RATIONAL POINTS ON HIGHER-DIMENSIONAL VARIETIES

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Introduction

The aim of these lectures is to give examples of algebraic varieties for which the local-to-global approach to (the existence and approximation of) rational points can be made to work, and to sketch bits of relevant theories.

The basic paradigm of such a variety is a smooth plane conic $C \subset \mathbb{P}^2_{\mathbb{Q}}$:

$$ax^2 + by^2 + cz^2 = 0, \tag{0.1}$$

where $a, b, c \in \mathbb{Q}^*$. Legendre's theorem [10, 4.3.2 Thm. 8 (ii)] gives a necessary and sufficient condition for the solubility of (0.1) in $\mathbb{Q}$. It implies that the Hasse principle holds for smooth plane conics over $\mathbb{Q}$. By the classical stereographic projection, a conic C with a rational point is isomorphic to the projective line, so describing all rational points on it is straightforward. Equally straightforward is to approximate local points $M_p \in C(\mathbb{Q}_p)$ for finitely many primes p, by a rational point $M \in C(\mathbb{Q})$ (reduce to the affine line and then use the independence of valuations in a number field; in the case of $\mathbb{Q}$ this is just the Chinese remainder theorem).

Minkowski and Hasse generalised this to arbitrary number fields k and to non-degenerate quadratic forms in any number of variables. Going from dimension 1 to dimension 2 uses a reciprocity law from class field theory, but going from dimension n to dimension $n + 1$ becomes easy when $n \geq 3$. The idea is to consider conics, quadrics or some other varieties for which the Hasse principle and weak approximation are already established, in a family parameterised by $\mathbb{P}^1_k$, and try to deduce the Hasse principle and weak approximation for the total space of this family.

Developing this idea we encounter the Brauer–Manin obstruction to the Hasse principle and weak approximation. Indeed, Iskovskih's counterexample

$$x^2 + y^2 = (t^2 - 2)(3 - t^2)$$

shows that the Hasse principle does not hold for 1-parameter families of conics over $\mathbb{Q}$. So we won't be able to prove the existence of rational points unless certain conditions provided by the elements of the Brauer–Grothendieck group of the variety are satisfied.

So we start Lecture One of this mini-course with an introduction to the Brauer group and the Brauer–Manin obstruction.

The arithmetic of surfaces and threefolds that can be represented as 1-parameter families of conics and quadrics will be the main subject of Lecture Two. Rational points have to ultimately come from somewhere. In easier cases this can be done by algebraic methods, but in general one will have to use counting results established by difficult analytic methods.

Finally, in Lecture Three we want to go beyond conic bundles which, being geometrically rational surfaces, belong to a very special class of varieties. We give a very short survey of some recent results for K3 surfaces. We also discuss Enriques surfaces (which are not geometrically simply connected). This discussion will lead us to the refinement of the Brauer–Manin obstruction, the so called *étale Brauer–Manin obstruction*. However, there exist counterexamples to the Hasse principle not detected even by this strongest general obstruction we know today. The first such was constructed by Poonen. We give a simple construction due to Colliot-Thélène, Pál and the lecturer which is based on Poonen's trick.

1. Brauer–Grothendieck group

Let k be a field with an algebraic closure $\bar{k}$ and $\Gamma = \mathrm{Gal}(\bar{k}/k)$.

1.1. Two definitions the Brauer group.

References: [4], [11], [14]

Recall that $\mathrm{Br}(k)$ consists of the equivalence classes of central simple k-algebras (CSA). A k-algebra A is a CSA if $A \otimes_k \bar{k}$ is isomorphic to a matrix algebra $M_n(\bar{k})$ for some positive integer n. In other words, a CSA is a $\bar{k}/k$-form of $M_n(k)$ for some n. A key example of a CSA over $\mathbb{R}$ not isomorphic to a matrix algebra is the algebra of Hamilton's quaternions $\mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij$, where $i^2 = j^2 = -1$ and $ij = -ji$. If k is a field of characteristic different from 2 and $a, b \in k^*$, then the quaternion algebra $Q(a,b) = k \oplus ki \oplus kj \oplus kij$, where $i^2 = a$, $j^2 = b$ and $ij = -ji$, is a CSA[1]. CSAs A and B are equivalent if $M_n(A) \cong M_\ell(B)$ for some positive integers n and ℓ (this is called Brauer, or Morita, equivalence). We denote by $[A]$ the equivalence class of A. We write $(a,b) \in \mathrm{Br}(k)$ for the class of the quaternion algebra $Q(a,b)$. It can be shown that $Q(a,b) \otimes_k Q(a,b) \cong M_4(k)$ (see the appendix), hence $(a,b) \in \mathrm{Br}(k)[2]$.

The group structure on $\mathrm{Br}(k)$ is defined by the tensor product of algebras; the neutral element is the class of matrix algebras; the inverse element is given by the opposite algebra (with inverted order of multiplication). Using the Skolem–Noether theorem (which says that all automorphisms of a matrix algebra are inner, i.e. are conjugations by invertible matrices) one canonically identifies the $\bar{k}/k$-forms of $M_n(k)$ with the (non-abelian) Galois cohomology *set* $H^1(k, \mathrm{PGL}_n)$. The exact sequence

$$1 \to \bar{k}^* \to \mathrm{GL}_n(\bar{k}) \to \mathrm{PGL}(\bar{k}) \to 1$$

gives rise to the boundary map $H^1(k, \mathrm{PGL}_n) \to H^2(k, \bar{k}^*)$. The image of the class of a CSA under this map is zero if and only if this CSA is a matrix algebra. The classical construction of a CSA from a "system of factors" (a 2-cocycle) gives the surjectivity of $\mathrm{Br}(k) \to H^2(k, \bar{k}^*)$, so this map is an isomorphism of groups.

For any field extension K/k we the *restriction*

$$\mathrm{res}_{K/k} : \mathrm{Br}(k) \to \mathrm{Br}(K)$$

defined by sending $[A]$ to $[A \otimes_k K]$. If K/k is a *finite separable* extension, then we also have the *corestriction* map

$$\mathrm{cores}_{K/k} : \mathrm{Br}(K) \to \mathrm{Br}(k)$$

[1] See the appendix for basic properties of quaternion algebras.

defined as the composite map (assuming $K \subset \bar{k}$)

$$H^2(K, \bar{k}^*) = H^2(k, (\bar{k} \otimes_k K)^*) \to H^2(k, \bar{k}^*).$$

The isomorphism here is obtained by Shapiro's lemma from the fact that $(\bar{k} \otimes_k K)^*$ is the direct sum of $\bar{k}^*$ numbered by different embeddings of K into $\bar{k}$, so this $\mathrm{Gal}(\bar{k}/k)$-module is induced from the $\mathrm{Gal}(\bar{k}/K)$-module $\bar{k}^*$. The arrow is defined by the norm from K to k. The composition $\mathrm{cores}_{K/k}\mathrm{res}_{K/k} = [K : k]$ is easily seen to be the multiplication by the degree $[K : k]$ on $\mathrm{Br}(k)$. Any CSA becomes isomorphic to a matrix algebra already over a finite extension of k, so we see that every element of $\mathrm{Br}(k)$ has finite order. In other words, $\mathrm{Br}(k)$ **is a torsion group**.

The group $\mathrm{PGL}_n(k)$ is also the automorphism group of the projective space $\mathbb{P}_k^{n-1}$. Thus $H^1(k, \mathrm{PGL}_n)$ also classifies the twisted forms of $\mathbb{P}_k^{n-1}$ (up to isomorphism), i.e. the algebraic varieties over k that are $\bar{k}/k$-forms of $\mathbb{P}_k^{n-1}$. They are called *Severi–Brauer varieties*. The easiest case is when $n = 1$, then we are simply talking about *conics*. The conic associated to $Q(a, b)$ is $ax^2 + by^2 = z^2$. We denote it by $C(a, b)$.

Exercise Let F be the field of functions on $C(a, b)$. Then $Q(a, b) \otimes_k F \cong M_2(F)$, so that the image of (a, b) under the restriction map $\mathrm{Br}(k) \to \mathrm{Br}(F)$ is zero. (*Hint*: show that $Q(a, b) \otimes_k F$ has zero divisors.)

There are two ways to define the Brauer group of a scheme X. One way is to define the analogue of CSAs – the Azumaya algebras – as vector bundles on X with a fibre-wise structure of CSAs. Azumaya algebras A and B are called equivalent if $A \otimes_X \mathrm{End}(V)$ is isomorphic to $B \otimes_X \mathrm{End}(W)$ for some vector bundles V and W on X.

In these lectures we follow a different approach which is based on étale cohomology. Define $\mathrm{Br}(X) = H^2(X, \mathbb{G}_m)$, the second étale cohomology group of X with coefficients in the étale sheaf defined by the multiplicative group scheme $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$. The Brauer group of Azumaya algebras $\mathrm{Br}_{Az}(X)$ is a torsion group (when X has finitely many connected components, i.e. X is quasi-compact, see [9, Prop. 4.2.7]). It naturally maps to $\mathrm{Br}(X)$. By a theorem of Gabber (see de Jong's paper) the map $\mathrm{Br}_{Az}(X) \to \mathrm{Br}(X)_{\mathrm{tors}}$ is an isomorphism, at least when X has an ample invertible sheaf. (This means that X is quasi-compact and there exists an invertible sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules with the following property: for each $x \in X$ there is an $s \in H^0(X, \mathcal{L}^{\otimes n})$ for some $n \geq 1$ such that $s(x) \neq 0$ and the open subset $s \neq 0$ is affine.) When X is regular and integral, we shall see that $\mathrm{Br}(X)$ is naturally a subgroup of the Brauer group of the residue field at the generic point of X, hence $\mathrm{Br}(X)$ is also a torsion subgroup. In particular, **the two definitions of the Brauer group of a scheme coincide for regular quasi-projective schemes over the spectrum of a Noetherian ring, e.g. for regular quasi-projective varieties over a field.**

1.2. **Residues and purity.** *References*: [8], [9]

Let X be a *regular* integral scheme with the generic point $j : \mathrm{Spec}(F) \hookrightarrow X$. If $D \subset X$ is an irreducible divisor, we denote its field of rational functions by $k(D)$. There is an exact sequence of sheaves in étale topology [9, Example III.2.22], which decribes the embedding of the group of invertible regular functions into the group of non-zero rational functions as the kernel of the divisor map:

$$0 \to \mathbb{G}_{m,X} \to j_*\mathbb{G}_{m,F} \to \oplus_{D\in X^1}\ i_{D*}\mathbb{Z}_{k(D)} \to 0, \tag{1.1}$$

where $i_D : \mathrm{Spec}(k(D)) \hookrightarrow X$ is the embedding of the generic point of an irreducible divisor on X, and the direct sum ranges over all such divisors. This sequence exists because X is regular, and hence Weil divisors are the same thing as Cartier divisors (i.e. any divisor is locally given by one equation).

Let $f : Y \to X$ be a morphism and let $\mathcal{F}$ be a sheaf on Y. We shall often use the spectral sequence of composed functors

$$H^p(X, R^qf_*\mathcal{F}) \Rightarrow H^{p+q}(Y, \mathcal{F}),$$

see [9, Thm. III.1.8], and refer to it as the 'usual' spectral sequence.

It immediately implies that $H^1(X, i_{D*}\mathbb{Z}_{k(D)})$ is a subgroup of $H^1(k(D), \mathbb{Z}) = 0$; this group is zero because the first cohomology group of a profinite group with coefficients in $\mathbb{Z}$ is trivial. (Similarly one shows that $R^1i_{D*}\mathbb{Z}_{k(D)} = 0$. Indeed, let $\bar{x}$ be a geometric point of X and let $\mathcal{O}^{\mathrm{sh}}_{\bar{x}}$ be the strict Henselisation of the local ring of X at $\bar{x}$. The stalk of $R^1i_{D*}\mathbb{Z}_{k(D)}$ at $\bar{x}$ is H^1 of the inverse image of the sheaf $\mathbb{Z}_{k(D)}$ on $\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{\bar{x}}) \times_X \mathrm{Spec}(k(D))$, see [9, Thm. III.1.5]. If $\bar{x} \notin D$, this fibred product is empty, so the stalk is 0. If $\bar{x} \in D$ this fibred product is the spectrum of the field of fractions of the strictly Henselian local ring of $\bar{x}$ in D, so H^1 is zero.) Therefore, the long exact sequence of cohomology groups attached to (1.1) gives

$$0 \to \mathrm{Br}(X) \to H^2(X, j_*\mathbb{G}_{m,F}) \to \oplus_{D\in X^1}\ H^2(X, i_{D*}\mathbb{Z}_D). \tag{1.2}$$

By Hilbert's theorem 90 we have $R^1j_*\mathbb{G}_{m,F} = 0$. (Again, the stalk at $\bar{x}$ is H^1 of $\mathbb{G}_m$ on $\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{\bar{x}}) \times_X \mathrm{Spec}(F)$, which is the spectrum of the field of fractions $F^{\mathrm{sh}}_{\bar{x}}$ of $\mathcal{O}^{\mathrm{sh}}_{\bar{x}}$.) Then the usual spectral sequence implies that $H^2(X, j_*\mathbb{G}_{m,F})$ is a subgroup of $H^2(F, \mathbb{G}_{m,F}) = \mathrm{Br}(F)$. Thus $\mathrm{Br}(X)$ **is naturally a subgroup of** $\mathrm{Br}(F)$. This also shows that if $U \subset X$ is a dense open subset, then the canonical map $\mathrm{Br}(X) \to \mathrm{Br}(U)$ **is injective**. Since $\mathrm{Br}(F)$ is a torsion group, $\mathrm{Br}(X)$ **is also a torsion group**.

By Tsen's theorem $\mathrm{Br}(F) = 0$ if X is a curve over an algebraically closed field. Hence $\mathrm{Br}(X) = 0$ **when** X **is an integral regular curve over an algebraically closed field**.

If X is a scheme of dimension 1 such that the residue fields at its closed points are perfect (for example, finite) and the residue field at the generic point has characteristic zero, we have $R^2j_*\mathbb{G}_{m,F} = 0$. (Indeed, the stalk of this sheaf at $\bar{x}$ is $H^2(F^{\mathrm{sh}}_{\bar{x}}, \mathbb{G}_m) = \mathrm{Br}(F^{\mathrm{sh}}_{\bar{x}})$. By a theorem of Lang the field of fractions of a Henselian DVR with an algebraically closed residue field is a C_1-field – at least when it has characteristic zero – so its Brauer group is zero.) Since $R^1i_{x*}\mathbb{Z} = 0$, where x is a closed point of X, the usual spectral sequence shows that $H^2(X, i_{x*}\mathbb{Z})$ is a subgroup of $H^2(k(x), \mathbb{Z}) = H^1(k(x), \mathbb{Q}/\mathbb{Z})$. (The last isomorphism follows from the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

since $H^i(k, \mathbb{Q}) = 0$, $i > 0$, for any field k, as this group is a torsion group where the multiplication by any integer is an isomorphism.) Hence $H^2(X, j_*\mathbb{G}_{m,F}) = \mathrm{Br}(F)$ and (1.2) gives an exact sequence

$$0 \to \mathrm{Br}(X) \to \mathrm{Br}(F) \to \oplus_x \, H^1(k(x), \mathbb{Q}/\mathbb{Z}), \tag{1.3}$$

where x ranges over the closed points of X. The map $\mathrm{Br}(F) \to H^1(k(x), \mathbb{Q}/\mathbb{Z})$ is called the *residue* at D. We see that $\mathrm{Br}(X)$ is the kernel of all residue maps associated to the closed points of X.

For a ring R one write $\mathrm{Br}(R) = \mathrm{Br}(\mathrm{Spec}(R))$.

Let R be a local ring with the maximal ideal $\mathfrak{m}$ and the fraction field F. We assume that the residue field $k = R/\mathfrak{m}$ is perfect. By the functoriality of étale cohomology the embedding of the closed point $\mathrm{Spec}(k) \to \mathrm{Spec}(R)$ gives rise to the *specialisation* map $\mathrm{Br}(R) \to \mathrm{Br}(k)$. Azumaya's theorem says that when R is Henselian (for example, complete), this map is an isomorphism. So for $X = \mathrm{Spec}(R)$, where R is a Henselian DVR, (1.3) simplifies further. In this case the residue map in (1.3) has a section given by the choice of a generator π of $\mathfrak{m}$, so we obtain a split short exact sequence (Witt's theorem)

$$0 \to \mathrm{Br}(k) \to \mathrm{Br}(F) \to H^1(k, \mathbb{Q}/\mathbb{Z}) \to 0. \tag{1.4}$$

Here is how one can compute the residue and show that (1.4) is split. Let us assume that the DVR R is complete. By Lang's theorem the Brauer group of the maximal unramified extension F_{nr} is zero. By Hilbert's theorem 90 the usual spectral sequence $H^p(\mathrm{Gal}(F_{\mathrm{nr}}/F), H^q(F_{\mathrm{nr}}, \bar{F}^*)) \Rightarrow H^{p+q}(F, \bar{F}^*)$ gives an isomorphism $H^2(\mathrm{Gal}(F_{\mathrm{nr}}/F), F_{\mathrm{nr}}^*) \tilde{\longrightarrow} \mathrm{Br}(F)$. We have $\mathrm{Gal}(F_{\mathrm{nr}}/F) \cong \mathrm{Gal}(\bar{k}/k) = \Gamma_k$. Combined with the map given by the valuation $F_{\mathrm{nr}}^* \to \mathbb{Z}$ this defines a map

$$\mathrm{Br}(F) \tilde{\longrightarrow} H^2(k, F_{\mathrm{nr}}^*) \longrightarrow H^2(k, \mathbb{Z}) = H^1(k, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(\Gamma_k, \mathbb{Q}/\mathbb{Z}),$$

which coincides with the residue as defined above. We also see that a choice of a generator of $\mathfrak{m}$ defines a section of the second arrow and hence of the composed map.

If the characteristic of k is not 2, the residue map becomes

$$\mathrm{Br}(F)[2] \longrightarrow H^1(k, \mathbb{Q}/\mathbb{Z})[2] = \mathrm{Hom}(\Gamma_k, \mathbb{Z}/2) = k^*/k^{*2}.$$

The above description of residue can be used to show that if $\mathfrak{m} = (\pi)$ and $u \in R^*$, then the residue of the class (π, u) is zero is the image of u in k^*/k^{*2}. If $v \in R^*$, then the residue of (u, v) is zero. It is known that $(a, b) \in \mathrm{Br}(F)$ is a multiplicative function in each of the two variables, so the computation of residue in the general case can be reduced to these particular cases using $(a^2, b) = (a, -a) = 0$ for any $a, b \in F^*$.

Now let k be a finite field. Since the Brauer group of a finite field is zero and its absolute Galois group is $\hat{\mathbb{Z}}$ (with Frobenius as a topological generator), in the case when $F = K_v$ is the completion of a global field K at a non-Archimedean place v we obtain an isomorphism $\mathrm{Br}(K_v)\tilde{\longrightarrow}\mathbb{Q}/\mathbb{Z}$. Up to sign this map coincides with the local invariant isomorphism inv_v provided by class field theory. If K_v is the field of p-adic numbers $\mathbb{Q}_p$ and $u \in \mathbb{Z}_p^*$, then $\mathrm{inv}_p(p,u) = 1/2$ if the Legendre symbol $\left(\frac{u}{p}\right) = -1$, and $\mathrm{inv}_p(p,u) = 0$ is $\left(\frac{u}{p}\right) = 1$.

The analogue of (1.3) is also valid when X is a smooth integral variety over a field k of characteristic zero. Let $Y \subset X$ be a closed subset which is *smooth*[2] *of codimension* d at every point $y \in Y$. We have a long exact sequence of cohomology with support:

$$\ldots \to H^n_Y(X,\mathbb{G}_m) \to H^n(X,\mathbb{G}_m) \to H^n(X\setminus Y,\mathbb{G}_m) \to H^{n+1}_Y(X,\mathbb{G}_m) \to \ldots$$

Using local purity for the smooth pair (X,Y) (see [9, VI, §5]) one shows that $H^2_Y(X,\mathbb{G}_m) = H^3_Y(X,\mathbb{G}_m) = 0$ if $d \geq 2$, so in this case we obtain an isomorphism

$$\mathrm{Br}(X)\tilde{\longrightarrow}\mathrm{Br}(X\setminus Y). \tag{1.5}$$

If $d = 1$, we have $H^2_Y(X,\mathbb{G}_m) = 0$ and

$$H^3_Y(X,\mathbb{G}_m) = \oplus_D H^2(k(D),\mathbb{Z}) = \oplus_D H^1(k(D),\mathbb{Q}/\mathbb{Z}),$$

where D ranges over the irreducible components of Y, so there is an exact sequence

$$0 \to \mathrm{Br}(X) \to \mathrm{Br}(X\setminus Y) \to \oplus_D H^1(k(D),\mathbb{Q}/\mathbb{Z}). \tag{1.6}$$

For an arbitrary closed subset $Y \subset X$ we define Y_0 as the smooth locus of the union of irreducible components of Y of codimension 1 in X. Then $Y\setminus Y_0$ has codimension at least 2. Define Y_1 as the smooth locus of the union of irreducible components of $Y\setminus Y_0$ of codimension 2, and so on. Then Y is the disjoint union of locally closed smooth subsets Y_n for $n \geq 0$, where the codimension of Y_n in X is $n+1$ (we allow Y_n to be empty). The set Y_0 is open in Y, so $Y\setminus Y_0$ is closed in Y and hence in X. The open subset $U = X\setminus(Y\setminus Y_0)$ contains Y_0 as a smooth closed subset. By applying (1.5) we obtain an isomorphism $\mathrm{Br}(X) = \mathrm{Br}(U)$, and a similar isomorphism $\mathrm{Br}(X\setminus Y) = \mathrm{Br}(U\setminus Y_0)$. Now (1.6) gives

$$0 \to \mathrm{Br}(X) \to \mathrm{Br}(X\setminus Y) \to \oplus_D H^1(k(D),\mathbb{Q}/\mathbb{Z}), \tag{1.7}$$

where D ranges over the irreducible components of Y of codimension 1 in X. Passing to the inductive limit over closed subsets $Y \subset X$ we deduce the exact sequence

$$0 \to \mathrm{Br}(X) \to \mathrm{Br}(k(X)) \to \oplus_D\ H^1(k(D),\mathbb{Q}/\mathbb{Z}), \tag{1.8}$$

where D ranges over the irreducible divisors of X.

The embedding of the generic point $i_D : \mathrm{Spec}(k(D)) \to X$ factors as

$$\mathrm{Spec}(k(D)) \to \mathrm{Spec}(\widehat{\mathcal{O}_{X,D}}) \to \mathrm{Spec}(\mathcal{O}^{\mathrm{h}}_{X,D}) \to \mathrm{Spec}(\mathcal{O}_{X,D}) \to X,$$

[2]There is a mistake in GBIII,§6, where the smoothness of Y is not mentioned.

where $\widehat{\mathcal{O}_{X,D}}$ is the completion of the local ring $\mathcal{O}_{X,D}$ (the point is that the Henselisation and the completion of a local ring don't affect the residue field). Each residue map $\mathrm{Br}(k(X)) \to H^1(k(D), \mathbb{Q}/\mathbb{Z})$ can be computed at the level of the local ring $\mathcal{O}_{X,D}$ which is a DVR with the residue field $k(D)$ and the field of fractions $k(X)$. Thus the residue map in (1.8) factors through the residue map in the exact sequence (1.4) attached to the Henselisation $\mathcal{O}^{\mathrm{h}}_{X,D}$ (or the completion $\widehat{\mathcal{O}_{X,D}}$).

Finally we note that the Brauer group is a birationl invariant of smooth projective varieties in characteristic zero.

1.3. **Algebraic and transcendental parts of** $\mathrm{Br}(X)$**.** *References*: [8]

Let X be a smooth geometrically integral variety over a field k with a separable closure $\bar{k}$. We assume that $\bar{k}[X]^* = \bar{k}^*$, for example X projective. We write $\overline{X} = X \times_k \bar{k}$ and $\Gamma = \mathrm{Gal}(\bar{k}/k)$. The group $\mathrm{Br}(\overline{X})$ is called the *geometric* Brauer group of X. Define

$$\mathrm{Br}_0 = \mathrm{Im}[\mathrm{Br}(k) \to \mathrm{Br}(X)], \quad \mathrm{Br}_1(X) = \mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})].$$

The subgroup $\mathrm{Br}_1(X) \subset \mathrm{Br}(X)$ is called the *algebraic* Brauer group of X, and the quotient $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is called the *transcendental* Brauer group of X. A particular case of our usual spectral sequence for the structure morphism $X \to \mathrm{Spec}(k)$ is the spectral sequence $H^p(k, H^q(\overline{X}, \mathbb{G}_m)) \Rightarrow H^{p+q}(X, \mathbb{G}_m)$. It gives rise to the exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})^\Gamma \to \mathrm{Br}(k) \to \mathrm{Br}_1(X) \to H^1(k, \mathrm{Pic}(\overline{X})) \to \ldots, \quad (1.9)$$

whose next terms are $H^3(k, \bar{k}^*) \to H^3(X, \mathbb{G}_m)$. The last map, as well as the map $\mathrm{Br}(k) \to \mathrm{Br}_1(X)$, has a retraction if X has a k-point, because a k-point is naturally a section of the structure morphism $X \to \mathrm{Spec}(k)$. We also see that in this case $\mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\overline{X})^\Gamma$ is an isomorphism.

Raising functions to the ℓ^n-th power, where ℓ is a prime not equal to the characteristic of k, gives rise to the Kummer exact sequence

$$1 \to \mu_{\ell^n} \to \mathbb{G}_m \to \mathbb{G}_m \to 1.$$

At the level of H^2 it gives an exact sequence

$$0 \to \mathrm{Pic}(\overline{X})/\ell^n \to H^2(\overline{X}, \mu_{\ell^n}) \to \mathrm{Br}(\overline{X})[\ell^n] \to 0.$$

(This in particular implies that $\mathrm{Br}(\mathbb{A}^n_{\bar{k}}) = \mathrm{Br}(\mathbb{P}^n_{\bar{k}}) = 0$, because the affine space has trivial cohomology groups in positive dimension, whereas the cohomology of the projective space is spanned by the classes of projective subspaces of smaller dimension.) Since $\mathrm{Pic}^0(\overline{X})$ is divisible by powers of ℓ, this gives rise to

$$0 \to \mathrm{NS}(\overline{X}) \otimes \mathbb{Z}_\ell \to H^2(\overline{X}, \mathbb{Z}_\ell(1)) \to T_\ell \mathrm{Br}(\overline{X}) \to 0,$$

where $\mathrm{NS}(\overline{X})$ is the Néron–Severi group of $\overline{X}$, and $T_\ell \mathrm{Br}(\overline{X}) = \varprojlim \mathrm{Br}(\overline{X})[\ell^n]$ is the Tate module of the Brauer group of $\overline{X}$. The Tate module is a free $\mathbb{Z}_\ell$-module. We deduce an isomorphism of abelian groups $T_\ell \mathrm{Br}(\overline{X}) \cong \mathbb{Z}_\ell^{b_2 - \rho}$, where $b_2 = \mathrm{rk}\, H^2(\overline{X}, \mathbb{Q}_\ell)$ is the second Betti number of $\overline{X}$ and $\rho = \mathrm{rk}\, \mathrm{NS}(\overline{X})$ is the rank of the Néron–Severi group. Therefore, the divisible subgroup $\mathrm{Br}(\overline{X})\{\ell\}_{\mathrm{div}}$ is isomorphic to $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{b_2-\rho}$.

If we repeat the argument at the level of H^3, we see that the Kummer sequence identifies $\mathrm{Br}(\overline{X})\{\ell\}/\mathrm{Br}(\overline{X})\{\ell\}_{\mathrm{div}}$ with the kernel of a map from $H^3(\overline{X},\mathbb{Z}_\ell)$ to the ℓ-adic Tate module of $H^3(\overline{X},\mathbb{G}_m)$. Since the former is torsion-free, we get an isomorphism

$$\mathrm{Br}(\overline{X})\{\ell\}/\mathrm{Br}(\overline{X})\{\ell\}_{\mathrm{div}} \xrightarrow{\sim} H^3(\overline{X},\mathbb{Z}_\ell)_{\mathrm{tors}}.$$

(The same argument as the one showing that the torsion subgroup of $\mathrm{NS}(\overline{X})\{\ell\}$ coincides with the torsion subgroup of $H^2(\overline{X},\mathbb{Z}_\ell(1))$.) All in all, we obtain a short exact sequence of Galois modules (see [GBIII, §8])

$$0 \longrightarrow (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{b_2-\rho} \longrightarrow \mathrm{Br}(\overline{X})\{\ell\} \longrightarrow H^3(\overline{X},\mathbb{Z}_\ell(1))_{\mathrm{tors}} \longrightarrow 0. \qquad (1.10)$$

If the ground field k has characteristic zero, for almost all ℓ the group $H^3(\overline{X},\mathbb{Z}_\ell(1))$ is torsion-free: this is a consequence of the comparison theorem between étale cohomology and Betti cohomology, see [9, Thm. III.3.12]. Then we obtain an exact sequence

$$0 \longrightarrow (\mathbb{Q}/\mathbb{Z})^{b_2-\rho} \longrightarrow \mathrm{Br}(\overline{X}) \longrightarrow \bigoplus_\ell H^3(\overline{X},\mathbb{Z}_\ell(1))_{\mathrm{tors}} \longrightarrow 0, \qquad (1.11)$$

where the direct sum is a finite abelian group. Hence (1.11) represents $\mathrm{Br}(\overline{X})$ as an extension of a finite group by a divisible group. When $k\subset\mathbb{C}$, this finite group is isomorphic to the torsion subgroup of the Betti cohomology group $H^2(X(\mathbb{C}),\mathbb{Z})$.

1.4. **Exercises.** 1. Use Tsen's theorem and (1.9) to find $\mathrm{Br}(\mathbb{A}^n_k)$ and $\mathrm{Br}(\mathbb{P}^n_k)$.

2. Let k be a field of characteristic zero. Use Tsen's theorem, Hilbert's theorem 90 and the exact sequence of Γ-modules

$$0\to \bar{k}(t)^*/\bar{k}^*\to \mathrm{Div}(\mathbb{P}^1_{\bar{k}})\to \mathrm{Pic}(\mathbb{P}^1_{\bar{k}})\to 0$$

to deduce Faddeev's exact sequence

$$0\to \mathrm{Br}(k)\to \mathrm{Br}(k(t))\to \bigoplus_{p\in(\mathbb{P}^1_k)^1} \mathrm{Hom}(\Gamma_{k_p},\mathbb{Q}/\mathbb{Z})\to \mathrm{Hom}(\Gamma_k,\mathbb{Q}/\mathbb{Z})\to 0. \qquad (1.12)$$

Here $(\mathbb{P}^1_k)^1$ means the set of codimension 1 points of $\mathbb{P}^1_k$, i.e. the closed points. These are the point ∞ for which $k_\infty = k$, and the zero sets of irreducible monic polynomials $p(t)\in k[t]$ for which $k_p = k[t]/(p(t))$. The third arrow is the direct sum of residue maps, and the fourth arrow is the sum of so called transfer maps.

The sequence (1.12) treats all closed points equally. If we take out the point at infinity, then we can rewrite (1.12) as

$$0\to \mathrm{Br}(k)\to \mathrm{Br}(k(t))\to \bigoplus_{p(t)} \mathrm{Hom}(\Gamma_{k_p},\mathbb{Q}/\mathbb{Z})\to 0, \qquad (1.13)$$

where the sum is over all monic irreducible polynomials in $k[t]$.

For 2-torsion we have $H^1(k_p, \mathbb{Q}/\mathbb{Z})[2] = k_p^*/k_p^{*2}$, so we obtain

$$0 \to \mathrm{Br}(k)[2] \to \mathrm{Br}(k(t))[2] \to \oplus_p\, k_p^*/k_p^{*2} \to 0. \tag{1.14}$$

Remark One can construct a section of the third arrow in (1.14) as follows. Let $\theta \in k_p$ be a root of $p(t) = 0$, so that $p(t) = N_{k(D)/k}((t-\theta))$. Take any $a \in k_p^*$ and consider the quaternion algebra $A = Q(a, t-\theta)$ over $k_p(t)$. Prove that its only non-trivial residue in the affine line is at $t = \theta$; it is the class of a in k_p^*/k_p^{*2}. From the definitions of the corestriction and residue maps one can deduce that the residue of $\mathrm{cores}_{k_p/k}[A] \in \mathrm{Br}(k(t))[2]$ is non-zero only at the closed point $p(t) = 0$, where it is the class of a in k_p^*/k_p^{*2}.

3. Prove that the Brauer group of a smooth geometrically integral curve C with a k-point is isomorphic to $\mathrm{Br}(k) \oplus H^1(k, J)$, where J is the Jacobian of C and the splitting is given by the choice of a k-point.

4. Convince yourself that if X is a K3 surface or an abelian variety, then $\mathrm{Br}(\overline{X}) \cong (\mathbb{Q}/\mathbb{Z})^{b_2-\rho}$ (in this case the $\mathbb{Z}_\ell$-cohomology is torsion-free).

5. Compute the Brauer group of a conic $C(a,b)$ by making the terms and maps in (1.9) explicit. (*Hint*: use the exercise from Subsection 1.1.) Use this exact sequence to prove that a conic with a k-point is isomorphic to $\mathbb{P}^1_k$.

6. Compute the Brauer group of a smooth projective quadric of dimension $d \geq 3$. (*Hint*: The geometric Picard group is generated by the hyperplane section.)

7† Compute the Brauer group of a smooth projective quadric of dimension 2. (*Hint*: Consider separately the cases when the discriminant is a square or not. In the second case argue as in the previous exercise.)

8†† Compute the Brauer group of a genus 1 curve without a k-point assuming that $H^3(k, \bar{k}^*) = 0$. (*Hint*: You can assume that the curve is an n-covering of its Jacobian for some n.)

2. Brauer–Manin obstruction and the arithmetic of conic bundles

2.1. **Smooth proper models of conic bundles.** *References*: [14]

Consider a conic over the field $k(t)$, where $char(k)$ is not 2. We can diagonalise a quadratic form that defines it,

$$a_0(t)x_0^2 + a_1(t)x_1^2 + a_2(t)x_2^2 = 0, \quad a_0, a_1, a_2 \in k(t), \tag{2.1}$$

and multiply the coefficients by a common multiple so that $a_0(t), a_1(t), a_2(t) \in k[t]$ and $a_0(t)a_1(t)a_2(t)$ is a separable polynomial. Let us treat $(x_0 : x_1 : x_2)$ as homogeneous coordinates in the projective plane and define the surface $X' \subset \mathbb{A}^1 \times \mathbb{P}^2$ by the above equation. A straightforward calculation with partial derivatives in local coordinate shows that X' is smooth (this can be done over $\bar{k}$).

We would like to build a projective surface, with a morphism to $\mathbb{P}^1_k$. Without loss of generality we can assume that the degrees $d_i = \deg(a_i)$ *have the same parity*. Indeed, replacing t by $t - c$ for some $c \in k$ we can assume that

$a_0(0)a_1(0)a_2(0) \neq 0$. Now let $t = 1/v$. This gives an equation similar to (2.1) with coefficients that are polynomials in v whose degrees have the same parity. We now assume this and revert to denoting the parameter by t.

Let T be another variable. The coefficients of the equation

$$T^{d_0}a_0(1/T)X_0^2 + T^{d_1}a_1(1/T)X_1^2 + T^{d_2}a_2(1/T)X_2^2 = 0 \qquad (2.2)$$

are polynomials in T (reciprocal polynomials to the original coefficients). Let $X'' \subset \mathbb{A}^1 \times \mathbb{P}^2$ be the closed subset it defines, where T is a coordinate in $\mathbb{A}^1$, and $(X_0 : X_1 : X_2)$ are homogeneous coordinates in the plane. Note that X'' is smooth since the product of three coefficients is a separable polynomial in T. We think of these two affine lines as affine pieces of $\mathbb{P}^1$, so that the coordinates are related by $T = 1/t$. Let n_i be the integer such that $2n_i = d_i$ or $2n_i = d_i+1$, where $i = 0, 1, 2$. The substitution $x_i = T^{n_i}X_i$ tranforms (2.1) in (2.2). Hence the restrictions of X' and X'' to the intersection of two affine pieces of $\mathbb{P}^1$ are isomorphic and we can glue X' and X'' to define a smooth surface X. It comes equipped with the morphism $\pi : X \to \mathbb{P}^1$ that extends the first projections $X' \to \mathbb{A}^1$ and $X'' \to \mathbb{A}^1$. The fibres of π are projective conics. Note that the fibre at $t = \infty$ $(T = 0)$ is a smooth conic.

This is sometimes called a *standard model*, in these notes a conic bundle is always assumed to be a standard model. (A standard model is not unique.) So by definition any conic bundle is smooth and proper.

Structure of bad fibres. 1. The degenerate fibres are the fibres over the roots of $a_0(t)a_1(t)a_2(t)$. (If we allow the case when the parities of the degrees of coefficients are not the same, the fibre $t = \infty$ is also singular).

2. Each degenerate fibre is geometrically a pair of transversal lines meeting at one point.

3. If $p(t)$ is a monic irreducible factor of $a_0(t)$, then the components of the fibre of π at the closed point $p(t) = 0$ are defined over the extension of $k_p = k[t]/(p(t))$ given by the square root of the image of $-a_1(t)a_2(t)$ in $k[t]/(p(t))$. Similarly for the prime factors of a_1 and a_2, and, in the unequal parity case, for the fibre at infinity. We denote this image by α_p.

Definition 2.1. *The discriminant of $\pi : X \to \mathbb{P}^1_k$ is the product of monic irreducible polynomials $p(t)$ dividing $a_0(t)a_1(t)a_2(t)$ such that α_p is not a square in k_p (equivalently, the fibre X_p is an irreducible and singular). The degree of the discriminant is called rhe rank of $\pi : X \to \mathbb{P}^1_k$.*

Note that the generic fibre of π is the conic $a_0(t)x_0^2 + a_1(t)x_1^2 + a_2(t)x_2^2 = 0$ over the field $k(t)$. This is the conic attached to the quaternion algebra

$$Q = Q(-a_0a_1, -a_0a_2) = Q(-a_0a_1, -a_1a_2).$$

Lemma 2.2. *The class of α_p in k_p^*/k_p^{*2} equals the residue $Res_p(Q)$. We have $\prod_p N_{k_p/k}(\alpha_p) \in k^{*2}$.*

Proof The first statement follows from the explicit description of the residue of quaternion algebras. For the second statement note that by Exercise 2 the element $[Q] - \sum_p A_p \in \mathrm{Br}(k(t))$, where $A_p = \mathrm{cores}_{k_p/k}(\alpha_p, t-\theta_p)$, has trivial residues at all the closed points of $\mathbb{A}^1_k$. By the exact sequence (1.14) $[Q] - \sum_p A_p \in \mathrm{Br}(k)$, and so this element has trivial residue at ∞. Let us assume without loss of generality, as we always do, that the fibre of $\pi : X \to \mathbb{P}^1_k$ at ∞ is smooth. Then the residue of Q at ∞ is trivial, so that the sum of residues of A_p for all p is trivial too. A calculation shows that the residue of A_p at ∞ is $N_{k_p/k}(\alpha_p)$, hence the second statement. $\square$

2.2. **The Brauer group of a conic bundle.**

Theorem 2.3. *Let $\pi : X \to \mathbb{P}^1_k$ be a conic bundle, where k is a field of characteristic different from 2. Let Q be the quaternion algebra defined by the generic fibre of π. Then $\mathrm{Br}(X)$ is isomorphic to the subgroup of $\mathrm{Br}(k(t))$ consisting of the elements whose residue at each closed point $M \in \mathbb{P}^1_k$ is 0 or $Res_M(Q)$, modulo the subgroup generated by $[Q] \in \mathrm{Br}(k(t))$.*

Proof Since the Picard group of a conic is isomorphic to $\mathbb{Z}$ as a Galois module, the Brauer group of the generic fibre is the surjective image of $\mathrm{Br}(k(t))$. In fact, by the solution to Exercise 4 it is the quotient of $\mathrm{Br}(k(t))$ by the subgroup generated by $[Q]$. So we need to determine which elements $A \in \mathrm{Br}(k(t))$ come from $\mathrm{Br}(X)$ when pulled back to $\mathrm{Br}(k(X))$. By the exact sequence (1.8) this is equivalent to the triviality of the residues of A at each divisor of X. Since $Res_M(A) \neq 0$ for only finitely many closed points $M \in \mathbb{P}^1_k$, we have $A \in \mathrm{Br}(U)$ where U is a dense open subset of $\mathbb{P}^1_k$. Thus $\pi^*A \in \mathrm{Br}(\pi^{-1}(U))$, so only vertical divisors $D \subset X$ can give rise to non-zero residues of π^*A. They are of two kinds: smooth and singular fibres of π.

Suppose $M \in \mathbb{P}^1_k$ is such that the fibre X_M is a smooth conic. Then we have an extension of local rings $\mathcal{O}_{\mathbb{P}^1_k,M} \subset \mathcal{O}_{X,X_M}$ such that the maximal ideal of $\mathcal{O}_{\mathbb{P}^1_k,M}$ generates the maximal ideal of $\mathcal{O}_{X,X_M}$, in other words, the valuations are compatible. In this situation the explicit description of the residue in the case of a complete local ring shows that $Res_{k(X_M)}(\pi^*A)$ is the image of $Res_M(A)$ under the natural map $k_M^*/k_M^{*2} \to k(X_M)^*/k(X_M)^{*2}$. But if a constant from k_M is a square in the function field $k(X_M)$ of the smooth conic X_M, then it is already a square in k_M. It follows that we must have $Res_M(A) = 0$. Note that in this case $Res_M(Q) = 0 \in k_M^*/k_M^{*2}$.

If X_M is singular and $Res_M(Q) = 0$, then X_M is a singular conic which is a union of two projective lines meeting at a point. The above argument can be applied to any of these components, with the same conclusion.

Finally, if X_M is singular but $\alpha_M = Res_M(Q) \neq 0$, then X_M consists of two conjugate projective lines individually defined over $k_M(\sqrt{\alpha_M})$. The same arguments as above show that $Res_{k(X_M)}(\pi^*A)$ is the image of $Res_M(A)$ under the natural map $k_M^*/k_M^{*2} \to k(X_M)^*/k(X_M)^{*2}$. Now $k(X_M) = k_M(\sqrt{\alpha_M})(x)$, where x is an independent variable. It follows that $Res_{k(X_M)}(\pi^*A) = 0$ if and only if $Res_M(A) = 0$ or $Res_M(A) = \alpha_M$. QED

Corollary 2.4. *In the notation of the theorem $\mathrm{Br}(X)$ modulo the image of $\mathrm{Br}(k)$ is a finite abelian group of exponent 2. It is the quotient of the subspace* *of the $\mathbb{F}_2$-vector space with the basis given by the closed points $M \in \mathbb{P}^1_k$ with $Res_M(Q) \neq 0$, defined by the condition*

$$\prod_M N_{k_M/k}(Res_M(Q))^{n_M} = 1 \in k^*/k^{*2},$$

modulo the 1-dimensional subgroup generated by the vector $(1, \dots, 1)$.

2.3. **Brauer–Manin obstruction.** *References*: [13]

From now on k will be a number field. Artin–Hasse–Brauer–Noether theorem says that a CSA over k is trivial if and only if it is trivial over each completion k_v of k. Thus $\mathrm{Br}(k)$ injects into the direct sum of $\mathrm{Br}(k_v)$ for all completions of k. The cokernel of this injective map is identified by the sum of local invariants with $\mathbb{Q}/\mathbb{Z}$, so there is an exact sequence

$$0 \to \mathrm{Br}(k) \to \bigoplus_v \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z} \to 0,$$

where the third arrow is $\sum \mathrm{inv}_v$ (including the local invariants at the real places). If X is a projective variety we write $X(\mathbb{A}_k) = \prod_v X(k_v)$. The Brauer–Manin pairing is

$$X(\mathbb{A}_k) \times \mathrm{Br}(X)/\mathrm{Br}(k) \to \mathbb{Q}/\mathbb{Z},$$

defined by sending the adelic point (M_v) and $A \in \mathrm{Br}(X)$ to $\sum_v \mathrm{inv}_v(A(M_v))$. Its left kernel is called the Brauer–Manin set and is denoted by $X(\mathbb{A}_k)^{\mathrm{Br}}$. We have $X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}}$.

Note that the map $X(k_v) \to \mathbb{Q}/\mathbb{Z}$ that sends a point P to $\mathrm{inv}_v(P)$ is continuous for the topology of k_v and takes its values in $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$, and so is locally constant. We obtain that the topological closure of $X(k)$ is also contained in the Brauer–Manin set.

The best we can possibly hope for is when $X(k)$ is dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$. Such results have been obtained for conic bundles. When the rank is small, best possible results have been proved already in the 80's and early 90's.

Theorem 2.5 (Colliot-Thélène, Sansuc, Swinnerton-Dyer, Salberger, S.)**.** *Let k be a number field, and let $\pi : X \to \mathbb{P}^1_k$ be a conic bundle of rank at most 5. Then the closure of $X(k)$ in the space of adèles of X is the Brauer–Manin set of X.*

This is also true for some specific conic bundles of degree 6 (Swinnnerton-Dyer), but the general case of rank 6 and higher is open. For the proofs see [13, Ch. 7]. Colliot-Thélène and Sansuc conjectured that $X(k)$ is dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$ for any smooth, projective, *geometrically rational* surface X over a number field k.

Example (Iskovskikh, Sansuc) Here is an explicit example of the Brauer–Manin obstruction. Consider the conic bundle $\pi : X_c \to \mathbb{P}^1_{\mathbb{Q}}$

$$x^2 + 3y^2 = (c - t^2)(t^2 - c + 1)z^2 \tag{2.3}$$

where $c \in \mathbb{Z}$, $c \neq 0$, $c \neq 1$. One sees immediately that $X_c(\mathbb{R}) \neq \emptyset$ if and only if $c > 1$. The local solubility of (2.3) in $\mathbb{Q}_p$ for any prime p imposes no restriction on c. This is easily seen for $p \neq 3$ by setting $t = p^{-1}$ and using the fact that a unit is a norm for an unramified extension. For $p = 3$ the solubility of (2.3) is established by a case by case computation.

Consider the classes $(-3, c-t^2)$ and $(-3, t^2-c+1)$ in $\mathrm{Br}(\mathbb{Q}(t))[2]$. Then

$$A = \pi^*(-3, c-t^2) = \pi^*(-3, t^2-c+1) \in \mathrm{Br}(\mathbb{Q}(X_c)),$$

because $(c-t^2)(t^2-c+1)$ is a rational function on X_c which is a norm for the quadratic extension of $\mathbb{Q}(X_c)$ given adjoining the square root of -3 (see the appendix for details). By Theorem 2.3 this algebra defines a class in $\mathrm{Br}(X_c)$ and by Corollary 2.4 this class generates $\mathrm{Br}(X_c)$ modulo the image of $\mathrm{Br}(\mathbb{Q})$. Thus to compute the Brauer–Manin obstruction we only need to compute the sum $\sum_v \mathrm{inv}_v(A(P_v))$, where $P_v \in X_c(\mathbb{Q}_v)$.

Statement 1*: If $v \neq 3$, then* $\mathrm{inv}_v(A(P_v)) = 0$ *for any point* $P_v \in X_c(\mathbb{Q}_v)$. This value is locally constant in the v-adic topology, hence we may assume that P_v is not contained in the fibre at infinity or in any of the singular fibres, that is, $(c-t^2)(t^2-c+1) \neq 0$. We must prove that $c-t^2$ is locally a norm for the extension $\mathbb{Q}(\sqrt{-3})/\mathbb{Q}$. For $\mathbb{Q}_v = \mathbb{R}$ this easily follows since $c - t^2 > 0$. For a finite $v \neq 3$ we only have to consider the case when p is inert for $\mathbb{Q}(\sqrt{-3})/\mathbb{Q}$. We have two possibilities: $v(t) < 0$ and $v(t) \geq 0$. In the first case $v(c-t^2)$ is even, hence this is the product of a unit, which is a norm for the unramified extension $\mathbb{Q}_v(\sqrt{-3})$, and an even power of a uniformizer, which is trivially a norm for any quadratic extension. Since $(c-t^2) + (t^2-c+1) = 1$, in the second case either $v(c-t^2) = 0$, then $c-t^2$ is a norm, or $v(t^2-c+1) = 0$. Then from the equation of X_c it follows that $c-t^2$ is a norm multiplied by a unit, hence is a norm.

Statement 2*: For $v = 3$ and $c = 3^{2n+1}(3m+2)$ we have* $\mathrm{inv}_3(A(P_3)) = \frac{1}{2}$ *for any point $P_3 \in X_c(\mathbb{Q}_3)$, whereas for other values of c the local invariant takes both values* 0 *and* $\frac{1}{2}$. This purely local computation is omitted here.

Conclusion. When the sum of local invariants is never 0, the Manin obstruction tells us that no $\mathbb{Q}$-point can exist on X_c. This happens for $c = 3^{2n+1}(3m+2)$, whereas X_c has adelic points for any $c > 1$. Theorem 2.5 implies that in all the other cases for $c \in \mathbb{Z}$, $c > 1$, the surface X_c contains a $\mathbb{Q}$-point.

2.4. **Digression: the Green–Tao–Ziegler theorem.** In a series of papers Green–Tao [5, 6] and Green–Tao–Ziegler [7] proved the generalized Hardy–Littlewood conjecture in the finite complexity case. The following qualitative statement is [5, Cor. 1.9].

Theorem 2.6 (Green, Tao, Ziegler)**.** *Let $L_1(x,y), \dots, L_r(x,y) \in \mathbb{Z}[x,y]$ be pairwise non-proportional linear forms, and let $c_1, \dots, c_r \in \mathbb{Z}$. Assume that for each prime p, there exists $(m,n) \in \mathbb{Z}^2$ such that p does not divide $L_i(m,n) + c_i$ for any $i = 1, \dots, r$. Let $K \subset \mathbb{R}^2$ be an open convex cone containing a point $(m,n) \in \mathbb{Z}^2$ such that $L_i(m,n) > 0$ for $i = 1, \dots, r$. Then there exist infinitely many pairs $(m,n) \in K \cap \mathbb{Z}^2$ such that $L_i(m,n) + c_i$ are all prime.*

A famous particular case is the system of linear forms $x, x+y, \ldots, x+(r-1)y$. In this case the result is the existence of arithmetic progressions in primes of arbitrary length r.

We shall use the following easy corollary of Theorem 2.6. For a finite set of rational primes S we write $\mathbb{Z}_S = \mathbb{Z}[S^{-1}]$. This is the set of rational numbers whose denominators are divisibel only by the primes from S.

Proposition 2.7. *Suppose that we are given $(\lambda_p, \mu_p) \in \mathbb{Q}_p^2$ for p in a finite set of primes S, and a positive real constant C. Let $e_1, \ldots, e_r$ be pairwise different elements of $\mathbb{Z}_S$. Then there exist infinitely many pairs $(\lambda, \mu) \in \mathbb{Z}_S^2$ and pairwise different primes $p_1, \ldots, p_r$ not in S such that*

(1) $\lambda > C\mu > 0$;
(2) (λ, μ) *is close to* (λ_p, μ_p) *in the p-adic topology for* $p \in S$;
(3) $\lambda - e_i\mu = p_i u_i$, *where* $u_i \in \mathbb{Z}_S^*$, *for* $i = 1, \ldots, r$.

This follows from Theorem 2.6 via the Chinese remainder theorem. The proof is boring and so we omit it here; it can be found in [3].

Proposition 2.8. *Let $e_1, \ldots, e_r$ be pairwise different rational numbers. Let S be a finite set of primes containing 2 and the prime factors of the denominators of $e_1, \ldots, e_r$. Suppose that we are given $\tau_p \in \mathbb{Q}_p$ for $p \in S$ and a positive real constant C. Then there exist pairwise different primes $p_1, \ldots, p_r$ not in S and $\tau \in \mathbb{Q}$ such that*

(1) τ *is arbitrarily close to* τ_p *in the p-adic topology, for* $p \in S$;
(2) $\tau > C$;
(3) $\mathrm{val}_p(\tau - e_i) \leq 0$ *for any* $p \notin S \cup \{p_i\}$, $i = 1, \ldots, r$;
(4) $\mathrm{val}_{p_i}(\tau - e_i) = 1$ *for any* $i = 1, \ldots, r$;
(5) *for any integer α divisible only by the primes from S and such that*
$$\sum_{p \in S} \mathrm{inv}_p(\alpha, \tau_p - e_i) = 0 \in \mathbb{Q}/\mathbb{Z}$$
for some i, the prime p_i splits in $\mathbb{Q}(\sqrt{\alpha})/\mathbb{Q}$.

Proof. By increasing the list of e_i's, we may assume $r \geq 2$. We then apply Proposition 2.7 to $(\lambda_p, \mu_p) = (\tau_p, 1)$ for $p \in S$. This produces $(\lambda, \mu) \in \mathbb{Z}_S^2$ such that $\tau = \lambda/\mu$ satisfies all the properties in the proposition. Indeed, (1) and (2) are clear. For $p \notin S$ we have $\mathrm{val}_p(\mu) \geq 0$. For $p \neq p_i$ we have $\mathrm{val}_p(\lambda - e_i\mu) = 0$, so that
$$\mathrm{val}_p(\tau - e_i) = \mathrm{val}_p(\lambda - e_i\mu) - \mathrm{val}_p(\mu) \leq 0,$$
which proves (3). From the assumption $r \geq 2$ we can deduce that $\mathrm{val}_{p_i}(\mu) = 0$ for $i = 1, \ldots, r$. Otherwise $\mathrm{val}_{p_i}(\mu) > 0$, which would imply $\mathrm{val}_{p_i}(\lambda) > 0$; we would obtain $\mathrm{val}_{p_i}(\lambda - e_j\mu) > 0$ and therefore $p_i = p_j$ for some $j \neq i$, thus contradicting the hypothesis that the primes $p_1, \ldots, p_r$ are pairwise different. This proves (4). Since (λ, μ) is close to $(\tau_p, 1)$ in the p-adic topology for $p \in S$, by continuity we have
$$\sum_{p \in S} \mathrm{inv}_p(\alpha, \lambda - e_i\mu) = 0.$$

We also have $\lambda - e_i\mu > 0$, hence $\mathrm{inv}_{\mathbb{R}}(\alpha, \lambda - e_i\mu) = 0$. By global reciprocity law (essentially, Gauss reciprocity) this implies

$$\sum_{p \notin S} \mathrm{inv}_p(\alpha, \lambda - e_i\mu) = 0.$$

Since α is odd and comprime to the primes in S, we have $\mathrm{inv}_p(\alpha, \lambda - e_i\mu) = 0$ for any prime $p \notin S \cup \{p_i\}$, because in this case $\mathrm{val}_p(\lambda - e_i\mu) = 0$. Thus

$$\mathrm{inv}_{p_i}(\alpha, \lambda - e_i\mu) = 0.$$

But $\mathrm{val}_{p_i}(\lambda - e_i\mu) = 1$ hence p_i must split in $\mathbb{Q}(\sqrt{\alpha})$, so that (5) is proved. □

2.5. **Conic bundles over $\mathbb{Q}$ with degenerate $\mathbb{Q}$-fibres.** In the rest of this lecture we assume that $Res_M(Q) \neq 0$ only for k-points M, i.e. for closed points with the residue field k. Let $e_1, \ldots, e_n \in k$ be the coordinates of these points. As usual, we assume that the fibre at ∞ is smooth. By a slight abuse of notation we represent $Res_{e_i}(Q)$ by $\alpha_i \in k^*$ defined up to a square. Define $A_i = (\alpha_i, t - e_i) \in \mathrm{Br}(k(t))$. It is clear that A_i is ramified at e_i and ∞, but nowhere else. From Theorem 2.3 we see that $\mathrm{Br}(X)$ modulo $\mathrm{Br}(k)$ is the quotient by $\mathbb{Z}/2[Q]$ of the subgroup of $\mathbb{Z}/2A_1 \oplus \ldots \oplus \mathbb{Z}/2A_n$ given by the condition

$$\prod_{i=1}^{n} \alpha_i^{n_i} = 1 \in k^*/k^{*2}. \tag{2.4}$$

Here is the main theorem of this lecture. By a $\mathbb{Q}$-fibre of $\pi : X \to \mathbb{P}^1$ we understand a closed fibre above a $\mathbb{Q}$-point of $\mathbb{P}^1$.

Theorem 2.9 (Browning, Matthiesen, AS, based on Green, Tao, Ziegler)**.** *Let X be a conic bundle over $\mathbb{Q}$ such that all fibres that are irreducible singular conics are $\mathbb{Q}$-fibres. Then the closure of $X(k)$ in the space of adèles of X is the Brauer–Manin set of X.*

Note that the rank in this statement can be arbitrary. This result is interesting because it implies the existence of many (in fact, a Zariski dense set of) solutions in $\mathbb{Q}$ besides the obvious ones (singular points of the sigular fibres). We give a proof following [3].

Proof. Let X_i be the fibre above the point $e_i \in \mathbb{A}^1(\mathbb{Q})$, for $i = 1, \ldots, r$. We have seen that every element of $\mathrm{Br}(X)$ is of the form $\sum n_i \pi^* A_i + A_0$ for some n_i satisfying (2.4) and $A_0 \in \mathrm{Br}(\mathbb{Q})$.

Assume that $X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}} \neq \emptyset$, otherwise there is nothing to prove. Pick any $(M_p) \in X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}}$, where M_0 is a point in $X(\mathbb{R})$. By a small continuous deformation we can assume that M_p does not belong to any of the fibres $X_1, \ldots, X_r$. Here we use the fact $X(\mathbb{Q}_p)$ is a smooth p-adic manifold, so each point has an open neighbourhood homeomorphic to a disc.

We include the real place in the finite set of places S where we need to approximate. The set of real points M_0 in a small neighbourhood of M_0 for which $\pi(M_0) \in \mathbb{P}^1(\mathbb{Q})$ is dense in this neighbourhood, and so it is enough to approximate adelic points (M_p) such that $\pi(M_0) \in \mathbb{P}^1(\mathbb{Q})$. By a change of variables we then assume that $\pi(M_0)=\infty$. By another small deformation of M_p for each prime p we can further assume that $\pi(M_p)\neq\infty$ when $p\neq 0$.

We include in S the primes of bad reduction for X. We ensure that $e_i\in\mathbb{Z}_S$ for each $i=1,\dots,r$, $e_i-e_j\in\mathbb{Z}_S^*$ for all $i\neq j$, and no prime outside of S is ramified in any of the fields $\mathbb{Q}(\sqrt{\alpha_i})$.

We claim that if $A=\sum_{i=1}^r n_iA_i$ has trivial residue at ∞, which happens if and only if $\pi^*A\in \mathrm{Br}(X)$, then $\mathrm{inv}_p\big(A(\pi(N_p))\big)=0$ for any $p\notin S$ and any $N_p\in X(\mathbb{Q}_p)$. Indeed, each $\alpha_i\in\mathbb{Z}_p^*$ so this is clear if $\pi(N_p)$ reduces modulo p to any point other than $e_1,\dots,e_r$, because A can have non-zero residues only at these points. If $\pi(N_p)$ reduces modulo p to some e_i, then N_p reduces to a *smooth* $\mathbb{F}_p$-point of the singular fibre X_i. But any smooth $\mathbb{F}_p$-point of X_i belongs to one of the two geometric irreducible components of X_i, hence these components are actually defined over $\mathbb{F}_p$, so $\mathbb{Q}(\sqrt{\alpha_i})$ is split at p. Now the reduction of α_i modulo p is a square, and this means $\mathrm{inv}_p\big(A(\pi(N_p))\big)=0$.

Lemma 2.10 (Harari's 'formal lemma')**.** *There exist a prime $\ell\notin S$ and a point $M_\ell\in X(\mathbb{Q}_\ell)$ such that for each $i=1,\dots,r$ we have*

$$\sum_{p\in S\cup\{\ell\}}\mathrm{inv}_p\big(A_i(\pi(M_p))\big)=0. \tag{2.5}$$

Proof of lemma. By the last paragraph before the lemma the fact that $(M_p)\in X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}}$ implies

$$\sum_{p\in S}\mathrm{inv}_p\Big(\sum_{i=1}^r n_iA_i(\pi(M_p))\Big)=0$$

whenever we have $\pi^*(\sum_{i=1}^r n_iA_i)\in\mathrm{Br}(X)$, which is equivalent to the condition $\prod_{i=1}^r\alpha_i^{n_i}\in k^{*2}$.

If $\sum_{p\in S}\mathrm{inv}_p\big(A_i(\pi(M_p))\big)=0$ for all i, we are done.

If (2.5) does not hold, then for some $B=\sum_{i=1}^r m_iA_i$ we have

$$\sum_{p\in S}\mathrm{inv}_p\big(B(\pi(M_p))\big)=1/2.$$

Let K be the extension of $\mathbb{Q}$ obtained by adjoining the square roots of all $\prod_{i=1}^r\alpha_i^{s_i}$ for which $\sum_{p\in S}\mathrm{inv}_p\big(\sum_{i=1}^r s_iA_i(\pi(M_p))\big)=0$. It is clear that $\mathbb{Q}(\sqrt{b})$ is not contained in K. Using Dirichlet's theorem on primes in an arithmetic progression we can find a prime ℓ outside of S such that each ℓ is totally split in K and inert in $\mathbb{Q}(\sqrt{b})$.

The surface X has good reduction outside of S, so for any $p\notin S$ the fibre at infinity X_∞ is a smooth conic. Any smooth conic over $\mathbb{F}_p$ contains $\mathbb{F}_p$-points. By Hensel's lemma any such point lifts to a $\mathbb{Q}_p$-point of X. We choose such a point M_ℓ for the prime ℓ. The point $\pi(M_\ell)\in\mathbb{P}^1_{\mathbb{Q}_\ell}$ reduces to ∞ modulo ℓ, so we have $\mathrm{inv}_\ell\big(B(\pi(M_\ell))\big)=1/2$ as the Legendre symbol $\big(\frac{b}{\ell}\big)=-1$.

By our choice of ℓ we see that $\sum_{p\in S} \mathrm{inv}_p\big(\sum_{i=1}^r s_iA_i(\pi(M_p))\big) = 0$ implies

$$\sum_{p\in S\cup\{\ell\}} \mathrm{inv}_p\Big(\sum_{i=1}^r s_iA_i(\pi(M_p))\Big) = 0$$

(the residue of such a sum reduces to a square modulo ℓ because K is totally split at ℓ). We also have

$$\sum_{p\in S\cup\{\ell\}} \mathrm{inv}_p\big(B(\pi(M_p))\big) = 0.$$

Thus (2.5) holds for all linear combinations of $A_1,\ldots,A_n$. $\square$

End of proof of the theorem Let τ_p be the coordinate of $\pi(M_p)$, where p is a prime in S_1. An application of Proposition 2.8 produces $\tau\in\mathbb{Q}$ which is an arbitrarily large real number, and is close to τ_p in the p-adic topology for the primes $p\in S_1$.

Let us prove that $X_\tau(\mathbf{A}_\mathbb{Q})\neq\emptyset$. By the inverse function theorem we have $X_\tau(\mathbb{R})\neq\emptyset$ and $X_\tau(\mathbb{Q}_p)\neq\emptyset$ for $p\in S_1$. Thus it remains to consider the following two cases.

$\mathbb{Q}_v=\mathbb{Q}_p$, where $p=p_i$, $i=1,\ldots,r$. Since $\mathrm{val}_{p_i}(\tau-e_i)=1$, the reduction of τ modulo p_i equals the reduction of e_i. For each given value of i the field $\mathbb{Q}(\sqrt{\alpha_i})$ are split at p_i. Hence the reduction of X_τ is the union of two projective lines over $\mathbb{F}_{p_i}$ meeting in one point. Hence it contains smooth $\mathbb{F}_{p_i}$-points. By Hensel's lemma any such point gives rise to a $\mathbb{Q}_{p_i}$-point in X_τ.

$\mathbb{Q}_v=\mathbb{Q}_p$, where $p\notin S_1\cup\{p_1,\ldots,p_r\}$. We have $\mathrm{val}_p(\tau-e_i)\leq 0$ for each $i=1,\ldots,r$, and hence the reduction of τ modulo p is a point of $\mathbb{P}^1(\mathbb{F}_p)$ other than the reduction of any of $e_1,\ldots,e_r$. Hence the reduction of X_τ is a smooth conic over $\mathbb{F}_p$, so it has smooth $\mathbb{F}_p$-points. By Hensel's lemma any such point gives rise to a $\mathbb{Q}_p$-point in X_τ.

By Legendre's theorem $X_\tau(\mathbb{Q})\neq\emptyset$, so we have found a $\mathbb{Q}$-point in X.

To prove weak approximation recall that we can make τ very close to τ_p for $p\in S$. Since $X_\tau\simeq\mathbb{P}^1_k$ we can approximate in this fibre, and so find $\mathbb{Q}$-points on X as close as we like to M_p for $p\in S$. $\square$

Very recently Harpaz and Wittenberg, using a result of Matthiesen, generalised Theorem 2.9 by allowing the fibres of $\pi: X\to\mathbb{P}^1_\mathbb{Q}$ to be arbitrary geometrically rational varieties for which $\mathbb{Q}$-rational points are dense in the Brauer–Manin set.

Colliot-Thélène conjectured that for any number field if X is a smooth projective rationally connected variety (for example, a geometrically rational or unirational variety), then $X(k)$ is dense in the Brauer–Manin set of X. This is a very strong and almost completely open conjecture that has some amazingly strong consequences like the inverse Galois problem.

2.6. **Exercises.** 1. Use Tsen's theorem to prove that any conic bundle $\pi: X\to\mathbb{P}^1_{\bar k}$ over an algebraically closed field $\bar k$ has a section, that is, there is a morphism $\sigma:\mathbb{P}^1_{\bar k}\to X\times_k\bar k$ such that the composition $\pi\sigma$ is the identity map.

Show that $\bar{k}(X)$ is a purely transcendental extension of $\bar{k}$, in other words X is geometrically rational.

2. Deduce Corollary 2.4 from Theorem 2.3 (compare with Exercise 2 from Lecture 1).

3. Let $X \to \mathbb{P}^1_k$ be a standard model of the conic bundle with the affine equation

$$x^2 - ay^2 = (t - e_1) \dots (t - e_n) z^2,$$

where n is even, $a \notin k^{*2}$, $e_i \in k$, $e_i \neq e_j$. Prove that the cokernel of the natural map $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is isomorphic to $(\mathbb{Z}/2)^{n-2}$. Decide what happens for n odd.

4. Let X be a variety over a number field k such that $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$.

(a) Prove that the natural map $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is injective.

(b) Deduce from (a) the Hasse–Minkowski theorem that a conic over a number field satisfies the Hasse principle.

$5^{\dagger\dagger}$ Use Proposition 2.7 to deduce the following result of Lilian Matthiesen. Let $a_i \in \mathbb{Q}^*$, $c_i \in \mathbb{Q}^*$ and $e_i \in \mathbb{Q}$, for $i = 1, \dots, r$, be such that $e_i \neq e_j$ for $i \neq j$. Then the variety $W \subset \mathbb{A}^{2r+2}_{\mathbb{Q}}$ defined by

$$c_i(u - e_i v) = x_i^2 - a_i y_i^2 \neq 0, \quad i = 1, \dots, r, \tag{2.6}$$

satisfies the Hasse principle and weak approximation. (*Hint*: A proof can be found in [3].)

3. Beyond rational varieties

3.1. **K3 surfaces.** A *K3 surface* X over a field k is a geometrically simply connected surface with trivial canonical class. Examples of K3 surfaces are smooth quartic surfaces in $\mathbb{P}^3_k$, double covers of $\mathbb{P}^2_k$ ramified in a smooth sextic curve, complete intersections of three quadrics in $\mathbb{P}^5_k$. A Kummer surface is another example of a K3 surface. This is the minimal desingularisation of the quotient of an abelian surface (e.g. the product of two elliptic curves) by the antipodal involution $P \mapsto -P$. More generally, we shall call Kummer surfaces the desingularisations of quotients of 2-coverings of abelian surfaces by the antipodal involution. Such Kummer surfaces feature in Theorem 3.3 below; they may have no rational points.

Theorem 3.1 (Zarhin, S.)**.** *Let k be a field of characteristic not equal to 2 finitely generated over its prime subfield. The cokernel of the natural map $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is finite if char$(k) = 0$, and finite modulo p-primary torsion if char$(k) = p$.*

The proof uses the Kuga–Satake abelian variety of X as interpreted by Deligne and the various versions of the Tate conjecture proved by Faltings, Zarhin, Madapusi Pera (using Kisin's work on integral models of Shimura varieties).

Corollary 3.2. *For a K3 surface X over a number field k the Brauer–Manin set $X(\mathbb{A}_k)^{\mathrm{Br}}$ is an open and closed subset of $X(\mathbb{A}_k) = \prod X(k_v)$.*

It is quite easy to show that the cokernel of $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is finite for geometrically rational varieties, so in analogy with the conjecture of Colliot-Thélène and Sansuc one is tempted to state the following conjecture.

Conjecture 1. $X(k)$ is a dense subset of $X(\mathbb{A}_k)^{\mathrm{Br}}$.

In particular, if $\mathrm{Br}(X) = \mathrm{Br}(k)$, then $X(k)$ should be dense in $X(\mathbb{A}_k)$ (and so satisfy the Hasse principle and weak approximation). Some numerical evidence for the Hasse principle part of this conjecture has been obtained by Bright. Computations in the direction of weak approximation on Kummer surfaces have been done by Elsenhans and Jahnel.

Conjecture 1 implies the following Conjecture 2.

Conjecture 2. If $X(k)$ is non-empty, then $X(k)$ is Zariski dense, in particular, infinite.

Can something be proved for K3 surfaces with a pencil of curves of genus 1, essentially by doing descent on these curves in a family? For surfaces fibred into genus 1 curves over $\mathbb{P}^1_k$ there is some theoretical evidence for the Hasse principle with the Brauer–Manin obstruction (Colliot-Thélène, S., Swinnerton-Dyer). One difficulty here is that the analogue of the Minkowski–Hasse theorem fails for curves of genus 1: torsors of a given elliptic curve E do not in general satisfy the Hasse principle. The obstruction lies in the Shafarevich–Tate group $\text{Ш}(E)$ formed by the classes of everywhere locally soluble torsors of E. However, if $\text{Ш}(E)$ is finite, then the Cassels–Tate pairing on this group is non-degenerate, so the class of a torsor is zero precisely when it is contained in the kernel of the Cassels–Tate pairing. In his seminal talk at the ICM in Nice Manin interpreted this pairing in terms of the pairing between (a part of) the Brauer group and adelic points on the curve of genus 1. Establishing the finiteness of $\text{Ш}(E)$ (in the cases when it is known) is beyond these lectures. As this is a necessary first step we shall simply assume this finiteness in the following result, which is a particular case of a more general statement.

Theorem 3.3 (Harpaz, S.). *Let $g_1(x)$ and $g_2(x)$ be irreducible polynomials of degree 4 over k, each with the Galois group S_4, such that there exist odd places $w_1 \neq w_2$ with $g_i(x) \in \mathcal{O}_{w_j}[x]$ and $\mathrm{val}_{w_j}(\mathrm{discr}(g_i)) = \delta_{ij}$ for $i, j \in \{1, 2\}$.*

Assume $|\text{Ш}\{2\}| < \infty$ for the Jacobian of the curve $y^2 = cg_i(x)$ for any $c \in k^*$, $i = 1, 2$, when its 2-Selmer rank is 1.

If the K3 surface X given by the affine equation $z^2 = g_1(x)g_2(y)$ is everywhere locally soluble, then $X(k)$ is Zariski dense in X.

If C_i is the genus 1 curve $y^2 = g_i(x)$, then its Jacobian E_i is the elliptic curve $u^2 = f_i(t)$, where f_i is the *cubic resolvent* of g_i. There is a morphism $C_i \to E_i$, which makes C_i into a 2-covering of E_i. The surface X can be obtained as the minimal desingularisation of the quotient of $C_1 \times C_2$ by the involution that changes the sign of y simultaneously for both curves. So X is a Kummer surface in the sence alluded to above.

Much less is known about weak approximation on K3 surfaces.

Theorem 3.4 (Swinnerton-Dyer, Pannekoek). *Let $V_c \subset \mathbb{P}^3_{\mathbb{Q}}$ be the quartic surface given by the equation*

$$x_0^4 + cx_1^4 = x_2^4 + cx_3^4,$$

where $c = 2, 4, 6, 10, 12, 14, 18, 20, 22$ (and some other rational numbers, e.g. $c = 2/3$) the set $V_c(\mathbb{Q})$ is dense in $V_c(\mathbb{Q}_2)$.

The proof is based on the observation that V_c has two pencils of genus 1 curves and exploits the group structure on the fibres of both pencils which contain rational points. These pencils are the inverse images of two families of lines on the quadric (isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$)

$$y_0^2 + cy_1^2 = y_2^2 + cy_3^2$$

under the morphism that squares each of the coordinates.

The best known result allows one to approximate local points by global points simultaneously at three different completions.

Theorem 3.5 (Pannekoek). *Let p and q be distinct primes not equal to 3. There exist (in fact, infinitely many non-isomorphic) Kummer K3 surfaces $X = \mathrm{Kum}(E \times E)$, where E is an elliptic curve over $\mathbb{Q}$, such that $X(\mathbb{Q})$ is dense in $X(\mathbb{R}) \times X(\mathbb{Q}_p) \times X(\mathbb{Q}_q)$.*

3.2. **Enriques surfaces and the étale Brauer–Manin obstruction.** A smooth projective surface Y is called an *Enriques surface* if there exists an unramified double cover $f : X \to Y$, where X is a K3 surface.

Here is a carefully constructed example. Let $X \subset \mathbb{P}^5_{\mathbb{Q}}$ be the complete intersection of the following quadrics:

$$\begin{aligned} x_0x_1 + 5x_2^2 &= y_0^2 \\ (x_0 + x_1)(x_0 + 2x_1) &= y_0^2 - 5y_1^2 \\ 12x_0^2 + 111x_1^2 + 13x_2^2 &= y_2^2. \end{aligned}$$

One checks that X is smooth, so it is a K3 surface. It is easy to see that the involution ι that changes the signs of x_0, x_1, x_2 and does not alter y_0, y_1, y_2 has no fixed points. This implies that $Y = X/\iota$ is an Enriques surface.

Theorem 3.6 (Balestrieri, Berg, Manes, Park, Viray). *We have $Y(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} \neq \emptyset$ but $Y(\mathbb{Q}) = \emptyset$.*

To prove that $Y(\mathbb{Q}) = \emptyset$ one has to use an obstruction which is finer than the Brauer–Manin obstruction. In this case it is the obstruction given by the Brauer–Manin conditions on *all* K3 covers of Y to which adelic points can be lifted. These covers are obtained as quadratic twists of each other, so let us write them as $f_a : X_a \to Y$, where $a \in \mathbb{Q}^*$. If $a/b \in \mathbb{Q}^{*2}$, then $X_a \simeq X_b$. Let us define the *étale* Brauer–Manin set as follows:

$$Y(\mathbb{A}_{\mathbb{Q}})^{\text{ét,Br}} = \bigcup_{a \in \mathbb{Q}^*} f_a(X_a(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}}).$$

The point is that the Brauer–Manin conditions on the K3 covers of Y give stronger constraints on rational points inside the adelic space of Y than the Brauer–Manin conditions on Y. In fact, in the above example one has $Y(\mathbb{A}_\mathbb{Q})^{\text{ét,Br}} = \emptyset$.

3.3. **Counterexamples to the Hasse principle not explained by the étale Brauer–Manin obstruction.** Sarnak and Wang constructed the first example of a variety over a number field for which no known obstruction explains the failure of the Hasse principle, though their example is conditional on the Bombieri–Lang conjecture. They exhibited a smooth hypersurface $Z \subset \mathbb{P}^6_\mathbb{Q}$ of degree 1130 with $\mathbb{Q}$-points such that $Z_\mathbb{C}$ is a hyperbolic complex manifold. The Bombieri–Lang conjecture then says that $Z(\mathbb{Q})$ is finite. From this, using the estimate of Lang–Weil, one easily deduces that infinitely many smooth fibres $X \subset \mathbb{P}^5_\mathbb{Q}$ of a sufficiently general pencil of hyperplane sections of Z are everywhere locally soluble, yet only finitely many contain rational points. On the other hand, the geometry of X is such that $\mathrm{Pic}(\overline{X}) \simeq \mathbb{Z}$ as a Galois module, $\mathrm{Br}(\overline{X}) = 0$ and the fundamental group of $\overline{X}$ is trivial. Thus $X(\mathbb{A}_\mathbb{Q})^{\text{ét,Br}} \neq \emptyset$ though $X(\mathbb{Q}) = \emptyset$.

We finish this lecture by descring a simple recent example due to Colliot-Thélène, Pál and the lecturer. It is based on a trick of Poonen.

Let us call a *quadric bundle* a surjective flat morphism $f : X \to B$ of smooth, projective, geometrically integral varieties over a field k, where B is a curve, the generic fibre of which is a smooth quadric of dimension at least 1, and all geometric fibres are reduced. We denote by $k(B)$ the function field of B, and by $X_{k(B)}$ the generic fibre of $f : X \to B$.

Lemma 3.7. *Let $f : X \to B$ be a quadric bundle over a field k of characteristic zero. If all the fibres of f are geometrically integral, then the natural map $\mathrm{Br}(B) \to \mathrm{Br}(X)$ is surjective.*

Lemma 3.8. *Let $f : X \to B$ be a quadric bundle over a field k of characteristic zero. Then any torsor $X' \to X$ of a finite k-group scheme G is the inverse image under f of a torsor $B' \to B$ of G.*

Proof. By our definition of quadric bundles, the morphism f is flat and all its geometric fibres are connected and reduced. The generic geometric fibre of f is simply connected. By SGA1, Cor. X.2.4, this implies that each geometric fibre of such a fibration is simply connected. The result then follows from SGA1, Cor. IX.6.8. □

Let k be a number field with a real place. We fix a real place v, so we can think of k as a subfield of $k_v = \mathbb{R}$.

Let C be a smooth, projective, geometrically integral curve over k such that $C(k)$ consists of just one point, $C(k) = \{P\}$. (By the work of Mazur–Rubin we can take C to be an elliptic curve over k.) Let $\Pi \subset C(\mathbb{R})$ be an open interval containing P. Let $f : C \to \mathbb{P}^1_k$ be a surjective morphism that is unramified at P. Choose a coordinate function t on $\mathbb{A}^1_k = \mathbb{P}^1_k \setminus f(P)$ such that f is unramified above $t = 0$. We have $f(P) = \infty$. Take any $a > 0$ in k such that a is an interior point of the interval $f(\Pi)$ and f is unramified above $t = a$.

Let w be a finite place of k. There exists a quadratic form $Q(x_0, x_1, x_2)$ of rank 3 that represents zero in all completions of k other than k_v and k_w, but not in k_v or k_w. We can assume that Q is positive definite over $k_v = \mathbb{R}$. Choose $n \in k$ with $n > 0$ in k_v and $-nQ(1,0,0) \in k_w^{*2}$. Let $Y_1 \subset \mathbb{P}^3_k \times \mathbb{A}^1_k$ be given by $Q(x_0, x_1, x_2) + nt(t-a)x_3^2 = 0$, and let $Y_2 \subset \mathbb{P}^3_k \times \mathbb{A}^1_k$ be given by $Q(X_0, X_1, X_2) + n(1-aT)X_3^2 = 0$. We glue Y_1 and Y_2 by identifying $T = t^{-1}$, $X_3 = tx_3$, and $X_i = x_i$ for $i = 0, 1, 2$. This produces a quadric bundle $Y \to \mathbb{P}^1_k$ with exactly two degenerate fibres (over $t = a$ and $t = 0$), each given by the quadratic form $Q(x_0, x_1, x_2)$ of rank 3. Define $X = Y \times_{\mathbb{P}^1_k} C$. This is a quadric bundle $X \to C$ with geometrically integral fibres.

Proposition 3.9 (Colliot-Thélène–Pál–S.)**.** *In the above notation we have* $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ *and* $X(k) = \emptyset$*. Moreover,* $X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}} \neq \emptyset$*.*

In all known unconditional counterexamples to the Hasse principle which cannot be explained by the Brauer–Manin obstruction the variety has non-trivial geometric fundamental group. It would be very interesting to construct such a counterexample on a geometrically simply connected variety, like in the example of Sarnak and Wang.

3.4. **Exercises.** 1. Prove Lemma 3.7.

2. Prove Proposition 3.9 in the following steps.

(a) Show that for $k = \mathbb{Q}$ one can take $Q(x_0, x_1, x_2) = x_0^2 + x_1^2 + x_2^2$, $k_w = \mathbb{Q}_2$ and $n = 7$.

(b) Show that $X(k) = \emptyset$.

(c) Show that $X(\mathbf{A}_k) \neq \emptyset$.

(d) Show that $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$. (*Hint*: Use Lemma 3.7 and the fact that the value of an element of $\mathrm{Br}(X)$ at a real point is a continuous function with values in the discrete set $\{0, 1/2\}$, hence a locally constant function.)

(e)† Show that $X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}} \neq \emptyset$. (*Hint*: Use Lemma 3.8.)

4. Appendix

4.1. **Quaternion algebras.** Let k be a field of characteristic not equal to 2.

To elements $a, b \in k^*$ one can attach a non-commutative k-algebra (a ring containing k). The *quaternion algebra* $Q(a, b)$ is defined as the 4-dimensional vector space over k with basis 1, i, j, ij and the multiplication table $i^2 = a$, $j^2 = b$, $ij = -ji$.

Example. If $k = \mathbb{R}$ and $a = b = -1$ we obtain Hamilton's quaternions $\mathbb{H}$. This is a division algebra: the set of units coincides with the set of non-zero elements.

Is the same true for $Q(a, b)$? Define the conjugation and the norm, in the usual way.

Define a pure quaternion as an element q such that $q \notin k$ but $q^2 \in k$. It follows that pure quaternions are exactly the elements of the form $yi + zj + wij$ (just square $x + yi + zj + wij$, then there are some cancellations, and if $x \neq 0$, then $y = z = w = 0$). This gives an intrinsic definition of the conjugation and the norm because any quaternion z is uniquely written as the sum of a pure quaternion and a scalar.

Exercise. If q is a pure quaternion such that q^2 is not a square in k, then $1, q$ span a quadratic field which is a maximal subfield of Q.

Lemma 4.1. *If $c \in k^*$ is a norm from $k(\sqrt{a})^*$, then $Q(a,b) \cong Q(a,bc)$.*

Proof. Write $c = x^2 - ay^2$, then set $J = xj + yij$. Then J is a pure quaternion, so $Ji = -iJ$ and $J^2 = -N(J) = bc$. One checks that $1, i, J, iJ$ is a basis, so we are done. □

When a quaternion algebra is a division algebra? Since $N(q) = q\bar{q}$, if q is a unit, then $N(q) \in k^*$. If $N(q) = 0$, then $q\bar{q} = 0$, so q is a zero divisor. Thus the units are exactly the elements with non-zero norm. The norm on $Q(a,b)$ is the diagonal quadratic form $\langle 1, -a, -b, ab\rangle$, and this leads us to the following criterion.

Proposition 4.2. *Let $a, b \in k^*$. Then the following statements are equivalent:*
(i) *$Q(a,b)$ is not a division algebra;*
(ii) *$Q(a,b)$ is isomorphic to the matrix algebra $M_2(k)$;*
(iii) *the diagonal quadratic form $\langle 1, -a, -b\rangle$ represents zero;*
(iv) *the diagonal quadratic form $\langle 1, -a, -b, ab\rangle$ represents zero;*
(v) *b is in the image of the norm homomorphism $k(\sqrt{a})^* \to k^*$.*

Proof. The equivalence of all of these is clear when $a \in k^{*2}$. Indeed, to prove the equivalence with (ii) we can assume that $a = 1$. The matrix algebra is spanned by

$$1 = Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}, \quad ij = \begin{pmatrix} 0 & b \\ -1 & 0 \end{pmatrix},$$

and so is isomorphic to $Q(1,b)$.

Now assume that a is not a square. Then (i) is equivalent to (iv) since $N(q) = q\bar{q}$. (iv) implies (v) because the ratio of two non-zero norms is a norm. (v) implies (iii) which implies (iv). So (iii), (iv) and (v) are equivalent (i). The previous lemma shows that under the assumption of (v) the algebra $Q(a,b)$ is isomorphic to $Q(a,a^2) = Q(a,1)$, so we use the result of the beginning of the proof. □

If the conditions of this theorem are satisfied one says that $Q(a,b)$ is *split*. If K is an extension of k such that $Q(a,b) \otimes_k K$ is split, then one says that K *splits* $Q(a,b)$.

We see that the quaternion algebra $Q(a,b)$, where $a, b \in k^*$ is a *form* of the 2×2-matrix algebra, which means that $Q(a,b) \otimes_k \bar{k} \cong M_2(\bar{k})$. For example, $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$.

Proposition 4.3. *Any quaternion algebra Q split by $k(\sqrt{a})$ contains this field and can be written as $Q = (a,c)$ for some $c \in k^*$. Conversely, if Q contains $k(\sqrt{a})$, then Q is split by $k(\sqrt{a})$.*

Proof. If the algebra Q is split, take $c = 1$. Assume it is not. Then $N(q_0 + q_1\sqrt{a}) = 0$ for some non-zero $q_0, q_1 \in Q$. Hence

$$N(q_0) + aN(q_1) + 2\sqrt{a}B(q_0, q_1) = 0,$$

where B is the bilinear form associated to the quadratic form N. This implies that $N(q_0) + aN(q_1) = 0$ and $2B(q_0, q_1) = q_0\bar{q}_1 + q_1\bar{q}_0 = 0$. Set $I = q_0/q_1$. We have $\bar{I} = -I$, hence I is a pure quaternion. Therefore, $I^2 = -N(I) = a$. The conjugation by I has order exactly 2 since $I \notin k$ (i.e. I is not in the centre of Q). Hence the -1-eigenspace is non-zero, so we can find $J \in Q$ such that $IJ + JI = 0$. One then checks that $1, I, J, IJ$ is a basis, hence $Q = (a, c)$, where $c = J^2$.

The converse follows from the fact that $k(\sqrt{a}) \otimes k(\sqrt{a})$ contains zero devisors (the norm form $x^2 - ay^2$ represents zero over $k(\sqrt{a})$). Hence the same is true for $Q \otimes k(\sqrt{a})$. □

Corollary 4.4. *The quadratic fields that split Q are exactly the quadratic subfields of Q.*

4.2. **Conics.** Define *the conic attached to the quaternion algebra $Q(a, b)$* as the plane algebraic curve $C(a, b) \subset \mathbb{P}^2_k$ (a closed subset of the projective plane) given by the equation

$$ax^2 + by^2 = z^2.$$

It has a k-point if and only if $Q(a, b)$ is split. An intrinsic definition is this: $C(a, b)$ is the conic

$$-ax^2 - by^2 + abz^2 = 0,$$

which is just the expression for the norm of pure quaternions.

Facts about conics. 1. Since the characteristic of k is not 2, every conic can be given by a diagonal quadratic form, and so is attached to some quaternion algebra.

2. The projective line is isomorphic to a conic $xz - y^2 = 0$ via map $(X : Y) \mapsto (X^2 : XY : Y^2)$.

3. If a conic C has a k-point, then $C \cong \mathbb{P}^1_k$. (The projection from a k-point gives rise to a rational parameterisation of C, which is a bijection.)

4. Thus the function field $k(C)$ is a purely transcendental extension of k if and only if C has a k-point.

Exercise. 1. Check that $Q(a, 1 - a)$ and $Q(a, -a)$ are split.

2. Check that if $k = \mathbb{F}_q$ is a finite field, then all quaternion algebras are split. (If $char(k)$ is not 2, write $ax^2 = 1 - by^2$ and use a counting argument for x and y to prove the existence of a solution in $\mathbb{F}_q$.)

3. $Q(a, b)$ is split over k if and only if $Q(a, b) \otimes_k k(t)$ is split over $k(t)$. (Take a $k(t)$-point on $C(a, b)$ represented by three polynomials not all divisible by t, and reduce modulo t.)

4. $Q(a, b)$ is split over $k(C(a, b))$. (Consider the generic point of the conic.)

Theorem 4.5 (Tsen)**.** *If k is algebraically closed, then every quaternion algebra over $k(t)$ is split.*

Proof. We only prove that every quaternion algebra over $k(t)$ is split. For this it is enough to show that any conic over $k(t)$ has a point. We can assume that the coefficients of the corresponding quadratic form are polynomials of degree at most m. We look for a solution (X, Y, Z) where X, Y and Z are polynomials in t (not all of them zero) of degree n for some large integer n. The coefficients of these polynomials can be thought of as points of $\mathbb{P}^{3n+2}$. The solutions bijectively correspond to the points of a closed subset of $\mathbb{P}^{3n+2}$ given by $2n+m+1$ homogeneous quadratic equations. Since k is algebraically closed this set is non-empty when $3n+2 \geq 2n+m+1$, by a standard result from algebraic geometry. (If an irreducible variety X is not contained in a hypersurface H, then $\dim(X \cap H) = \dim(X) - 1$. This implies that on intersecting X with r hypersurfaces the dimension drops at most by r, see [12, Ch. 1]). $\square$

Theorem 4.6 (Witt)**.** *Two quaternions algebras are isomorphic if and only if the conics attached to them are isomorphic.*

Proof. Since C_Q is defined intrinsically in terms of Q, it remains to prove that if $C_Q \cong C_{Q'}$ then $Q \cong Q'$. If Q is split, then $C_Q \cong \mathbb{P}^1_k$, hence $C_{Q'} \cong \mathbb{P}^1_k$. Thus Q' is split by the field of functions $k(\mathbb{P}^1_k) \cong k(t)$. Then Q' is split by Exercise 3 above.

Now assume that neither algebra is split. Write $Q = Q(a, b)$ so that $C_Q = C(a, b)$. The conic $C_Q \cong C_{Q'}$ has a $k(\sqrt{a})$-point, hence Q' is split by $k(\sqrt{a})$. By Proposition 4.3 we can write $Q' = Q(a, c)$ for some $c \in k^*$. By Exercise 4 above Q' is split by the function field $k(C_{Q'}) \cong k(C(a, b))$. By Proposition 4.2 this implies that c is contained in the image of the norm map

$$c \in \mathrm{Im}[k(C(a,b))(\sqrt{a})^* \longrightarrow k(C(a,b))^*].$$

Let $\sigma \in \mathrm{Gal}(k(\sqrt{a})/k) \cong \mathbb{Z}/2$ be the generator. Then we can write $c = f\sigma(f)$, where f is a rational function on the conic $C(a,b) \times_k k(\sqrt{a})$. One can replace f with $f\sigma(g)g^{-1}$ for any $g \in k(C(a,b))(\sqrt{a})^*$ without changing c. Our aim is to show that c is a product of a norm from $k(\sqrt{a})^*$ and a power of b. The power of b is odd because Q' is not split over k, so this is enough to prove the theorem.

The group Div of divisors on $C(a,b) \times_k k(\sqrt{a}) \cong \mathbb{P}^1_{k(\sqrt{a})}$ is freely generated by the closed points of $C(a,b) \times_k k(\sqrt{a})$. This is a module of $\mathbb{Z}/2 = \langle \sigma \rangle$ with a σ-stable basis. The divisors of functions are exactly the divisors of degree 0. The divisor $D = \mathrm{div}(f)$ is an element of Div satisfying $(1+\sigma)D = 0$. Hence there is $G \in Div$ such that $D = (1-\sigma)G$. Let $P = (1 : 0 : \sqrt{a})$. If $n = \deg(G)$ the divisor $G - nP \in Div$ has degree 0, and thus $G - nP = \mathrm{div}(g)$ for some $g \in k(\sqrt{a})(C(a,b))^*$. We have

$$\mathrm{div}(f\sigma(g)g^{-1}) = D + \sigma G - G + n(P - \sigma P) = n(P - \sigma P) = n \ \mathrm{div}\left(\frac{z - \sqrt{a}x}{y}\right).$$

It follows that

$$f\sigma(g)g^{-1} = e\left(\frac{z - \sqrt{a}x}{y}\right)^n,$$

where $e \in k(\sqrt{a})^*$. Thus

$$c = f\sigma(f) = N(e)\left(\frac{z^2 - ax^2}{y^2}\right)^n = N(e)b^n. \qquad \square$$

4.3. **Central simple algebras and the Brauer group.** A k-algebra A is called a *central simple algebra* if and only if A is a form of a matrix algebra, that is, $A \otimes_k \bar{k} \cong M_n(\bar{k})$ for some positive integer n. Equivalently, the centre of A is k (A is "central") and A has no non-trivial two-sided ideals (A is "simple").

Recall that if V and W are vector spaces over k, then $V \otimes_k W$ is the linear span of vectors $v \otimes w$, $v \in V$, $w \in W$, subject to the axioms

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \quad v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2,$$

and

$$c(v \otimes w) = (cv) \otimes w = v \otimes (cw) \quad \text{for any} \quad c \in k.$$

This turns $V \otimes_k W$ into a k-vector space. Note that $(V \otimes U) \otimes W$ is canonically isomorphic to $V \otimes (U \otimes W)$.

If (e_i) is a basis of V, and (f_j) is a basis of W, then $(e_i \otimes f_j)$ is a basis of $V \otimes_k W$. Now, if V and W are k-algebras, then $V \otimes_k W$ is a k-algebra with multiplication $(x \otimes y) \cdot (x' \otimes y') = (xx') \otimes (yy')$.

Properties. 1. $M_n(k)$ is a c.s.a.

2. $M_m(k) \otimes_k M_n(k) \cong M_{mn}(k)$. Hence the set of c.s.a. is closed under $\otimes$.

3. $Q(a,b) \otimes_k Q(a,b') \cong Q(a,bb') \otimes_k M_2(k)$. (Proof: The span of $1 \otimes 1$, $i \otimes 1$, $j \otimes j'$, $ij \otimes j'$ is $A_1 = Q(a,bb')$. The span of $1 \otimes 1$, $1 \otimes j'$, $i \otimes i'j'$, $-b(i \otimes i')$ is $A_2 = Q(b', -a^2b') \cong M_2(k)$. The canonical map $A_1 \otimes_k A_2 \to Q(a,b) \otimes_k Q(a,b')$ defined by the product, is surjective. The kernel of a homomorphism is a two-sided ideal, hence it is zero so that this map is an isomorphism.)

4. $Q(a,b) \otimes_k Q(a,b) \cong M_4(k)$. (This follows from parts 2 and 3.)

Two c.s.a. A and B are *equivalent* if there are n and m such that $A \otimes_k M_n(k) \cong B \otimes_k M_m(k)$. The relation is transitive by Property 2. The equivalence class of k consists of the matrix algebras of all sizes.

Theorem 4.7. *The tensor product turns the set of equivalence classes of c.s.a. into an abelian group, called the Braeur group* $\mathrm{Br}(k)$.

Proof. The neutral element is the class of k and the inverse element of A is the equivalence class of the *opposite* algebra A°. Indeed, $A \otimes_k A^\circ$ is a c.s.a., and there is a non-zero homomorphism $A \otimes_k A^\circ \to \mathrm{End}_k(A)$ that sends $a \otimes b$ to $x \mapsto axb$. It is injective since a c.s.a. has no two-sided ideals, and hence is an isomorphism by the dimension count. $\square$

We denote by $(a,b) \in \mathrm{Br}(k)$ the class of the quaternion algebra $Q(a,b)$.

We write the group operation in $\mathrm{Br}(k)$ additively. By Property 4 we have $(a,b) \in \mathrm{Br}(k)[2]$ and $(a,b) + (a,b') = (a,bb')$. We also have $(a,-a) = (a,1-a) = 0$ for any $a,b,b' \in k$ for which these symbols are defined.

Examples. $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2$ and $\mathrm{Br}(\mathbb{F}_q) = 0$. (proofs are omitted) The full version of Tsen's theorem states (with a similar proof) that if k is algebraically closed, then $\mathrm{Br}(k(t)) = 0$.

Wedderburn's theorem *For any c.s.a. A there is a unique division algebra D such that $A \cong D \otimes_k M_n(k) = M_n(D)$.*

Skolem–Noether theorem *Let B be a simple algebra and let A be a c.s.a. Then all non-zero homomorphisms $B \to A$ can be obtained from one another by a conjugation in A.*

This generalises the fact that any automorphism of $M_n(k)$ is inner.

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