INTEGRAL POINTS ON CURVES

(1) We study the integral points of $\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$. Recall that these are the set of morphisms $\operatorname{Spec} \mathbb{Z} \to \mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$.

(a) Let's first look at the integral points of $\mathbb{P}^1_{\mathbb{Q}}$. In the case of $\mathbb{Q}$-points, the informal description such as $[2 : 3]$ would have denoted the image of the single point $\operatorname{Spec} \mathbb{Q}$. However, $\operatorname{Spec} \mathbb{Z}$ has many points, and hence $[2 : 3]$ now refers to the image of the generic point. Describe (in as much detail as you can stomach) where the rest of the points of $\operatorname{Spec} \mathbb{Z}$ should go to. Where does (7) map to? How about (2) and (3)?

(b) Now, look at the integral points of $\mathbb{P}^1_{\mathbb{Q}} - \{\infty\}$. Which integral points of $\mathbb{P}^1_{\mathbb{Q}}$ are no longer integral points on $\mathbb{P}^1_{\mathbb{Q}} - \{\infty\}$?

(c) Describe the integral points on $\mathbb{P}^1_{\mathbb{Q}} - \{0, \infty\}$.

(d) Finally, describe the integral points on $\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$. What happens if $\mathbb{Q}$ is replaced by a number field K, and $\mathbb{Z}$ by its ring of integers $\mathcal{O}_K$?

(2) Show that the diophantine equation

$$a_n x^n + a_{n-1} x^{n-1} y + \cdots + a_0 y^n = c$$

has only finitely many integral solutions, using Roth's theorem.