CIRCLE METHOD: PROBLEMS

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1. Let X and Y be large real numbers, and for the sake of concreteness suppose that $X^{1/2} \leqslant Y \leqslant X^2$. Also, let $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\beta \in \mathbb{R}$. Write

$$g(\alpha) = \sum_{1 \leqslant x \leqslant X} \sum_{1 \leqslant y \leqslant Y} e(\alpha x y^2).$$

(a) By applying the relation

$$g(\beta + a/q) = \sum_{r=1}^{q} \sum_{\substack{1 \leqslant x \leqslant X \\ x \equiv r \pmod{q}}} \sum_{s=1}^{q} \sum_{\substack{1 \leqslant y \leqslant Y \\ y \equiv s \pmod{q}}} e\left((\beta + a/q)\, x y^2 \right),$$

deduce that

$$g(\beta + a/q) = \sum_{r=1}^{q} \sum_{s=1}^{q} e(ars^2/q) V(\beta; r, s),$$

where

$$V(\beta; r, s) = \sum_{(1-r)/q \leqslant u \leqslant (X-r)/q} \sum_{(1-s)/q \leqslant v \leqslant (Y-s)/q} e\left(\beta (qu + r)(qv + s)^2 \right).$$

(b) Hence, by applying the mean value theorem, deduce that

$$g(\beta + a/q) = q^{-2} S(q, a) w(\beta) + O\left(q(X + Y)(1 + |\beta| X Y^2) \right),$$

where

$$S(q, a) = \sum_{r=1}^{q} \sum_{s=1}^{q} e(ars^2/q) \quad \text{and} \quad w(\beta) = \int_1^X \int_1^Y e(\beta \theta \phi^2) \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

(c) Let $\delta = 1/1000$ and put $Q = X^\delta$. When $0 \leqslant a \leqslant q \leqslant Q$ and $(a, q) = 1$, put $\mathfrak{M}(q, a) = \{\alpha \in [0, 1) : |\alpha - a/q| \leqslant Q(XY^2)^{-1}\}$. Define $\mathfrak{M}$ to be the union of these intervals, and put $\mathfrak{m} = [0, 1) \setminus \mathfrak{M}$. Show that for each positive integer t, and for any $c_1, \ldots, c_t \in \mathbb{Z}$, one has

$$\int_{\mathfrak{M}} g(c_1 \alpha) \ldots g(c_t \alpha) \, \mathrm{d}\alpha = \mathfrak{S}(Q) \mathfrak{J}(Q) + o(X^{t-1} Y^{t-2}),$$

where

$$\mathfrak{S}(Q) = \sum_{1 \leqslant q \leqslant Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-2t} S(q, c_1 a) \ldots S(q, c_t a)$$

and

$$\mathfrak{J}(Q) = \int_{-QX^{-1}Y^{-2}}^{QX^{-1}Y^{-2}} w(c_1 \beta) \ldots w(c_t \beta) \, \mathrm{d}\beta.$$

2. Recall that

$$g(\alpha)=\sum_{1\leqslant x\leqslant X}\sum_{1\leqslant y\leqslant Y}e(\alpha xy^2).$$

(a) By imitating the argument underlying the proof of Weyl's inequality, but making use of Cauchy's inequality in the shape

$$|g(\alpha)|^2\leqslant X\sum_{1\leqslant x\leqslant X}\Bigg|\sum_{1\leqslant y\leqslant Y}e(\alpha xy^2)\Bigg|^2,$$

show that whenever $\alpha\in\mathbb{R}$, and $r\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfy $(r,q)=1$ and $|\alpha-r/q|\leqslant q^{-2}$, then one has

$$|g(\alpha)|\ll(XY)^{1+\varepsilon}(q^{-1}+X^{-1}+Y^{-1}+q(XY^2)^{-1})^{1/2}.$$

(b) (i) Apply divisor function estimates to show that when $n\neq 0$, the number of solutions of the Diophantine equation $xy^2=n$ is $O(n^\varepsilon)$. Hence prove that

$$\int_0^1|g(\alpha)|^2\,\mathrm{d}\alpha\ll(XY)^{1+\varepsilon}.$$

(ii) By imitating the argument underlying the proof of Hua's lemma, but making use of Cauchy's inequality again in the above shape, show that

$$\int_0^1|g(\alpha)|^4\,\mathrm{d}\alpha\ll(X^3Y^2)^{1+\varepsilon}.$$

(c) Define $\mathfrak{m}$ as in question 1, and suppose that $t\geqslant 5$. Establish that for any non-zero integers $c_1,\dots,c_t$, one has

$$\int_{\mathfrak{m}}g(c_1\alpha)\dots g(c_t\alpha)\,\mathrm{d}\alpha=o(X^{t-1}Y^{t-2}).$$

Conclude that

$$\int_0^1g(c_1\alpha)\dots g(c_t\alpha)\,\mathrm{d}\alpha=\mathfrak{S}(Q)\mathfrak{J}(Q)+o(X^{t-1}Y^{t-1}).$$

3. Let
$$f(\alpha)=\sum_{1\leqslant x\leqslant P} e(\alpha x^k).$$
Recall **Weyl's inequality**. Thus, when $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfy $(a,q)=1$ and $|\beta-a/q|\leqslant q^{-2}$, one has
$$f(\beta)\ll P^{1+\varepsilon}(q^{-1}+P^{-1}+qP^{-k})^{2^{1-k}}.$$

Carry out the following procedure to establish that, whenever $\alpha\in\mathbb{R}$, $b\in\mathbb{Z}$ and $r\in\mathbb{N}$ satisfy $(b,r)=1$, then one has
$$f(\alpha)\ll P^{1+\varepsilon}\left(\frac{r+P^k|r\alpha-b|}{P^k}+P^{-1}+\frac{1}{r+P^k|r\alpha-b|}\right)^{2^{1-k}}.$$

(i) Apply Dirichlet's theorem on Diophantine approximation to show that there exist $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ with $1\leqslant q\leqslant 2r$ and $|q\alpha-a|\leqslant (2r)^{-1}$. By considering $|a/q-b/r|$, show that either $a/q=b/r$, or else $q^{-1}\leqslant 2|r\alpha-b|$.

(ii) When $a/q\neq b/r$, obtain the desired conclusion from Weyl's inequality.

Assume in the remaining parts of this problem that $a/q=b/r$.

(iii) Show that when $\alpha=b/r$, then the desired conclusion follows from Weyl's inequality. Assume henceforth that $\alpha\neq b/r$.

(iv) Use Dirichlet's theorem to find $c\in\mathbb{Z}$ and $s\in\mathbb{N}$ with
$$(c,s)=1,\quad s\leqslant 2|r\alpha-b|^{-1}\quad\text{and}\quad |s\alpha-c|\leqslant \tfrac{1}{2}|r\alpha-b|.$$
Show that $c/s\neq b/r$ by comparing $|\alpha-b/r|$ with $|\alpha-c/s|$.

(v) By considering $|b/r-c/s|$, show that $(2s)^{-1}\leqslant |r\alpha-b|$.

(vi) Use Weyl's inequality with the rational approximation c/s to α, and establish the claimed result.

3†. (Supplement) (a) Define
$$g(\alpha)=\sum_{1\leqslant x\leqslant X}\sum_{1\leqslant y\leqslant Y} e(\alpha xy^2).$$
Show that whenever $\alpha\in\mathbb{R}$, $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfy $(a,q)=1$, then one has
$$|g(\alpha)|\ll (XY)^{1+\varepsilon}\left(\frac{q+XY^2|q\alpha-a|}{XY^2}+X^{-1}+Y^{-1}+\frac{1}{q+XY^2|q\alpha-a|}\right)^{1/2}.$$

(b) Interpret $\lim_{Q\to\infty}\mathfrak{S}(Q)$ and $\lim_{Q\to\infty}\mathfrak{J}(Q)$ in terms of real and p-adic densities of solutions associated with the equation
$$c_1x_1y_1^2+\ldots+c_tx_ty_t^2=0,$$
first proving convergence of these limits. Hence obtain an asymptotic formula for the number of solutions of this equation with $1\leqslant x_i\leqslant X$ and $1\leqslant y_i\leqslant Y$ $(1\leqslant i\leqslant t)$.