)$ is strictly decreasing, and has a unique zero which we denote by $t_0 > 0$.

The following result is due to Käenmäki [8].

**Theorem 5** (Käenmäki)**.** *For any ergodic measure $\nu$ we have*

$$P(s) \geq h_\nu + E_\nu(s).$$

*Moreover,*

**(a):** *There exists an ergodic measure $\mu$ on $\Sigma$ such that*

$$0 = P(t_0) = h_\mu + E_\mu(t_0), \tag{23}$$

**(b):** $s(A_1, \ldots, A_m) = t_0$.

We are now in a position to prove Proposition 1, assuming Theorem 5.

*Proof of Proposition 1.* It follows from (21) that the function $s \to E_\mu(s)$ is strictly decreasing. Therefore it follows from (22) and (23) that $t_0 = D(\mu)$. Using part (b) of Theorem 5 the result follows. □

## 4. Generalized Projection Scheme for self affine IFS

In [18, p. 542], Solomyak presented a generalized projection scheme applicable to self-conformal IFS. In this section we construct an analogous scheme suitable for self-affine IFS. In the next section we give the details of the proofs.

Assume that we are given a contractive IFS $\mathcal{F}$ of the form (1). For every distinct $\mathbf{i}, \mathbf{j} \in \Sigma$ we write $\alpha_k(\mathbf{i} \wedge \mathbf{j})$ for the $k$-th largest singular

value of $A_{\mathbf{i}\wedge\mathbf{j}}$ if $1 \le k \le d$ and $\alpha_0(\mathbf{i}\wedge\mathbf{j}) := \infty$, $\alpha_{d+1}(\mathbf{i}\wedge\mathbf{j}) := 0$. Let $J_\ell := [\alpha_{\ell+1}(\mathbf{i}\wedge\mathbf{j}), \alpha_\ell(\mathbf{i}\wedge\mathbf{j})]$. We define a function $Z_{\mathbf{i}\wedge\mathbf{j}} : [0,\infty) \to [0,1]$ (see Figure 2) by
(24)

$$Z_{\mathbf{i}\wedge\mathbf{j}}(\rho) := \prod_{k=1}^{d} \frac{\min\{\rho, \alpha_k(\mathbf{i}\wedge\mathbf{j})\}}{\alpha_k(\mathbf{i}\wedge\mathbf{j})} = \sum_{\ell=1}^{d} \frac{\rho^\ell}{\alpha_1(\mathbf{i}\wedge\mathbf{j})\cdots\alpha_\ell(\mathbf{i}\wedge\mathbf{j})} \cdot \mathbb{1}_{J_\ell} + \mathbb{1}_{J_0}.$$

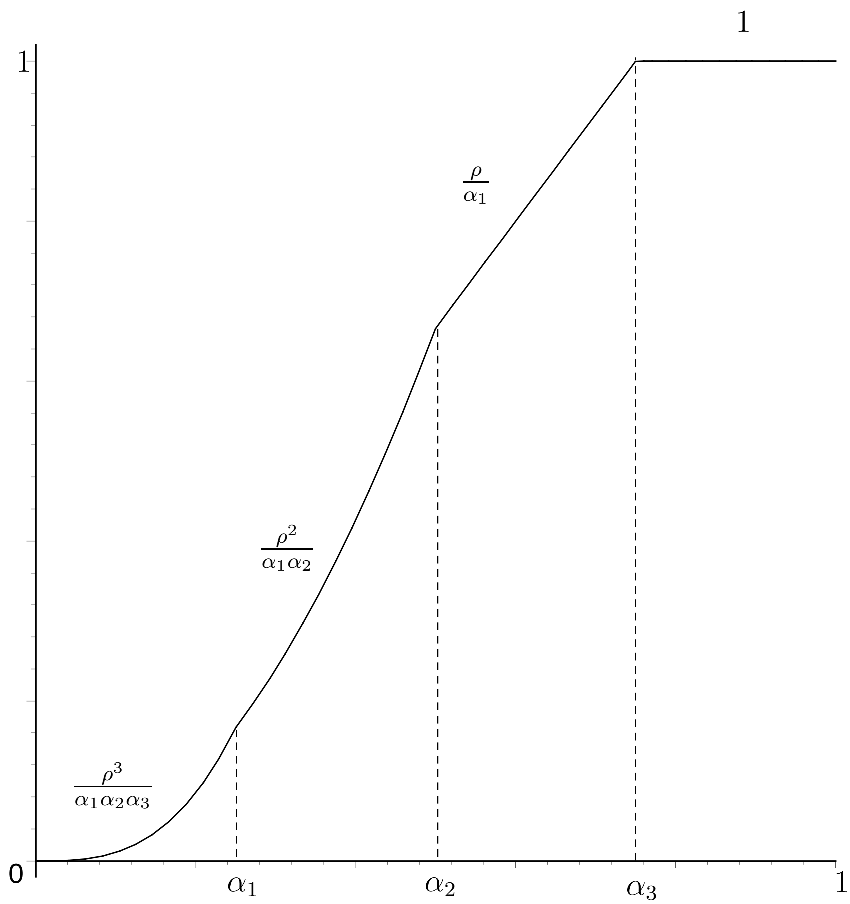


FIGURE 2. The function $\rho \to Z_{\mathbf{i}\wedge\mathbf{j}}(\rho)$ for $\alpha_k := \alpha_k(\mathbf{i}\wedge\mathbf{j})$

Note that if all the maps of $\mathcal{F}$ are similarities then the similarity ratio of $f_{\mathbf{i}\wedge\mathbf{j}}$ is $\lambda_{\mathbf{i}\wedge\mathbf{j}}$ and $Z_{\mathbf{i}\wedge\mathbf{j}}(\rho) = \min\left\{1, \frac{\rho^d}{\lambda^d_{\mathbf{i}\wedge\mathbf{j}}}\right\}$.

The motivation for the next Theorem is that we want to consider a one-parameter family of self-affine IFS defined by random perturbations of $\mathcal{F}$. The parameter space is a compact metric space $U$ on which we are given a probability measure $\mathcal{M}$. We want to give a dimension estimate for $\mathcal{M}$ almost all parameters $u \in U$.

**Definition 4.** *Consider a contractive IFS $\mathcal{F}$ of the form (1). Let $U$ be compact metric space (parameter space) with a finite Borel measure $\mathcal{M}$. Assume we are also given a continuous map $\Pi : U \times \Omega \to \mathbb{R}^d$ (the natural projection), where $\Omega \subset \Sigma = \{1, \cdots, m\}^{\mathbb{Z}^+}$ is compact and $\sigma$-invariant. For any $u \in U, \mathbf{i} \in \Omega$ we write $\Pi^u(\mathbf{i}) := \Pi(u, \mathbf{i})$. We make the following two definitions.*

***Self-affine Hölder condition:*** *There exists a constant $K > 0$ such that for every $u \in U$ and $\mathbf{i} \in \Omega$ and for every $n \in \mathbb{N}$ we can find an isometry $G = G(\mathbf{i}, n) : \mathbb{R}^d \to \mathbb{R}^d$ with*

$$\Pi^u(\mathbf{i}|n) \subset K \cdot G\left([0, \alpha_1(\mathbf{i}|n)] \times \cdots \times [0, \alpha_d(\mathbf{i}|n)]\right) \tag{25}$$

*where $\mathbf{i}|n$ denotes the truncation of $\mathbf{i} \in \Sigma$ to a word of length $n$ (i.e., we scale the image of the isometry by the constant $K$).*

This condition is used only for upper bounds on the Hausdorff dimension.

***Self-affine transversality condition:*** *There is a constant $C > 0$ (independent of $\mathbf{i}, \mathbf{j}$) such that for all $\mathbf{i}, \mathbf{j} \in \Omega$, $\mathbf{i} \neq \mathbf{j}$ we have*

$$\mathcal{M}\left\{u \in U : |\Pi^u(\mathbf{i}) - \Pi^u(\mathbf{j})| < \rho\right\} < C \cdot Z_{\mathbf{i} \wedge \mathbf{j}}(\rho). \tag{26}$$

With this definition, we can now state the following theorem.

**Theorem 6.** *Consider a contractive IFS of the form (1) and a compact metric space $U$ with a finite Borel measure $\mathcal{M}$. Let $\Pi$ and $\Omega$ be as in Definition 4. Let $\nu$ be an ergodic measure on $\Omega$. If the self-affine Hölder condition and self-affine transversality condition hold then for $\mathcal{M}$-almost all $u \in U$ we have:*

- **(a):** $\dim_{\mathrm{H}}\left(\Pi^u(\Omega)\right) = \min\left\{d, s_\Omega(A_1, \ldots, A_m)\right\}$*; and*
- **(b):** *If $s_\Omega(A_1, \ldots, A_m) > d$ then $\mathcal{L}\mathrm{eb}_d\left(\Pi^u(\Omega)\right) > 0$.*

Note that the self-affine Hölder condition (25) in the self-similar case reduces to the formulae in [18, p.542, formulae (10)]. Similarly the self-affine transversality condition (26) in the self-similar case reduces to the usual transversality condition (see e.g. [18, p.542, formulae (11)]).

We also have the following measure theoretic result.

**Theorem 7.** *Consider a contractive IFS of the form (1) and a compact metric space $U$ with a finite Borel measure $\mathcal{M}$. Let $\Pi$ and $\Omega$ be* *as in Definition 4. If the self-affine Hölder condition and self-affine transversality condition hold then for $\mathcal{M}$-almost all $u \in U$ we have:*

**(a):** $\dim_{\mathrm{H}} \Pi_*^u(\nu) = \min\{d, D(\nu)\}$; *and*

**(b):** *If* $h_\nu > -(\lambda_1 + \cdots + \lambda_d)$ *then*

$$\Pi_*^u(\nu) \ll \mathcal{L}\mathrm{eb}_d.$$

To get the lower bound in Theorems 6 and 7 we need the following Proposition.

**Proposition 2.** *Consider a contractive IFS of the form (1) and a compact metric space $U$ with a finite Borel measure $\mathcal{M}$. Let $\Pi$ and $\Omega$ be as in Definition 4. Assume that the self-affine transversality condition holds. Consider a Radon measure $\mu$ on $\Omega$ and a positive number $s$ such that there exists $c_1 > 0$ which satisfies that for all finite words $\omega \in \Sigma^*$,*

$$\mu([\omega]) \leq c_1 \phi^s(A_\omega). \tag{27}$$

*Then for $\mathcal{M}$-almost all $u \in U$ we have:*

**(a):** *If $s \leq d$ then* $\dim_{\mathrm{H}}(\Pi_*^u \mu) \geq s$.*; and*

**(b):** *If $s > d$ then* $\Pi_*^u(\mu) \ll \mathcal{L}\mathrm{eb}_d$.

In turn, to prove part (a) of Proposition 2 we need the following Lemma.

**Lemma 5.** *Consider a contractive IFS of the form (1) and a compact metric space $U$ with a finite Borel measure $\mathcal{M}$. Let $\Pi$ and $\Omega$ be as in Definition 4. Assume that the self-affine transversality condition holds. Then for every $t \notin \mathbb{N}$, $0 < t < d$ there exists a constant $c_2 = c_2(t)$ (independent of $\mathbf{i}, \mathbf{j}$) such that for all $\mathbf{i}, \mathbf{j} \in \Omega$, $\mathbf{i} \neq \mathbf{j}$ we have*

$$\int_{u \in U} |\Pi^u(\mathbf{i}) - \Pi^u(\mathbf{j})|^{-t} \, d\mathcal{M}(u) < c_2 \cdot \left(\phi^t(A_{\mathbf{i}\wedge\mathbf{j}})\right)^{-1}. \tag{28}$$

*Proof.* Fix $t > 0$ and $\mathbf{i}, \mathbf{j} \in \Omega$. Observe that we can write

$$\begin{aligned} &\int_{u \in U} |\Pi^u(\mathbf{i}) - \Pi^u(\mathbf{j})|^{-t} \, d\mathcal{M}(u) \\ &= t \int_{\rho=0}^{\infty} \mathcal{M}\left\{u \in U : |\Pi^u(\mathbf{i}) - \Pi^u(\mathbf{j})| < \rho\right\} \rho^{-t-1} d\rho. \end{aligned}$$

Thus using (28) it is enough to prove that there exists a constant such that if we write

$$\mathcal{I}^t_{\mathbf{i}\wedge\mathbf{j}} := \int_{\rho=0}^{\infty} Z_{\mathbf{i}\wedge\mathbf{j}}(\rho) \rho^{-t-1} d\rho \tag{29}$$

then

$$\mathcal{I}^t_{\mathbf{i}\wedge\mathbf{j}} \le \text{const} \cdot \left[\alpha_1(\mathbf{i}\wedge\mathbf{j}) \cdots \alpha_{k-1}(\mathbf{i}\wedge\mathbf{j}) \cdot \alpha_k^{t-(k-1)}(\mathbf{i}\wedge\mathbf{j})\right]^{-1}, \tag{30}$$

where

$$k-1 < t \le k.$$

Henceforth, for notational simplicity, we shall write $Z(\rho), \alpha_k$ and $\mathcal{I}$ in place of $Z_{\mathbf{i}\wedge\mathbf{j}}(\rho), \alpha_k(\mathbf{i}\wedge\mathbf{j})$ and $\mathcal{I}^t_{\mathbf{i}\wedge\mathbf{j}}$. We can write the integral in (29) as a sum of the integrals over the intervals $[0,\alpha_k]$ and $[\alpha_k,\infty]$. More precisely, if we write

$$\mathcal{I}_1 := \int\limits_{\rho=0}^{\alpha_k} Z(\rho)\rho^{-t-1}d\rho \text{ and } \mathcal{I}_2 := \int\limits_{\rho=\alpha_k}^{\infty} Z(\rho)\rho^{-t-1}d\rho,$$

then we have

$$\mathcal{I}^t_{\mathbf{i}\wedge\mathbf{j}} = \mathcal{I}_1 + \mathcal{I}_2. \tag{31}$$

First we prove $\mathcal{I}_1$ is bounded by the expression on the Right Hand Side of (30) and then we show the same holds for $\mathcal{I}_2$. The proof then follows. Next observe that

$$0 < \rho < \alpha_i \text{ implies } \frac{\rho^i}{\alpha_1\cdots\alpha_i} < \frac{\rho^{i-1}}{\alpha_1\cdots\alpha_{i-1}}.$$

Thus

$$\mathcal{I}_1 < \int\limits_{\rho=0}^{\alpha_k} \frac{\rho^k}{\alpha_1\cdots\alpha_k}\rho^{-t-1}d\rho = \frac{1}{k-t}\left[\alpha_1\cdots\alpha_{k-1}\alpha_k^{t-(k-1)}\right]^{-1} \tag{32}$$

which verifies the required inequality with the constant $\frac{1}{k-t}$. We now bound

$$\mathcal{I}_2 = \sum_{\ell=0}^{k-1}\int\limits_{J_\ell} Z(\rho)\rho^{-t-1}d\rho. \tag{33}$$

Observe that

$$\int\limits_{J_\ell} Z(\rho)\rho^{-t-1}d\rho = \frac{1}{t-\ell}\left(\alpha_1\cdots\alpha_\ell\right)^{-1}\left[\alpha_{\ell+1}^{\ell-t} - \alpha_\ell^{\ell-t}\right],$$

where for $\ell=0$ we set $\alpha_0^{-t}=0$. Using that

$$\frac{1}{t-(\ell+1)}\left(\alpha_1\cdots\alpha_{\ell+1}\right)^{-1}\alpha_{\ell+1}^{\ell+1-t} > \frac{1}{t-\ell}\left(\alpha_1\cdots\alpha_\ell\right)^{-1}\alpha_{\ell+1}^{\ell-t}$$

one can see that the number which we add in the sum (33) for each $\ell$ is smaller than that which we subtract for $\ell+1$. Therefore

$$\mathcal{I}_2 < \frac{1}{t-(k-1)} \cdot \left(\alpha_1 \cdots \alpha_{k-1}\alpha_k^{t-(k-1)}\right)^{-1}.$$

This completes the proof of the required inequality for $\mathcal{I}_2$. It follows from this (31) and (32) that (30) holds for $\mathcal{I}$ with the constant $\frac{1}{t-(k-1)} + \frac{1}{k-t}$. □

We now use Lemma 5 to prove Proposition 2. This proposition is important in the proofs of all of the main results in this paper.

*The proof of Proposition 2.* We first prove part (a). Assume that $s < d$. It follows from the potential theoretic characterization of the Hausdorff dimension (see e.g. [5, Theorem 4.13]) that for every measure $\nu$ on $\mathbb{R}^d$ we have

$$\dim_{\mathrm{H}} \nu \geq \sup\left\{\alpha : \iint |x-y|^{-\alpha} d\nu(x) d\nu(y) < \infty\right\}. \tag{34}$$

Fix $t < s$. It suffices to prove that

$$\mathcal{I}^t := \int\limits_{u\in U} \iint\limits_{(\mathbf{i},\mathbf{j})\in\Omega\times\Omega} |\Pi^u(\mathbf{i}) - \Pi^u(\mathbf{j})|^{-t}\, d\mu(\mathbf{i}) d\mu(\mathbf{j}) d\mathcal{M}(u) < \infty.$$

since this, together with (34), implies that $\dim_{\mathrm{H}}(\Pi^u_* \mu) \geq t$ for $\mathcal{M}$-almost all $u \in U$. Using (27) and Lemma 5 we obtain

$$\begin{aligned} \mathcal{I}^t &< c_2 \sum_{k=0}^{\infty} \sum_{|\omega|=k} \iint\limits_{\mathbf{i}\wedge\mathbf{j}=\omega} \left(\phi^t(A_\omega)\right)^{-1} d\mu(\mathbf{i}) d\mu(\mathbf{j}) \\ &\leq c_2 c_1 \sum_{k=0}^{\infty} \sum_{|\omega|=k} \mu([\omega]) \phi^s(A_\omega) \left(\phi^t(A_\omega)\right)^{-1}. \end{aligned} \tag{35}$$

It follows from (14) and (15) that

$$\phi^s(A_\omega) \cdot \left(\phi^t(A_\omega)\right)^{-1} \leq b_{\mathcal{F}}^{k\cdot(s-t)}.$$

Using this and (35) we obtain that

$$\mathcal{I}^t \leq c_1 c_2 \sum_{k=0}^{\infty} b_{\mathcal{F}}^{k\cdot(s-t)} \underbrace{\sum_{|\omega|=k} \mu([\omega])}_{=1} < \infty.$$

Since $t$ can be chosen arbitrarily close to $s$ this completes the proof of part (a).

We next prove part (b). It follows from the definition (24) of $Z_{\mathbf{i}\wedge\mathbf{j}}(\rho)$ that for distinct $\mathbf{i},\mathbf{j}\in\Omega$ and $\rho>0$ we have the bound

$$Z_{\mathbf{i}\wedge\mathbf{j}}(\rho)\le\frac{\rho^d}{\phi^d(A_{\mathbf{i}\wedge\mathbf{j}})}.$$

Thus it follows from the self-affine transversality condition (26) that for all distinct $\mathbf{i},\mathbf{j}\in\Omega$ and $\rho>0$,

$$\mathcal{M}\left\{u\in U:|\Pi^u(\mathbf{i})-\Pi^u(\mathbf{j})|<\rho\right\}<C\cdot\frac{\rho^d}{\phi^d(A_{\mathbf{i}\wedge\mathbf{j}})}. \tag{36}$$

Furthermore, from (16) we obtain that for $\omega\in\Sigma^*$

$$\frac{\mu([\omega])}{\phi^d(A_\omega)}\le c_1\cdot\frac{\phi^s(A_\omega)}{\phi^d(A_\omega)}\le b_{\mathcal{F}}^{|\omega|(s-d)}. \tag{37}$$

To show absolute continuity of $\Pi^u_*\mu$ for $\mathcal{M}$ almost all $u$ we will follow a standard approach (introduced by Peres and Solomyak in [14]). In particular, it suffices to show that

$$I:=\iint\liminf_{r\to0}\frac{\Pi^u_*\mu(B(x,r))}{2r^d}\mathrm{d}\Pi^u_*\mu(x)\mathrm{d}\mathcal{M}(u)<\infty.$$

We apply Fatou's Lemma, lift to the shift space, and apply Fubini's Theorem. Finally, using inequality (36) we have that

$$\begin{aligned}I&\le\liminf_{r\to0}\frac{1}{2r^d}\iiint\mathbb{1}_{\{(\mathbf{i},\mathbf{j}):|\Pi^u(\mathbf{i})-\Pi^u(\mathbf{j})|\le r\}}\mathrm{d}\mu(\mathbf{i})\mathrm{d}\mu(\mathbf{j})\mathrm{d}\mathcal{M}(u)\\&\le\liminf_{r\to0}\frac{1}{2r^d}\iint\mathcal{M}\{u:|\Pi^u(\mathbf{i})-\Pi^u(\mathbf{j})|\le r\}\mathrm{d}\mu(\mathbf{i})\mathrm{d}\mu(\mathbf{j})\\&\le C\iint\frac{1}{\phi^d(A_{\mathbf{i}\wedge\mathbf{j}})}\mathrm{d}\mu(\mathbf{i})\mathrm{d}\mu(\mathbf{j}).\end{aligned}$$

To complete the proof we split the integral up into an infinite summation and use (37) to get the bound

$$\begin{aligned}I&\le C\sum_{k=0}^{\infty}\sum_{|\overline{\omega}|=k}\mu[\overline{\omega}]^2(\phi_d(A_{[\overline{\omega}]}))^{-1}\\&\le c_1C+C\sum_{k=0}^{\infty}b_{\mathcal{F}}^{k(s-d)}\sum_{|\overline{\omega}|=k}\mu(\overline{\omega})<\infty.\end{aligned}$$

This suffices to show the result (cf. [14]).

□

## 5. The proof of the Theorems

In this section we will prove Theorems 6 and 7, and then deduce from these Theorems 2 and 3.

*The proof of Theorem 6.* We first prove part (a). The upper bound follows immediately from the definition of the singular value function. To get the lower bound we fix an arbitrary $s < \min\{d, s_\Omega(A_1,\dots,A_m)\}$. In particular, $\mathcal{N}^s(\Omega) = \infty$ by (19). It then follows from Lemma 3 that we can find a measure $\mu$ which is supported on $\Omega$ satisfying (27), i.e.,

$$\mu([\omega]) \le c_1 \phi^s(A_\omega).$$

In particular, we now can apply Proposition 2 to deduce that for $\mathcal{M}$ almost all $u$ we have $\dim_{\mathrm{H}} \Pi^u(\Omega) \ge s$. This completes the proof of part (a) of the theorem.

Next we prove part (b). Since we are assuming that $d < s_\Omega(A_1,\dots,A_m)$ we can choose

$$d < s < s_\Omega(A_1,\dots,A_m).$$

Using definition (19) of $s_\Omega(A_1,\dots,A_m)$ it follows from Lemma 3 that there exists a measure $\mu$ supported by $\Omega$ and a constant $c_0 > 0$ such that for every $\omega \in \Sigma^*$ we have

$$\mu([\omega]) \le c_0 \cdot \phi^s(A_\omega).$$

Thus we can apply part (b) of Proposition 2 to deduce that for $\mathcal{M}$ a.e. $u \in U$ we have $\Pi^u_*(\mu) \ll \mathcal{L}\mathrm{eb}_d$ and thus $\mathcal{L}\mathrm{eb}_d\left(\Pi^u_*(\Omega)\right) > 0$. □

*Proof of Theorem 7.* We first prove part (b). It is apparent from the definitions of $Z_{\mathbf{i}\wedge\mathbf{j}}(\rho)$ and $\phi^d$ that

$$Z_{\mathbf{i}\wedge\mathbf{j}}(\rho) \le \frac{\rho^d}{\phi^d(A_{\mathbf{i}\wedge\mathbf{j}})}$$

for distinct $\mathbf{i}, \mathbf{j}$ (cf. Figure 2). Since we are assuming the self-affine transversality condition (26), there exists $C > 0$ such that

$$\mathcal{M}\{u \in U : |\Pi^u(\mathbf{i}) - \Pi^u(\mathbf{j})| \le \rho\} \le C \frac{\rho^d}{\phi^d(A_{\mathbf{i}\wedge\mathbf{j}})}.$$

Recall that by the Shannon-McMillan-Breiman Theorem we have that for $\nu$ almost all $\mathbf{i} \in \Omega$

$$\lim_{n\to\infty} \frac{1}{n} \log \nu([i_0,\dots,i_{n-1}]) = -h(\nu) \tag{38}$$

and that we have already shown that for $\nu$ almost all $\mathbf{i} \in \Omega$

$$\lim_{n\to\infty} \frac{1}{n} \log \alpha_k(A_{\mathbf{i}|n}) = \lambda_k(\nu),$$

which implies that for $\nu$ almost all $\mathbf{i} \in \Omega$

$$\lim_{n\to\infty} \sqrt[n]{\alpha_1(A_{\mathbf{i}|n}) \cdots \alpha_d(A_{\mathbf{i}|n})} = e^{\lambda_1(\nu)+\cdots+\lambda_d(\nu)}. \tag{39}$$

To show that $\nu_u := \Pi^u_* \nu$ is absolutely continuous it suffices to show that for an arbitrary $\varepsilon > 0$ we can find a measure $\nu_\varepsilon$ which is the restriction of $\nu$ to a set with measure greater than $1 - \varepsilon$ and for which $\Pi^u_* \nu_\varepsilon$ is absolutely continuous. For any $\varepsilon > 0$ we have that by Egorov's Theorem there exists a set $X_\varepsilon \subset \Omega$ such that $\nu(X_\varepsilon) > 1 - \varepsilon$ and the convergence in both (38) and (39) is uniform. We let $\nu_\varepsilon$ be $\nu$ restricted to this set. Choose $\delta > 0$ such that

$$h_\nu - \delta > -(\lambda_1 + \cdots + \lambda_d) + \delta \tag{40}$$

(where we write $\lambda_i$ for $\lambda_i(\nu)$ for convenience). By the uniform convergence in (38) and (39) we can choose $N \in \mathbb{N}$ such that for $n \geq N$ and $\mathbf{i} \in X_\varepsilon$

$$\nu([\mathbf{i}|n]) \leq e^{n(-h_\nu+\delta)} \text{ and } \phi^d(A_{\mathbf{i}|n}) \geq e^{n(\lambda_1+\cdots+\lambda_d-\delta)}.$$

It is now enough to check the absolute continuity of the measure $\Pi^u_* \nu_\varepsilon$. To see this we observe that by (40) there exists $d < s$ and a constant $c_1$ such that for every $\omega \in \Sigma^*$ we have $\nu_\varepsilon([\omega]) < c_1 \phi^s(A_\omega)$. Using Proposition 2 we obtain that $\Pi^u_* \nu_\varepsilon \ll \mathcal{L}\mathrm{eb}_d$. Since this holds for all $\varepsilon > 0$ this completes the proof of part (b).

We next turn to the proof of part (a). We can assume that $D(\nu) < d$. Let $k := k(\nu)$ be defined as in (13). In particular, $k < D(\nu) \leq k + 1$ and we can choose $k < s < D(\nu)$. Let $\varepsilon > 0$ satisfy

$$D(\nu) - s = \frac{2\varepsilon}{-\lambda_{k+1}}.$$

Then it follows from Lemma 4 that

$$E_\nu(s) > -h_\nu + 2\varepsilon.$$

Using Egorov's Theorem it follows from (20), the definition of $E_\nu(s)$, (17) and the Shannon-McMillan-Breiman Theorem that for every $\delta > 0$ there exists a set $H_\delta \subset \Omega$ with $\nu(H_\delta) > 1 - \delta$ and there exists an $N$ such that for all $n \geq N$ and for all $\mathbf{i} \in H_\delta$ we have

$$\nu(\mathbf{i}|n) \leq \exp(n(-h_\nu + \varepsilon)) < \exp(nE_\nu(s)) < \phi^s(A_{\mathbf{i}|n}).$$

In this way we see there is a $c > 0$ such that for all $\mathbf{i} \in H_\delta$ we have

$$\nu(\mathbf{i}|n) < c\phi^s(A_{\mathbf{i}|n}).$$

Let $\nu_\delta := \nu|_{H_\delta}$. Then by Proposition 2 we obtain that for $\mathcal{M}$ a.e. $u \in U$ we have

$$\dim_{\mathrm{H}}(\Pi_*^u \nu) \geq \dim_{\mathrm{H}}(\Pi_*^u \nu_\delta) \geq s.$$

Since $s < D(\nu)$ was arbitrary this gives the lower estimate in part (a) of the theorem.

To get the upper bound in part(a) assume that $D(\nu) < d$. We fix an arbitrary $u \in U$ and then we can prove the upper estimate for $\Lambda := \Pi^u(\Omega)$. Since $u$ is fixed, we write $\Pi$ instead of $\Pi^u$ to simplify the notation. Given any $\gamma > D(\nu)$, it is enough to prove that

$$\dim_{\mathrm{H}}(\Pi_* \nu) \leq \gamma.$$

We choose $\varepsilon > 0$ to satisfy

$$\gamma > \frac{h_\nu + \lambda_1 + \cdots \lambda_k}{-\lambda_{k+1} + \varepsilon} + k + \frac{(k+1)\varepsilon}{-\lambda_{k+1} + \varepsilon}. \tag{41}$$

By the self-affine Hölder condition (25) there is a $K > 0$ such that for every $\mathbf{i} \in \Omega$ and $n \in \mathbb{N}$ the set $\Pi(\mathbf{i}|n)$ can be covered by a rectangular box $B_{\mathbf{i}|n}$ with sides $K \cdot \alpha_1(\mathbf{i}|n), \ldots, K \cdot \alpha_d(\mathbf{i}|n)$. Without loss of generality, let us assume that $K = 1$. Let $k := k(\nu)$ be as in (13). For every $\mathbf{i} \in \Omega$ and $n \in \mathbb{N}$ we fix a subdivision of the box $B_{\mathbf{i}|n}$ into

$$N(\mathbf{i}|n) := \frac{\alpha_1(\mathbf{i}|n) \cdots \alpha_k(\mathbf{i}|n)}{\alpha_{k+1}^k(\mathbf{i}|n)}$$

boxes of sides

$$\ell(\mathbf{i}|n) := \underbrace{\alpha_{k+1}(\mathbf{i}|n), \ldots, \alpha_{k+1}(\mathbf{i}|n)}_{k+1}, \alpha_{k+2}(\mathbf{i}|n), \ldots, \alpha_d(\mathbf{i}|n).$$

Let $P_n(\mathbf{i})$ denote the box which contains $\Pi(\mathbf{i})$. We denote by $Q_n(\mathbf{i})$ the set in $\Omega$ which corresponds to $P_n(\mathbf{i})$. That is

$$Q_n(\mathbf{i}) := \{\mathbf{j} \in \Omega : \mathbf{j} \in [\mathbf{i}|n] \text{ and } \Pi(\mathbf{j}) \in P_n(\mathbf{i})\},$$

where $[\mathbf{i}|n]$ denotes the $n$th level cylinder. In general $Q_n(\mathbf{i}) \subsetneqq \Pi^{-1}(P_n(\mathbf{i}))$ for $\Pi$ is not $1-1$. Furthermore, let

$$A_\varepsilon^n := \left\{ \mathbf{i} \in \Omega : \nu(Q_n(\mathbf{i})) \geq \varepsilon \cdot \frac{\nu(\mathbf{i}|n)}{N(\mathbf{i}|n)} \right\}.$$

Observe that for every $n$ we have

$$\nu\left((A_\varepsilon^n)^c\right) \leq \varepsilon. \tag{42}$$

In particular,

$$\nu\left((A_\varepsilon^n)^c\right) = \sum_{|\omega|=n} \nu\left([\omega] \cap (A_\varepsilon^n)^c\right) \leq \sum_{|\omega|=n} N(\omega) \cdot \varepsilon \cdot \frac{\nu([\omega])}{N(\omega)} \leq \varepsilon.$$

Using (42) we see that the set

$$A_\varepsilon := \limsup_{n\to\infty} A_\varepsilon^n = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_\varepsilon^n$$

satisfies

$$\nu\left(A_\varepsilon\right) > 1-\varepsilon.$$

It follows from Egorov's Theorem that there exists a $G_\varepsilon \subset A_\varepsilon$ such that $\nu(G_\varepsilon) > 1-2\varepsilon$ and we have that

$$\frac{1}{n}\log\nu(\mathbf{i}|n) \to -h_\nu \text{ and } \frac{1}{n}\log\alpha_\ell(\mathbf{i}|n) \to \lambda_\ell(\nu),$$

where the convergence if uniform for all $1 \le \ell \le d$ and all $\mathbf{i} \in G_\varepsilon$. Thus, for every $\mathbf{i} \in G_\varepsilon$ we can find arbitrary large $n$ simultaneously satisfying the following four conditions:

**(i):** $\nu\left(Q_n(\mathbf{i})\right) > \varepsilon\frac{\nu(\mathbf{i}|n)}{N(\mathbf{i}|n)}$;
**(ii):** $e^{n(-h_\nu-\varepsilon)} < \nu(\mathbf{i}|n) < e^{n(-h_\nu+\varepsilon)}$;
**(iii):** $e^{n(\lambda_\ell-\varepsilon)} < \alpha_\ell(\mathbf{i}|n) < e^{n(\lambda_\ell+\varepsilon)}$ for every $1 \le \ell \le d$;
**(iv):** $\frac{\log\varepsilon}{\log\alpha_{k+1}(\mathbf{i}|n)} < \varepsilon$.

Fix $\mathbf{i} \in G_\varepsilon$ and choose $n$ satisfying (i)-(iv) above. We want to apply Frostman's Lemma to the measure $\Pi_*\nu$, and to this end we need to estimate

$$R_n(\mathbf{i}) := \frac{\log\Pi_*\nu\left[B\left(\Pi(\mathbf{i}),\alpha_{k+1}(\mathbf{i}|n)\right)\right]}{\log\alpha_{k+1}(\mathbf{i}|n)}.$$

In particular,

$$\begin{aligned} R_n(\mathbf{i}) &\le \frac{\log\nu(Q_n(\mathbf{i}))}{\log\alpha_{k+1}(\mathbf{i}|n)} \le \frac{\log\varepsilon}{\log\alpha_{k+1}(\mathbf{i}|n)} + \frac{\log\nu(\mathbf{i}|n)-\log N(\mathbf{i}|n)}{\log\alpha_{k+1}(\mathbf{i}|n)} \\ &\le \varepsilon + \frac{\log N(\mathbf{i}|n)-\log\nu(\mathbf{i}|n)}{-\log\alpha_{k+1}(\mathbf{i}|n)} \\ &\le \varepsilon + \frac{n\left(\lambda_1+\cdots\lambda_k+k\varepsilon-k\lambda_{k+1}+k\varepsilon\right)+n(h_\nu+\varepsilon)}{-n(\lambda_{k+1}-\varepsilon)} \\ &= \frac{h_\nu+\lambda_1+\cdots+\lambda_k}{-\lambda_{k+1}+\varepsilon} + k + \frac{(k+1)\varepsilon}{-\lambda_{k+1}+\varepsilon}. \end{aligned}$$

For $\varepsilon$ satisfying (41) we can find $n$ arbitrarily large such that

$$R_n(\mathbf{i}) < \gamma.$$

From this we deduce that for every $\mathbf{i} \in G_\varepsilon$ we can bound

$$\liminf_{n\to\infty}\frac{\log\Pi_*\nu\left[B\left(\Pi(\mathbf{i}),\alpha_{k+1}(\mathbf{i}|n)\right)\right]}{\log\alpha_{k+1}(\mathbf{i}|n)} < \gamma. \tag{43}$$

By construction, $G_{\varepsilon_1} \supset G_{\varepsilon_2}$ whenever $\varepsilon_1 < \varepsilon_2$ and thus we obtain that (43) holds for $\nu$-a.e. $\mathbf{i} \in \Omega$. Frostman's Lemma ([3], Theorem 3.3.14) now implies that, $\dim_{\mathrm{H}} \nu \le \gamma$. This completes the proof of the upper estimate since $\gamma > D(\nu)$ was chosen arbitrarily.

□

In order to apply Theorems 6 and 7 to deduce Theorems 2 and 3, we need to show that the self-affine transversality condition holds. This is the purpose of the next Lemma.

**Lemma 6.** *The self-affine transversality condition (26) holds for* $\Pi^{\mathbf{y}}$*,* $\mathbf{y} \in D^{\infty}$*, i.e., there exists a constant* $C > 0$ *such that for all distinct* $\mathbf{i}, \mathbf{j} \in \Sigma$ *we have*

$$\mathbb{P}\left\{\mathbf{y} \in D^{\infty} : |\Pi^{\mathbf{y}}(\mathbf{i}) - \Pi^{\mathbf{y}}(\mathbf{j})| < \rho\right\} < C \cdot Z_{\mathbf{i} \wedge \mathbf{j}}(\rho). \tag{44}$$

*Proof.* Fix $\mathbf{y} \in D^{\infty}$ and consider the natural projection $\Pi^{\mathbf{y}} : \Sigma \to \Lambda^{\mathbf{y}}$ defined in (8). It follows from (9) that for every $\mathbf{i}, \mathbf{j} \in \Sigma$ with $|\mathbf{i} \wedge \mathbf{j}| = n$ there exists a random variable $q_n(\mathbf{i}, \mathbf{j}, \mathbf{y})$ which is *independent* of $y_{i_0 \dots i_n}$ such that

$$|\Pi^{\mathbf{y}}(\mathbf{i}) - \Pi^{\mathbf{y}}(\mathbf{j})| = \left|A_{\mathbf{i} \wedge \mathbf{j}}\left[y_{i_0 \dots i_n} + q_n(\mathbf{i}, \mathbf{j}, \mathbf{y})\right]\right|. \tag{45}$$

Let us fix all of the terms in $\mathbf{y}$ except $y_{i_0 \dots i_n} \in D$. Using the product structure of $\mathbb{P}$ and the fact that the measure $\eta$ is absolutely continuous with respect to $\mathcal{L}\mathrm{eb}_d$ with bounded density we see that in order to verify (44) it is enough to prove that

$$\mathcal{L}\mathrm{eb}_d\left\{y_{i_0 \dots i_n} \in D : |\Pi^{\mathbf{y}}(\mathbf{i}) - \Pi^{\mathbf{y}}(\mathbf{j})| < \rho\right\} < C \cdot Z_{\mathbf{i} \wedge \mathbf{j}}(\rho).$$

Let us write $\mathrm{Box}_\rho := [-\rho, \rho]^d$ and let $r$ be the radius of the ball $D$. Let $\varphi$ be a rotation of $\mathbb{R}^d$ which sends the coordinate axes to the mutually orthogonal singular vectors of $A_{\mathbf{i} \wedge \mathbf{j}}^{-1}$ and let $\mathrm{Box}_{\mathbf{i} \wedge \mathbf{j}} := \varphi([-r, r]^d)$. It is clear that for all distinct $\mathbf{i}, \mathbf{j}$ we have $D \subset \mathrm{Box}_{\mathbf{i} \wedge \mathbf{j}}$. Then using (45) we get that

$$\mathcal{L}\mathrm{eb}_d\left\{y_{i_0 \dots i_n} \in D : |\Pi^{\mathbf{y}}(\mathbf{i}) - \Pi^{\mathbf{y}}(\mathbf{j})| < \rho\right\} \le \mathcal{L}\mathrm{eb}_d\left\{A_{\mathbf{i} \wedge \mathbf{j}}^{-1}\left(\mathrm{Box}_\rho\right) \cap \mathrm{Box}_{\mathbf{i} \wedge \mathbf{j}}\right\}.$$

Finally, it follows from elementary geometry that

$$\mathcal{L}\mathrm{eb}_d\left\{A_{\mathbf{i} \wedge \mathbf{j}}^{-1}\left(\mathrm{Box}_\rho\right) \cap \mathrm{Box}_{\mathbf{i} \wedge \mathbf{j}}\right\} \le C \cdot Z_{\mathbf{i} \wedge \mathbf{j}}(\rho),$$

which completes the proof of the Lemma. □

We are now in a position to complete the proofs of Theorems 2 and 3.

*Proof of Theorem 2.* Theorem 2 is an immediate consequence of Lemma 6 and Theorem 6. □

*Proof of Theorem 3.* Theorem 3 is an immediate consequence of Lemma 6 and Theorem 7. □

In order to deduce Theorem 4, we need the following Lemma. This is an adaptation of related results proved by Falconer [4] and Solomyak [18].

**Lemma 7.** *Consider a contractive self-affine IFS $\mathcal{F}$ of the form (1). Let $U$ be an arbitrary ball in $\mathbb{R}^{md}$ and let $\mathcal{M}$ be the measure $\mathcal{L}\mathrm{eb}_{md}$ restricted to $U$. Given $\mathbf{u} = (u_1, \ldots, u_m) \in U$ we define*

$$\Pi^u(\mathbf{i}) := \lim_{n\to\infty} (f_{i_0} + u_{i_0}) \circ \cdots \circ (f_{i_{n-1}} + u_{i_{n-1}})(0).$$

*Then the self-affine transversality condition (26) holds.*

*Proof.* Let $T$ be a $d \times d$ matrix with singular values $0 < \alpha_d \leq \ldots \leq \alpha_1 < 1$. We can bound

$$\mathcal{L}\mathrm{eb}\{x \in B(0,\delta) : |Tx| \leq \rho\} \leq C_0 \prod_{i=1}^{d} \min\left\{\frac{\rho}{\alpha_i}, \delta\right\} \tag{46}$$

for some constant $C_0 > 0$. This is easily seen by observing that $T^{-1}B(0,\rho)$ is contained in a box with sides of length $\frac{2r}{\alpha_i}$ aligned in the direction of the singular vector associated with $\alpha_i$. Fix distinct $\mathbf{i}, \mathbf{j} \in \Sigma$ and let

$$L := \mathcal{L}\mathrm{eb}_{md}\{\|\mathbf{u}\| \leq \delta : |\Pi^{\mathbf{u}}(\mathbf{i}) - \Pi^{\mathbf{u}}(\mathbf{i})| \leq \rho\}.$$

Let $k = |\mathbf{i} \wedge \mathbf{j}|$ and then we can write

$$\begin{aligned} |\Pi^{\mathbf{u}}(\mathbf{i}) - \Pi^{\mathbf{u}}(\mathbf{j})| &= A_{\mathbf{i}\wedge\mathbf{j}}(u_{i_{k+1}} - u_{j_{k+1}} + (f_{i_{k+1}} u_{i_{k+2}} + f_{i_{k+1}} f_{i_{k+2}} u_{i_{k+3}} + \cdots) \\ &- (f_{j_{k+1}} u_{j_{k+2}} + f_{j_{k+1}} f_{j_{k+2}} u_{j_{k+3}} + \cdots)) \\ &= A_{\mathbf{i}\wedge\mathbf{j}}(u_{i_{k+1}} - u_{j_{k+1}} + E(\mathbf{u})). \end{aligned}$$

It follows from [18, p. 540] that provided $\|A_i\| < \frac{1}{2}$ for all $1 \leq i \leq m$ then $\|E\| < 1$. Thus as in [4] the linear transformation from $\mathbb{R}^{md} \to \mathbb{R}^{md}$ defined by

$$(u_1, u_2 \ldots, u_m) \to (y, u_2 \ldots, u_m),$$

where $y = u_{i_{k+1}} - u_{j_{k+1}} + E(\mathbf{u})$ will be invertible. Thus there exists a constant $C_1 > 0$ such that

$$L \leq C_1 \mathcal{L}\mathrm{eb}_d\{(y, u_2, \ldots, u_d) \in B : \|f_{\mathbf{i}\wedge\mathbf{j}}(y)\| \leq \rho\}$$

where $B$ is the product of the interval $[-(2+m)\delta, (2+m)\delta]$ in the $y$ direction and the $d-1$ dimensional $\delta$-ball. It now follows from (46) that there exists $C_2 = C_2(\delta)$ and $C_3 = C_3(\delta)$ such that

$$L \leq C_2 \prod_{i=1}^{d} \min\left\{\frac{\rho}{\alpha_i}, \delta\right\} \leq C_3 Z_{\mathbf{i}\wedge\mathbf{j}}(\rho).$$

□

Finally, we can use Lemma 7 to prove Theorem 4.

*Proof of Theorem 4.* Theorem 4 immediately follows from Theorem 7, Theorem 6 and Lemma 7. □

## 6. An application in fractal image compression

In this section we will compute the almost sure Hausdorff dimension of certain randomly perturbed graph directed IFS which appear in the theory of fractal image compression.

The following graph directed Iterated Function System (GIFS) arose naturally in the field of fractal image compression. Given a large natural number $K$ we partition the unit square $Q$ into $M := 2^K \times 2^K$ subsquares of size $2^{-K}$ which we denote by $R_1, \ldots, R_M$. This is the family of *range squares*. We define the family of *domain squares* as the set of all $N := (2^K - 1) \times (2^K - 1)$ squares $D_1, \ldots, D_N$ which are the union of four range squares having a common vertex. We associate a domain square to every range square. More precisely, we are given a map

$$\varphi : \{1, \ldots, M\} \to \{1, \ldots, N\}. \tag{47}$$

This defines a directed graph $\Gamma = (\mathcal{V}, \mathcal{E})$ as follows The vertices are $\mathcal{V} = \{1, \ldots, M\}$. There is a directed edge $(k, \ell) \in \mathcal{E}$ from $k \in \mathcal{V}$ to $\ell \in \mathcal{V}$ if $R_\ell$ is one of the four squares of $D_{\varphi(k)}$. In this way $\Gamma$ is a directed graph with exactly four edges going out of every vertex.

We also assume that for each $1 \leq k \leq M$ and $\ell = \varphi(k)$ we have associated a self-affine map $S_k : (D_\ell \times [0,1]) \to (R_k \times [0,1])$ of the form

$$S_k(\mathbf{x}, z) := (T_k(\mathbf{x}), f_k(z)),$$

where $T_k : D_\ell \to R_k$ is onto, $f_k : [0,1] \to [0,1]$, and they satisfy

$$DT_k(\mathbf{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \text{ and } f_k(z) := \lambda_k \cdot z + t_k \tag{48}$$

where $0 < \lambda_k < 1$. See Figure 3

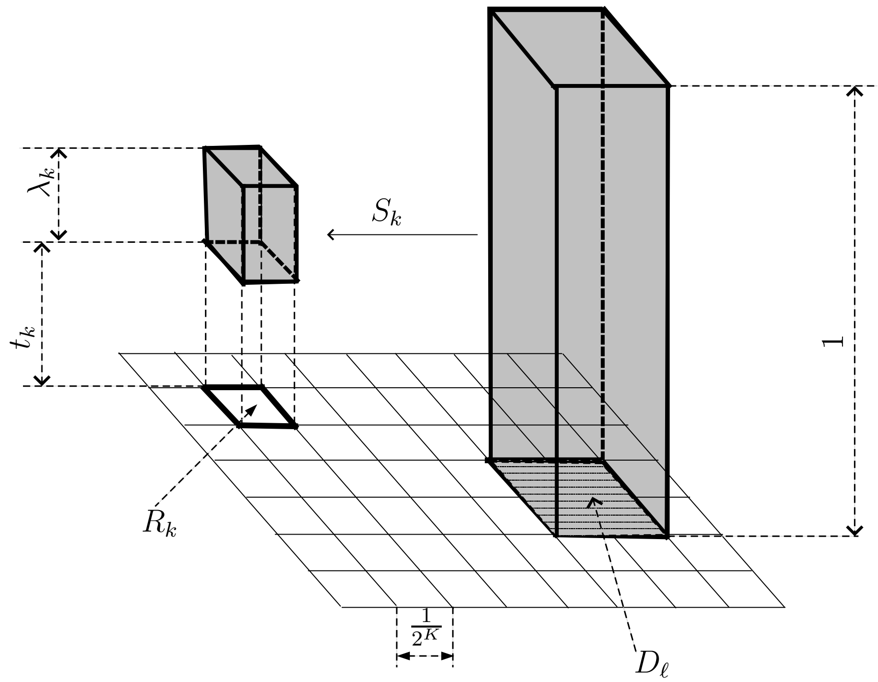


FIGURE 3. The function $S_k$

We define the graph directed sets $\Lambda_1, \ldots, \Lambda_M$ to be the unique family of compact sets satisfying

$$\Lambda_i = \bigcup_{(i,j)\in\mathcal{E}} S_i(\Lambda_j), \tag{49}$$

and the attractor of the GIFS is

$$\Lambda := \bigcup_{i=1}^{M} \Lambda_i. \tag{50}$$

Note that if we write $S_e := S_i$ whenever $(i, j) = e \in \mathcal{E}$ then (49) can be written as

$$\Lambda_i = \bigcup_{j=1}^{m} \bigcup_{e=(i,j)\in\mathcal{E}} S_e(\Lambda_j).$$

In particular, $\{S_i\}_{i=1}^{M}$ is a graph directed IFS in the sense of [12].

The attractor $\Lambda$ can also be written as

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{i_0\ldots i_{n-1}} S_{i_0\ldots i_{n-1}}(B),$$

where $B := [0,1]^3$ and the union is over admissible words $i_0, \ldots, i_{n-1}$. We suppose that at each application of the functions we make a random translational error in the vertical direction. More precisely, for every admissible $\mathbf{i}_n := (i_0, \ldots, i_{n-1})$ we consider

$$S_{\mathbf{i}_n}^{\mathbf{y}_{\mathbf{i}_n}} := \left(S_{i_0} + \overline{y}_{i_1}\right) \circ \cdots \circ \left(S_{i_{n-1}} + \overline{y}_{i_1 \ldots i_{n-1}}\right), \tag{51}$$

where $\overline{y}_{i_0 \ldots i_{n-1}} = (0, 0, y_{i_0 \ldots i_{n-1}}) \in \mathbb{R}^3$ and the numbers $y_{i_0 \ldots i_{n-1}}$ are chosen independently in every step, from an arbitrary small interval $I$ centered at the origin using an absolutely continuous distribution $\eta$ with bounded density. Gathering the random errors in an infinite vector as in (7) we can write,

$$\mathbf{y} := (y_1, \ldots, y_m, y_{11} \ldots, y_{m,m}, \ldots) \in I \times I \times \cdots =: I_\infty.$$

As in (10) we define the product measure on $I_\infty$ by

$$\mathbb{P} := \eta \times \eta \times \cdots. \tag{52}$$

It is our aim to compute the dimension of the attractor,

$$\Lambda_{\mathbf{y}} := \bigcap_{n=0}^{\infty} \bigcup_{\mathbf{i}_n} S_{\mathbf{i}_n}^{\mathbf{y}_{\mathbf{i}_n}}(B),$$

where $B = [0,1]^2 \times [-u, u] \subset \mathbb{R}^3$, where $u$ is sufficiently large, and the union is over admissible words $\mathbf{i}_n$, for $\mathbb{P}$ a.e perturbation $\mathbf{y}$.

**Theorem 8.** *Let $E$ be a $d \times d$ matrix such that $E_{ij} = \lambda_i$ if $(i,j) \in \mathcal{E}$ and $E_{ij} = 0$ otherwise. We denote by $\varrho(E)$ the spectral radius of the matrix $E$. Assume that the matrix $E$ is irreducible. Then for $\mathbb{P}$ a.e. $\mathbf{y} \in I_\infty$ we have*

$$\dim_{\mathrm{H}} \Lambda_{\mathbf{y}} = \max\left\{2, 1 + \frac{\log \varrho(E)}{\log 2}\right\}. \tag{53}$$

We remark that a related result for box dimension was obtained in [9] for almost all deterministic attractors of a similar type. The main difference between the GIFS considered here and the ones considered in [9] is that in [9] the authors considered deterministic GIFS with a restriction on the family of domain squares. This restriction allowed the use of a technique which cannot be used in this more general setting.

*Proof.* Let $\Omega := \left\{\mathbf{i} \in \Sigma : E_{i_k i_{k+1}} > 0\right\}$ and let $\Omega_n$ be the set of $n$ cylinders of $\Omega$.

*The upper estimate:* This is independent of the vertical translations. Let $\Lambda_n$ be the $n$-th approximation of the Cantor set $\Lambda$ consisting of $4^n$ boxes $B_{i_0 \ldots i_{n-1}}$, for $(i_0 \ldots i_{n-1}) \in \Omega_n$. Each of these has a face parallel to the $(x, y)$ plane which is a square of side $\frac{1}{2^n}$ and a vertical side of length

$2u\lambda_{i_0\dots i_{n-1}}$. Thus, for all $(i_0 \dots i_{n-1}) \in \Omega_n$ the box $B_{i_0\dots i_{n-1}}$ can be covered by $\frac{2u\lambda_{i_0\dots i_{n-1}}}{1/2^n}$ cubes of size $1/2^n$ where $\lambda_{i_0,\dots,i_{n-1}} = \lambda_{i_0} \cdots \lambda_{i_{n-1}}$. Therefore,

$$\mathcal{H}^s_{1/2^n}(\Lambda) \le \frac{2u}{2^{n(s-1)}} \left[ \underbrace{\sum_{i_0\dots i_{n-1}} \lambda^s_{i_0\dots i_{n-1}}}_{\|E^n\|_1} + \underbrace{\sum_{i_0\dots i_{n-1}} 1/2^n}_{2^n} \right] \tag{54}$$

where the summations are taken over all $(i_0 \dots i_{n-1}) \in \Omega_n$. This implies that whenever $\varrho(E) \le 2$ we have $\dim_{\mathrm{H}}(\Lambda) = 2$. If $\varrho(E) > 2$ it also follows immediately from (54) that $\dim_{\mathrm{H}}(\Lambda) \le 1 + \frac{\log \varrho(E)}{\log 2}$.

*The lower estimate:* In the case that $\varrho(E) \le 2$ we have already seen that $\dim_{\mathrm{H}}(\Lambda) = 2$. Therefore, for the rest of the proof we may assume that

$$\varrho(E) > 2.$$

If we could verify the self-affine transversality condition (26) then we would be able to apply Theorem 6 which would imply the required lower estimate. Unfortunately there is no way to check condition (26) for the GIFS under consideration. However, we will introduce another GIFS which is the same in the vertical direction and which is shrunk by a small amount in the direction of both the $x$ and $y$ axes. In this way there will be a gap in between any two shrunken range squares in the plane which will allows us to verify the self-affine transversality condition. Thus the Hausdorff dimension of its attractor gives a lower bound for the Hausdorff dimension of the original attractor.

Let us fix an arbitrary $0 < r < 1/2^K$. We use the same directed graph $\Gamma$ as above and let $\varphi$ be the same as in (47). For every $1 \le k \le M$ let $R^r_k$ be a square having the same center and sides parallel to the sides of $R_k$, but with sides of length $\frac{1}{2^K} - r$. Similarly, for $1 \le \ell \le N$ we define $D^r_\ell$ as the square having the same center as $D_\ell$ and sides of length $\frac{1}{2^{K-1}} - r$ parallel to the sides of $D_\ell$. For every $k$ and $\ell = \varphi(k)$ then we define the surjective affine map $S^r_k : D^r_\ell \to R^r_k$ of the form

$$S^r_k(\mathbf{x}, z) := (T^r_k(\mathbf{x}), f_k(z)),$$

where $f_k$ is defined as in (48) and $T^r_k : D^r_\ell \to R^\ell_k$ is surjective. Furthermore,

$$DT^r_k(\mathbf{x}) = \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix},$$

where

$$\beta = \beta(r) = \frac{1}{2} \cdot \frac{\frac{1}{M} - \frac{2}{r}}{\frac{1}{M} - \frac{1}{r}}.$$

Then as in (49) and in (50) there exists a unique family of compact nonempty sets $\Lambda_1^r, \ldots, \Lambda_m^r$ and $\Lambda^r$ satisfying

$$\Lambda_i^r = \bigcup_{(i,j)\in\mathcal{E}} S_j^r(\Lambda) \text{ and } \Lambda^r := \bigcup_{j=1}^{M} \Lambda_j^r.$$

Let us write $s_\Omega^{(r)}$ for the singularity dimension of this system (see (19)). To proceed we need the following result.

**Claim 1.** *If $\varrho(E) > 2$ and $0 < r < \frac{1}{2^K}$ is sufficiently small such that*

$$\frac{1}{\beta} < \varrho(E) \tag{55}$$

*then*

$$s_\Omega^{(r)} = 1 + \frac{\log \varrho(E)}{-\log \beta}.$$

*Proof of the Claim.* For $(i_0, \ldots, i_{n-1}) \in \Omega_n$ we have

$$DS^r_{i_0,\ldots,i_{n-1}} \equiv \begin{bmatrix} \beta^n & 0 & 0 \\ 0 & \beta^n & 0 \\ 0 & 0 & \lambda_{i_0\ldots i_{n-1}} \end{bmatrix}$$

Since the dimension of the attractor is larger than 2 it suffices to consider $s > 2$ for which we have that

$$\phi^s(DS^r_{i_0,\ldots,i_{n-1}}) = \max\left\{\lambda_{i_0\ldots i_{n-1}}\beta^{n(s-1)}, \lambda^{s-2}_{i_0\ldots i_{n-1}}\beta^{2n}\right\}.$$

Using that on the one hand

$$\sum_{i_0\ldots i_{n-1}} \lambda^{s-2}_{i_0\ldots i_{n-1}}\beta^{2n} < \sum_{i_0\ldots i_{n-1}} \beta^{2n} = (2\beta)^{2n} < 1$$

and on the other hand $\sum_{i_0\ldots i_{n-1}} \lambda_{i_0\ldots i_{n-1}} = \|E^n\|_1$ we obtain that

$$\beta^{n(s-1)}\|E^n\|_1 \leq \sum_{i_0\ldots i_{n-1}} \phi^s(DS^r_{i_0,\ldots,i_{n-1}}) \leq \beta^{n(s-1)}\|E^n\|_1 + 1,$$

which immediately implies that

$$\sum_n \sum_{i_0\ldots i_{n-1}} \phi^s(DS^r_{i_0,\ldots,i_{n-1}}) \begin{cases} < \infty, & \text{if } s > s_\Omega^{(r)}; \\ = \infty, & \text{if } s < s_\Omega^{(r)}. \end{cases}$$

Using (6) this completes the proof of the Claim. □

To complete the proof of the lower estimate in (53) the only thing we need to do is to verify the self-affine transversality condition 26 for the GIFS $\left\{S_j^r\right\}$. It would then follow from Theorem 6 that for $\mathbb{P}$ a.e. $\mathbf{y} \in I_\infty$ we have

$$\dim_{\mathrm{H}} \Lambda_{\mathbf{y}}^r = s_\Omega^{(r)}. \tag{56}$$

Since this holds for all sufficiently small $r > 0$ and since $s_\Omega = \lim_{r\to 0} s_\Omega^{(r)}$ this would complete the proof of the lower estimate.

Now we prove that the self-affine transversality condition holds for the GIFS $\left\{S_j^r\right\}$.

**Claim 2.** *For sufficiently small* $r > 0$ *there exists a constant,* $C > 0$, *such that for all distinct* $\mathbf{i}, \mathbf{j} \in \Omega$ *we have*

$$\mathbb{P}\left\{\mathbf{y} \in I_\infty : |\Pi_{\mathbf{y}}^r(\mathbf{i}) - \Pi_{\mathbf{y}}^r(\mathbf{j})| < \rho\right\} < C \cdot Z_{\mathbf{i}\wedge\mathbf{j}}(\rho).$$

*Where* $\Pi_{\mathbf{y}}^r : \Sigma \to \Lambda_{\mathbf{y}}^r$ *is the natural projection defined by:*

$$\Pi_{\mathbf{y}}^r(\mathbf{i}) := \lim_{n\to\infty} S_{\mathbf{i}_n}^{r,\mathbf{y}_{\mathbf{i}_n}}(0,0,0),$$

where

$$S_{\mathbf{i}_n}^{r,\mathbf{y}_{\mathbf{i}_n}} := \left(S_{i_0}^r + y_{i_0}\right) \circ \cdots \circ \left(S_{i_{n-1}}^r + \overline{y}_{i_0\dots i_{n-1}}\right). \tag{57}$$

*Proof.* We fix $0 < r < 1/2^K$ which is sufficiently small such that $\beta = \beta(r)$ satisfies (55). Henceforth we will not explicitly show the dependence of variables on $r$. For simplicity we write

$$A_{i_0,\dots,i_{n-1}} := DS_{i_0,\dots,i_{n-1}}^r \equiv \begin{bmatrix} \beta^n & 0 & 0 \\ 0 & \beta^n & 0 \\ 0 & 0 & \lambda_{i_0\dots i_{n-1}} \end{bmatrix}.$$

We can write the projection $\Pi_{\mathbf{y}}$ as

$$\Pi_{\mathbf{y}}(\mathbf{i}) = \widetilde{\Pi}(\mathbf{i}) + \widehat{\Pi}_{\mathbf{y}}(\mathbf{i}),$$

where

$$\widetilde{\Pi}(\mathbf{i}) := \mathbf{t}_{i_0} + \sum_{k=1}^{\infty} A_{i_0\dots i_{k-1}} \mathbf{t}_{i_k},$$

is the deterministic part and

$$\widehat{\Pi}_{\mathbf{y}}(\mathbf{i}) := \overline{y}_{i_0} + \sum_{k=1}^{\infty} A_{i_0\dots i_{k-1}} \overline{y}_{i_0\dots i_k},$$

is the random part. Furthermore, let $\mathbf{p}_{12}(\mathbf{i}) \in \mathbb{R}^2$ be the component of $\widetilde{\Pi}(\mathbf{i})$ in the $(x,y)$ plane and let $p_3(\mathbf{i})$ be the third component.

Given distinct $\mathbf{i}, \mathbf{j} \in \Omega$ set $\omega := \mathbf{i} \wedge \mathbf{j}$. Let $k := |\omega|$ and set

$$\gamma_k(\mathbf{i}, \mathbf{y}) := \lambda_{i_k} y_{i_0\dots i_{k+1}} + \lambda_{i_k i_{k+1}} y_{i_0\dots i_{k+2}} + \cdots.$$

Finally we denote

$$q_k(\mathbf{i},\mathbf{j},\mathbf{y}) := -y_{j_0\dots j_k} + \gamma_k(\mathbf{i},\mathbf{y}) - \gamma_k(\mathbf{j},\mathbf{y}) + p_3(\sigma^k\mathbf{i}) - p_3(\sigma^k\mathbf{j}),$$

which is independent of $y_{i_0\dots i_k}$. With this notation we can write

$$\begin{aligned}\Pi_{\mathbf{y}}(\mathbf{i}) - \Pi_{\mathbf{y}}(\mathbf{j}) &= A_{\overline{\omega}} \cdot \begin{bmatrix} \mathbf{p}_{12}(\sigma^k\mathbf{i}) - \mathbf{p}_{12}(\sigma^k\mathbf{j}) \\ y_{i_0\dots i_k} + q_k(\mathbf{i},\mathbf{j},\mathbf{y}). \end{bmatrix} \\ &= \begin{bmatrix} \beta^k\left(\mathbf{p}_{12}(\sigma^k\mathbf{i}) - \mathbf{p}_{12}(\sigma^k\mathbf{j})\right) \\ \lambda_{\overline{\omega}} y_{i_0\dots i_k} + \lambda_{\overline{\omega}} q_k(\mathbf{i},\mathbf{j},\mathbf{y}). \end{bmatrix}\end{aligned}$$

For all $\ell$ and for all $(n_1 \dots n_\ell) \in \Omega_\ell$ we fix $y_{n_1\dots n_\ell}$, with the exception of $y_{i_0\dots i_k}$, which is allowed to vary. For convenience, we denote our free variable $y_{i_0\dots i_k}$ by $y \in I$. Recall that the measure $\eta$ is absolutely continuous with bounded density. Thus it follows from (52) that it is sufficient to check that there exists a constant $c$ (independent of $\mathbf{i}$ and $\mathbf{j}$) such that for every $\rho > 0$ we have

$$\mathcal{L}\text{eb}\left\{ y \in I : \begin{bmatrix} \beta^k\left(\mathbf{p}_{12}(\sigma^k\mathbf{i}) - \mathbf{p}_{12}(\sigma^k\mathbf{j})\right) \\ \lambda_{\overline{\omega}} y_{i_0\dots i_k} + \lambda_{\overline{\omega}} q_k(\mathbf{i},\mathbf{j},\mathbf{y}). \end{bmatrix} \in \text{Box}_\rho \right\} < c \cdot Z_{\overline{\omega}}(\rho), \tag{58}$$

where $\text{Box}_\rho := [-\rho,\rho]^3$. Since we have assumed that $|\omega| = k$ the vectors $\mathbf{p}_{12}(\sigma^k(\mathbf{i}))$ and $\mathbf{p}_{12}(\sigma^k(\mathbf{j}))$ are in the same domain square, which we denote by $D^r_\ell$. However, these two vectors are in two different range squares $R^r_u$, $R^r_v$, say, contained in $D^r_\ell$. In particular, at least one of the two components of the vector $\mathbf{p}_{12}(\sigma^k\mathbf{i}) - \mathbf{p}_{12}(\sigma^k\mathbf{j})$ has absolute value greater than $2r$. Thus, whatever value $y = y_{i_0\dots i_k}$ takes we have

$$\text{if } \rho < 2r\beta^k \text{ then } \begin{bmatrix} \beta^k\left(\mathbf{p}_{12}(\sigma^k\mathbf{i}) - \mathbf{p}_{12}(\sigma^k\mathbf{j})\right) \\ \lambda_\omega y_{i_0\dots i_k} + \lambda_\omega q_k(\mathbf{i},\mathbf{j},\mathbf{y}). \end{bmatrix} \notin \text{Box}_\rho.$$

Hence, in this case (58) holds. Next we consider the case when $\rho > \beta^k$. If we additionally assume $\rho > \lambda_{\overline{\omega}}$ then $Z(\rho) = 1$ and (58) holds with $c = 1$. Therefore we may assume that

$$\beta^k < \rho < \lambda_\omega$$

holds. Then, by definition, $Z_{\omega(\rho)} = \frac{\rho}{\lambda_\omega}$. On the other hand, since $q_k(\mathbf{i},\mathbf{j},\mathbf{y})$ is independent of $y = y_{i_0\dots i_k}$ the Lebesgue measure of those $y$ for which the absolute value of the third component of the vector

$$\begin{bmatrix} \beta^k\left(\mathbf{p}_{12}(\sigma^k\mathbf{i}) - \mathbf{p}_{12}(\sigma^k\mathbf{j})\right) \\ \lambda_\omega y_{i_0\dots i_k} + \lambda_\omega q_k(\mathbf{i},\mathbf{j},\mathbf{y}). \end{bmatrix}$$

is smaller than $\rho$ is equal to $\frac{2\rho}{\lambda_\omega}$. Hence, (58) holds with constant $c = 2$. Finally, we assume that $2r\beta^k < \rho < \beta^k$. By an obvious case analysis one can see that for every $\omega$

$$\frac{Z_\omega(\beta^k)}{Z_\omega(2r\beta^k)} < \frac{1}{(2r)^3} =: c_1.$$

Using that the function $\rho \to Z_\omega(\rho)$ is monotone increasing we obtain that for all such $\rho$

$$\mathcal{L}\mathrm{eb}\left\{y \in I : \begin{bmatrix} \beta^k \left(\mathbf{p}_{12}(\sigma^k \mathbf{i}) - \mathbf{p}_{12}(\sigma^k \mathbf{j})\right) \\ \lambda_{\overline{\omega}} y_{i_0 \dots i_k} + \lambda_{\overline{\omega}} q_k(\mathbf{i}, \mathbf{j}, \mathbf{y}). \end{bmatrix} \in \mathrm{Box}_\rho \right\} < 2c_1 \cdot Z_{\overline{\omega}}(\rho).$$

Hence, (58) holds with constant $c = 2c_1$. This completes the proof of the Claim. □

We can now complete the proof of Theorem 8. Using Theorem 6 we obtain that for $\mathbb{P}$ a.e $\mathbf{y} \in I_\infty$ (56) holds. A simple computation shows that for every $r > 0$ and for every $\mathbf{y}$ we have $\dim_{\mathrm{H}} \Lambda_{\mathbf{y}} \geq \dim_{\mathrm{H}} \Lambda^r_{\mathbf{y}}$. This completes the proof of the Theorem. □

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