SOME HOMEWORK PROBLEMS

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1. Suppose that d is a non-square, positive integer. Let u, v be the smallest pair of positive integers for which $u^2 - dv^2 = 1$. Prove that if $x^2 - dy^2 = N$ then there exists a solution to

$$x_0^2 - dy_0^2 = N \quad \text{with} \quad |x_0| \leq \sqrt{|N| x_0},$$

and an integer $k \geq 0$ for which

$$x + \sqrt{d} y = (x_0 + \sqrt{d} y_0)(u + \sqrt{d} v)^k.$$

2. Using the identity

$$u(u+2v)^3 + v(-v-2u)^3 = (u+v)(u-v)^3,$$

show that there exist non-zero and coprime integers a, b, c for which there are infinitely many integer solutions x, y, z to

$$ax^3 + by^3 + cz^3 = 0.$$

Explain "geometrically" why this works.

3. We call m a *powerful number* if p^2 divides m, whenever p does, for every prime p.

a) Prove that there are infinitely many pairs of consecutive powerful numbers $m, m+1$.

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Use the *abc*-conjecture to prove that

b) There are only finitely many triples of consecutive powerful numbers $m-1, m, m+1$.

c) Show that if $p_1 = 1, p_2 = 4, p_3 = 8, p_4 = 9, \dots$ is the sequence of powerful numbers then $p_{n+2} - p_n \to \infty$ as $n \to \infty$.

d) There are only finitely many powerful Fibonacci numbers. (Hint: Study solutions to $x^2 - 5y^2 = \pm 4$).

4. Use the *abc*-conjecture to prove that there are only finitely many pairs of distinct positive integers x, y for which $x - j$ and $y - j$ have exactly the same prime factors for $j = -1, 0$ and 1. (Hint: Obtain a lower bound for the product of the distinct prime factors of $x^3 - x$.)

5. Use the *abc*-Roth conjecture to show that if f and g are two quadratic polynomials, such that fg has four distinct roots, and $r \in \mathbb{Q}$ such that $f(r)$ is a square and $g(r)$ is a cube, then r can be bounded as a function of coefficients of f and g.

6. Calculate a Belyi map from a given elliptic curve $y^2 = x^3 + ax + b$, with a, b integers, to $\mathbb{P}^1$. (Hint: Be careful with the points at ∞.)