EXPLICIT METHODS FOR MODULAR FORMS AND L-FUNCTIONS

TIM DOKCHITSER
BRISTOL 30.6-1.7.2014

Big picture

Varieties $\rightsquigarrow$ Galois representations $\rightsquigarrow$ L-functions $\leftrightsquigarrow$ Automorphic forms

Special case

Elliptic curves $E/\mathbb{Q}$ $\rightsquigarrow$ Tate module $T_\ell E$ $\rightsquigarrow$ $L(E,s)$ $\leftrightsquigarrow$ newforms on $\Gamma_0(N)$

§ ζ- and L-functions

(1) Riemann $\displaystyle \zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}} = \prod_p \frac{1}{F_p(p^{-s})}$; $F_p(T) = 1-T$

degree $d=1$

Dedekind ζ, K num. field

$$\zeta_K(s) = \sum_{0\neq \mathfrak{a}\subseteq \mathcal{O}_K} \frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \frac{1}{1-N\mathfrak{p}^{-s}} = \prod_p \frac{1}{F_p(p^{-s})} \;;\; F_p(T) = \prod_{\mathfrak{p}|p} 1-T^{f_{\mathfrak{p}}}$$

degree $[K:\mathbb{Q}]$

(2) $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ character

$\displaystyle L(\chi,s) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}$; $F_p(T) = 1-\chi(p)T$ degree 1

(3) $E/\mathbb{Q}$ ell. curve

$$L(E,s) = \sum_{n\geq 1}\frac{a_n}{n^s} = \prod_p \frac{1}{F_p(p^{-s})} \;;\; F_p(T) = \begin{cases} 1-a_pT+pT^2 & \text{if } E \text{ has good red. at } p\\ 1-T & \text{split mult.}\\ 1+T & \text{non-split mult.}\\ 1 & \text{additive}\end{cases}$$

degree 2

(4) $f\in S_k(\Gamma_0(N))$ cusp form

$\displaystyle f = \sum_{n\geq 1} a_n q^n \rightsquigarrow L(f,s) = \sum \frac{a_n}{n^s}$

Standard conjectures

each $L(s)$

- has meromorphic continuation to $\mathbb{C}$
- $\exists$ conductor N, weight k, Hodge numbers $\lambda_1,\ldots,\lambda_d\in\mathbb{Z}$, sign $w\in\mathbb{C}^\times$, $|w|=1$ (root number)

such that

$$\hat{L}(s) = \left(\frac{N}{\pi^d}\right)^{s/2} \Gamma\!\left(\frac{s+\lambda_1}{2}\right)\cdots\Gamma\!\left(\frac{s+\lambda_d}{2}\right) L(s)$$

satisfies functional equation

$$\hat{L}(k-s) = w\cdot \hat{L}(s)$$

	N	k	λ_i	w
Riemann	1	1	0	1
Dedekind	$\lvert\Delta_K\rvert$	1	r_1+r_2 0's, r_2 1's	1
χ	N	1	0 χ even, 1 χ odd	$\frac{\tau_{\bar\chi}}{\sqrt{N}}$
E, f	N	2	0,1	w

- zeroes of $L(s)$ are on $\mathrm{Re}\, s = \frac{k}{2}$ (RH)
- Special value conjectures at $s \in \mathbb{Z}$

  (BSD, Stark, Deligne-Beilinson, Bloch-Kato, ...)

__Goal__

num.fields $\to$
characters $\to$
ell.curves $\to$
cusp forms $\to$

Galois representations $\rho : G_{\mathbb{Q}} \to GL_d(\mathbb{C})$ $\longrightarrow$ $L(\rho, s) = \prod_p \frac{1}{F_p(p^{-s})}$

$F_p(T) = \det(1 - \mathrm{Frob}_p^{-1} T \mid \rho^{I_p})$
$N = N_\rho = \prod p^{n_p}$
$w = w_\rho = w_\infty \prod_p w_p$

Notation: $G_K = \mathrm{Gal}(\bar{K}/K)$.

§ Galois representations

(1) $[K : \mathbb{Q}] = d$

$\Sigma = \{\text{embeddings } K \hookrightarrow \mathbb{C}\} \circlearrowleft G_{\mathbb{Q}}$

$|\Sigma| = d$

$\mathbb{C}[\Sigma] \circlearrowleft G_{\mathbb{Q}} \rightsquigarrow \rho_K : G_{\mathbb{Q}} \to GL_d(\mathbb{C})$ Gal. rep.

(2) $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$

$\rho_\chi : G_{\mathbb{Q}} \longrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times$ Gal. rep.

__Note__ $K = \mathbb{Q}(\zeta_n)$
$\mathbb{C}[\Sigma]$ regular rep of $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$
$\rho_K = \bigoplus_\chi \rho_\chi$

Generally $\rho_K = \bigoplus$ Artin representations

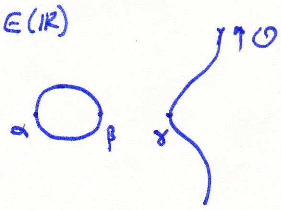
(3) $E/\mathbb{Q}$ ell. curve

$y^2 = x^3 + Ax + B$ $A, B \in \mathbb{Q}$

$E[n] := \{P \in E(\bar{\mathbb{Q}}) \mid nP = O\}$ n torsion

$\cong (\mathbb{Z}/n\mathbb{Z})^2 \circlearrowleft G_{\mathbb{Q}}$

E(ℝ)



$E(\mathbb{C}) \cong \mathbb{C}/\Lambda \cong (\mathbb{R}/\mathbb{Z})^2$ as Lie gp.

__Ex__ $E[2] = \{O, (\alpha, 0), (\beta, 0), (\gamma, 0)\} \circlearrowleft G_{\mathbb{Q}}$

For m, n coprime

$E[mn] \cong E[m] \times E[n]$

enough to look at $n = \ell^k$

$\rightarrow E[\ell^k] \xrightarrow{(\ell)} E[\ell^{k-1}] \xrightarrow{(\ell)} \dots \xrightarrow{(\ell)} E[\ell]$

<u>Def</u> Tate module $\quad T_\ell E = \varprojlim_k E[\ell^k] \qquad \cong (\mathbb{Z}_\ell)^2 \circlearrowleft G_\mathbb{Q}$

$V_\ell E = T_\ell E \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \qquad \cong (\mathbb{Q}_\ell)^2 \circlearrowleft G_\mathbb{Q}$

Linear action of $G_\mathbb{Q}$ on torsion pts $\Rightarrow$

$\bar{\rho}_n : G_\mathbb{Q} \longrightarrow Gl_2(\mathbb{Z}/n\mathbb{Z})$

$G_\mathbb{Q} \longrightarrow Gl_2(\mathbb{Z}_\ell)$

$\rho_\ell : G_\mathbb{Q} \longrightarrow Gl_2(\mathbb{Q}_\ell) \hookrightarrow Gl_2(\mathbb{C}) \qquad$ Gal. rep. (for every ℓ)

Weil pairing $\cancel{E}[n] \times E(n) \longrightarrow \mu_n$

$\det \rho_\ell = \chi_\ell$ cyclotomic character $\quad G_\mathbb{Q} \rightarrow \varprojlim \mathrm{Aut}\, \mu_{\ell^n} = \varprojlim (\mathbb{Z}/\ell^n\mathbb{Z})^\times = \mathbb{Z}_\ell^\times$

In general

$X/\mathbb{Q}$ non-sing. projective variety $\rightsquigarrow H^i = H^i_{ét}(X, \mathbb{Q}_\ell) \qquad$ étale coh. groups, $\mathbb{Q}_\ell$-vector spaces

- $H^i \otimes_{\mathbb{Q}_\ell} \mathbb{C} = H^i_{top}(X(\mathbb{C})) \qquad [\Rightarrow 0$ for $i > 2\dim X]$
- $H^i \circlearrowleft G_\mathbb{Q}$

<u>Ex</u> $K/\mathbb{Q}$ number field, $K = \mathbb{Q}[x]/f(x)$

$X = \operatorname{Spec} K \qquad$ roots of $f \quad \overset{1:1}{\longleftrightarrow} \quad$ embeddings $K \hookrightarrow \mathbb{C}$

Σ

$\circlearrowleft G_\mathbb{Q}$

$H^0 = \mathbb{Q}_\ell[\Sigma]$

$H^i = 0 \quad (i > 0)$

<u>Ex</u> $E/\mathbb{Q}$ ell. curve $\qquad E(\mathbb{C}) = \mathbb{C}/\Lambda$

$H^0 = \mathbb{Q}_\ell \qquad$ 1-dim

$H^1 = T_\ell E^* \qquad$ 2-dim

$H^2 = \chi_\ell \qquad$ 1-dim

$[\, E[\ell^n] = \ell^n$-coverings $E \to E$

$T_\ell E = \pi^1_{ét}(E)_\ell = H_1^{ét}(E, \mathbb{Z}_\ell) = (H^1)^* \,]$

Ex $C/\mathbb{Q}$ curve

$H^0 = \mathbb{Q}_\ell$ 1-dim

$H^1 = H^1(\mathrm{Jac}\, C) = T_\ell(\mathrm{Jac}\, C)^*$ 2g-dim.

$H^2 = \chi_\ell$ 1-dim

Ex $A/\mathbb{Q}$ abelian variety

$H^i = \bigwedge^i H^1 \quad ; \quad H^1 = (T_\ell A)^*$

(4) Ex f cusp form of weight k level N

$\left.\begin{array}{l} \underline{k=2} \quad \text{Eichler-Shimura} \\ \qquad \rho_f \subseteq H^1(X_0(N)) \\ \underline{k>2} \quad \text{Deligne} \\ \underline{k=1} \quad \text{Deligne-Serre} \end{array}\right\} \rho_f$ Gal. rep.

Summary: Algebraic varieties $X/\mathbb{Q}$ $\rightsquigarrow$ large source of Galois representations: $H^i_{et}(X, \mathbb{Q}_\ell)$ & their direct summands ("motives")

(Dirichlet characters, Artin reps, ρ_f)

§ Local invariants of Galois representations

Galois rep ρ $\rightsquigarrow$ L-function $L(\rho, s)$

local factors $F_p(T)$

conductor $N = \prod_p p^{n_p}$

root number $w = w_\infty \prod_p w_p$

Fix a prime p,

$\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \Rightarrow \underset{\substack{\text{decomposition}\\ \text{subgroup}}}{G_{\mathbb{Q}_p}} \xrightarrow{\text{Res}} G_{\mathbb{Q}}$

Study $\rho|_{G_{\mathbb{Q}_p}}$. $\quad (\Rightarrow F_p(T), n_p, w_p)$

Structure of $G_{\mathbb{Q}_p}$:

$$\begin{array}{l} \overline{\mathbb{Q}}_p \\ \quad | \; I = I_{\mathbb{Q}_p} \text{ inertia subgroup} \\ \mathbb{Q}_p^{nr} = \bigcup_{p \nmid n} \mathbb{Q}_p(\mu_n) \\ \quad | \; G_{\mathbb{F}_p} \\ \mathbb{Q}_p \end{array} \qquad G_{\mathbb{Q}_p}$$

$$1 \longrightarrow I \longrightarrow G_{\mathbb{Q}_p} \xrightarrow{\sigma \mapsto \bar\sigma} G_{\mathbb{F}_p} \longrightarrow 1$$

$$\langle x \mapsto x^p \rangle \cong \hat{\mathbb{Z}}$$

Def $I = \{\sigma \in G_{\mathbb{Q}_p} \mid \bar\sigma = 1\} \trianglelefteq G_{\mathbb{Q}_p}$

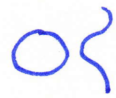
Frob_p = any elt of $G_{\mathbb{Q}_p}$ reducing to $x \mapsto x^p$

A $G_{\mathbb{Q}_p}$ module M is <u>unramified</u> if I acts trivially on M $\quad (\Rightarrow G_{\mathbb{F}_p}\text{-module})$

Main example: torsion points on elliptic curves

Ex $K = \mathbb{Q}_5$

$E_1 : y^2 = x^3 - 1$ $\qquad$ $\tilde{E}_1/\mathbb{F}_5 : y^2 = x^3 - 1$



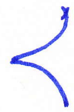
elliptic curve (good reduction)

$E_2 : y^2 = (x-1)(x^2-5)$ $\qquad$ $\tilde{E}_2/\mathbb{F}_5 : y^2 = x^3 - x^2$

multiplicative reduction

[$y^2 + x^2 = x^3$ tangent lines at $(0,0)$ $y = \sqrt{-1}\,x$, $y = -\sqrt{-1}\,x$ defined $/\mathbb{F}_5$ $\Rightarrow$ split multiplicative red.]

$E_3 : y^2 = x^3 - 5^2$ $\qquad$ $\tilde{E}_3/\mathbb{F}_5 : y^2 = x^3$



additive reduction

[becomes good or multiplicative after a finite extension]

Recall $E[2] = \{0, (\alpha,0), (\beta,0), (\gamma,0)\}$ $\quad$ α, β, γ roots of cubic

$E_1[2] = \{0, (1,0), (\zeta_3, 0), (\zeta_3^2, 0)\}$ $\quad$ unramified $\longleftarrow$

$E_2[2]$, $E_3[2]$ ramified $\longleftarrow$ involve $\sqrt{5}$, $\sqrt[3]{5}$

$$\begin{array}{c} \overline{\mathbb{Q}}_5 \\ | \\ \mathbb{Q}_5^{nr} \\ | \\ \mathbb{Q}_5(\zeta_3) \\ | \\ \mathbb{Q}_5 \end{array}$$

Computation $\Rightarrow$ for these 3 curves

	E_1	E_2	E_3
$\vert$	$\vert$		
$K(E[8])$	$K(\zeta_{16})$	$K(\sqrt[8]{5}, \zeta_8)$	$K(\zeta_{16}, \sqrt[3]{5})$
$K(E[4])$	$K(\zeta_{16})$	$K(\sqrt[4]{5})$	$K(\zeta_{16}, \sqrt[3]{5})$
$K(E[2])$	$K(\zeta_8)$	$K(\sqrt{5})$	$K(\sqrt[3]{5}, \zeta_8)$
K	K	K	K

NB $\mathbb{Q}_5(\zeta_3) = \mathbb{Q}_5(\zeta_8) = \mathbb{Q}_5(\zeta_{24}) = \mathbb{Q}_5(\sqrt{-3})$ unique quad. unramified ext. of $\mathbb{Q}_5$

Residue degree: Weil pairing $\Rightarrow$ $\mu_{2^n} \subseteq K(E[2^n])$

$\Rightarrow$ residue degree $\to \infty$

Ramification: Guess?

E_1 all unramified $\qquad$ [Yes! $\Leftrightarrow$ good reduction

E_2 regularly growing ramification $\qquad$ $\Leftrightarrow$ (pot.) mult. red.

E_3 finite ramification $\qquad$ $\Leftrightarrow$ additive pot. good. red.

§ Good reduction

Thm (Néron–Ogg–Shafarevich) $E/\mathbb{Q}_p$ ell. curve, $\ell \neq p$.

E has good reduction $\Leftrightarrow$ $E[\ell^n]$ unramified $\forall n \geq 1$

($\Leftrightarrow$ $T_\ell E$, $V_\ell E$ unramified)

[similarly $/K$, $K/\mathbb{Q}_p$ finite]

In this case

$\rho_\ell(I) = 1$

$\rho_\ell(\mathrm{Frob}_p) = 2\times 2$ matrix with char poly $T^2 - aT + p$, $a = p+1-\#\tilde{E}(\mathbb{F}_p)$

Here

$(V_\ell E^*)^I = V_\ell E^*$ (unramified), so $\boxed{F_p(T) = 1 - aT + pT^2}$

<u>Ex</u> $E_1/\mathbb{Q}_5$: $y^2 = x^3 - 1$ good red.

$\tilde{E}_1/\mathbb{F}_5$: $y^2 = x^3 - 1$ has 6 pts $O, (0,\pm 2), (1,0), (3,\pm 2)$

$T^2 - aT + q = T^2 + 5$, $\underline{F(T) = 1 + 5T^2}$

Here $\rho_\ell : G_{\mathbb{Q}_p} \longrightarrow GL_2(\mathbb{Q}_\ell) \hookrightarrow GL_2(\mathbb{C})$

$I \longrightarrow 1$

$\mathrm{Frob}_5 \longrightarrow \begin{pmatrix} 0 & -5 \\ 1 & 0 \end{pmatrix}$ in some basis.

§ Additive pot. good reduction

$\rightsquigarrow$ Exc.

Prop If $E/\mathbb{Q}_p$ has additive pot. good reduction, then $\boxed{F_p(T) = 1}$

<u>Pf</u>

$$\begin{array}{ccc} \overline{\mathbb{Q}}_p & & \\ | & & \\ K & & E/K \text{ good} \\ \text{fin. Galois } | & & \\ \mathbb{Q}_p & & E/\mathbb{Q}_p \text{ bad} \end{array}$$

$\overset{\text{N.O.S.}}{\Longrightarrow}$ on $V_\ell E$: I_K acts trivially, $I = I_{\mathbb{Q}_p}$ non-trivially $\Rightarrow$ $I_{\mathbb{Q}_p}$ acts through $I_{F/K}$ (finite group)

(in fact $C_2, C_3, C_4, C_6, C_3 \rtimes C_4, Q_8$ or $SL_2(\mathbb{F}_3)$)

$\dim V_\ell E^{*I} < 2 \Rightarrow 0$ or 1

If 1, then I acts as $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$

$\det = \chi_\ell^{-1} = 1$ on I_K $\rightarrow$ $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ $\leftarrow$ infinite order! contradiction.

So $V^I = \{0\}$, $F(T) = \det(\cdots \mid (V_\ell E^*)^I) = 1$ ∎

§ Multiplicative reduction

For $E/\mathbb{C}$

$$E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z} \xrightarrow[\cong]{\exp(2\pi i\,\cdot)} \mathbb{C}^\times/q^{\mathbb{Z}} \qquad q = e^{2\pi i \tau} \qquad \left[\cong \text{ given by transcendental fncs}\right]$$

as complex Lie groups.

<u>Thm</u> (Tate) If $E/\mathbb{Q}_p$ has multiplicative reduction, then

- $\exists!\ q \in p\mathbb{Z}_p$ s.t.

$$E \cong E_q : \quad y^2 + xy = x^3 + a_4(q)x + a_6(q) \qquad \leftarrow \text{Tate curve} \quad \left[\text{with Tate parameter } q\right]$$

$$a_4(q) = -5 \sum_{n\geq 1} \frac{n^3 q^n}{1-q^n}$$

$$a_6(q) = \frac{1}{12}\left[a_4 q + 7 \sum_{n\geq 1} \frac{n^5 q^n}{1-q^n}\right]$$

$$\Delta(q) = q \prod_{n=1}^{\infty} (1-q^n)^{24}$$

$$j(q) = \frac{1}{q} + 744 + 196884\, q + \dots$$

- $E(\overline{\mathbb{Q}}_p) \cong \overline{\mathbb{Q}}_p^\times / q^{\mathbb{Z}}$ as $G_{\mathbb{Q}_p}$-modules $\qquad [\Rightarrow E(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times/q^{\mathbb{Z}}]$

  given by power series $(x(q;u), y(q;u)) \longleftarrow u$

<u>Cor</u>

$$E[\ell] \cong \langle \mu_\ell, q^{1/\ell}\rangle \qquad \text{as } G_{\mathbb{Q}_p}\text{-modules}$$
$$E[\ell^n] \cong \langle \mu_{\ell^n}, q^{1/\ell^n}\rangle$$

<u>Ex</u> $E_2/\mathbb{Q}_5$: $y^2 = (x-1)(x^2-5)$

$$j = \frac{2^4}{5}, \quad \Delta = 2^{10}\cdot 5 \quad \Rightarrow \quad q = 5 \times \text{unit}.$$

$$\mathbb{Q}_5(E[2^n]) \cong \mathbb{Q}_5(\sqrt[2^n]{5}, \mu_{2^n})$$

Action of I on $V_\ell E$

$$\begin{pmatrix} 1 & \varphi \\ 0 & 1 \end{pmatrix} \qquad \varphi : I \longrightarrow \mathbb{Z}_\ell$$

$$\sigma \longmapsto \left(\frac{\sigma(p^{1/\ell^n})}{p^{1/\ell^n}}\right)_{n\geq 1} \in \varprojlim \mu_{\ell^n} \cong \mathbb{Z}_\ell \qquad \underline{\text{tame character}}$$

Action of Frob_p on $V_\ell E$

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

On $V_\ell E^*$

invariants

$$I : \begin{pmatrix} 1 & 0 \\ \varphi & 1 \end{pmatrix}$$

$$\mathrm{Frob}_p : \begin{pmatrix} 1/p & 0 \\ 0 & 1 \end{pmatrix}$$

$$F_p(T) = \det(1 - \mathrm{Frob}_p^{-1} T \mid (V_\ell E^*)^{I_p}) = \boxed{1-T}$$

§ Non-split multiplicative reduction

Unramified quad. twist of split mult. reduction: $I : \begin{pmatrix} 1 & 0 \\ \varphi & 1 \end{pmatrix}$ $\mathrm{Frob}_p : \begin{pmatrix} -1/p & 0 \\ 0 & -1 \end{pmatrix}$

$$\boxed{F_p(T) = 1 + T}$$

§ Additive reduction

Becomes good or multiplicative after a finite (ramified) extension.

Exc $\Rightarrow$ $(V_\ell E^*)^I = 0$ and so $\boxed{F_p(T) = 1}$

Summary $E/\mathbb{Q}_p$ ell. curve, $\ell \neq p$

$$F_p(T) = \det(1 - \mathrm{Frob}_p^{-1} T \mid (V_\ell E^*)^I) = \begin{cases} 1 - a_p T + p T^2 & E \text{ good} \\ 1 - T & E \text{ split mult.} \\ 1 + T & E \text{ nonsplit mult.} \\ 1 & E \text{ additive} \end{cases}$$

independent of ℓ

Same defn for $H^i(X)$ for any nonsingular projective X (or any other)

Galois representation $\rho : G_{\mathbb{Q}} \to GL_d(\mathbb{C})$: compatible system of ℓ-adic representations

Def $F_p(T) = \det(1 - \mathrm{Frob}_p^{-1} T \mid \rho^{I_p})$ local factor

$L(\rho, s) = \prod_p \frac{1}{F_p(p^{-s})}$ global L-function

Ex $E/\mathbb{Q}$ elliptic curve

$$L(H^0(E), s) = \prod_p \frac{1}{1 - p^{-s}} = \zeta(s)$$

$$L(H^1(E), s) = \prod_{p \mid \Delta} \frac{1}{1 - a_p p^{-s}} \prod_{p \nmid \Delta} \frac{1}{1 - a_p p^{-s} + p^{1-2s}} \quad \leftarrow L(E, s)$$

$$L(H^2(E), s) = \prod_p \frac{1}{1 - p \cdot p^{-s}} = \zeta(s-1)$$

Thm (Grothendieck Monodromy Thm) $X/\mathbb{Q}_p$ non-singular projective variety

After a fin. ext. $K/\mathbb{Q}_p$, I acts on $H^i(X, \mathbb{Q}_\ell)$ as $\mathrm{Id} + \varphi \cdot N$,

$\varphi : G_{\mathbb{Q}_p} \to \mathbb{Z}_\ell$ tame character as before, N nilpotent matrix.

Rmk Such representations are called Weil-Deligne representations, and Weil representations if $N = 0$.

Ex $E/\mathbb{Q}_p$ good (or potentially good) $\Rightarrow$ $N = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $V_\ell E$ Weil rep.

$E/\mathbb{Q}_p$ multiplicative (or pot. mult.) $\Rightarrow$ $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $V_\ell E$ Weil-Deligne.

8