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<u>Big picture</u> [$\geq$ Fermat, Langlands, BSD, ...]

$$
\begin{tikzcd}
\text{L-functions} & \text{Galois representations} \arrow[l] & \text{Varieties} \arrow[l] \\
\text{Modular and automorphic forms} \arrow[u] & &
\end{tikzcd}
$$

Homework to
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Plan:

Riemann ζ $\leftsquigarrow$ $\mathbb{Z}$ $\leftsquigarrow$ $\bullet$

Dedekind ζ $\zeta_K(s)$ $\leftsquigarrow$ permutation reps of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ $\leftsquigarrow$ $\circlearrowright \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

Artin L-functions $\leftsquigarrow$ pieces of $\uparrow$, all finite image reps.

$$\frac{\dim = 0}{\text{varieties}}$$

L-functions of varieties $\leftsquigarrow$ general Galois reps $\leftsquigarrow$ varieties ($\dim > 0$)

§1 Riemann ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \dots\right) \times \left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \dots\right) \times \dots$$

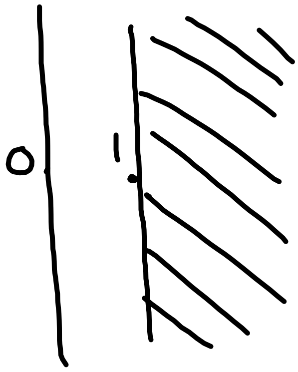
$$= \prod_p \frac{1}{1 - p^{-s}}$$

Encodes distribution of primes, e.g.

$$\sum \frac{1}{n} = \infty \Rightarrow \exists \infty \text{ primes.}$$

Viewed as fnc of a $\mathbb{C}$-variable $s = \sigma + it$

$$\left|\frac{1}{n^s}\right| = \frac{1}{n^\sigma}$$



<u>Thm</u> $\zeta(s)$ has meromorphic cont. to $\mathbb{C}$, only pole (simple) at $s=1$

and the completed ζ-function

$$\hat{\zeta}(s) = \frac{1}{\pi^{s/2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$ satisfies the functional equation

ξ or Λ

$$\hat{\zeta}(1-s) = \hat{\zeta}(s).$$

Proof Poisson summation formula:

$$f(n) : \mathbb{R} \longrightarrow \mathbb{C} \qquad \left(C^2, \ |f + f''| = O\left(\frac{1}{|n|^r + 1}\right) \text{ some } r > 1\right)$$

$$(\mathcal{F}f)(m) = \int_{-\infty}^{\infty} e^{2\pi i n m}$$

Fourier transform.

Then $$\sum_{n \in \mathbb{Z}} f(n) = \sum_{m \in \mathbb{Z}} (\mathcal{F}f)(m).$$

Apply to $f(n) = e^{-\pi x n^2}$

Jacobi Θ.

$$\Theta(x) = \sum_{n \in \mathbb{Z}} \underbrace{e^{-\pi x n^2}}_{f(n)} \overset{\text{Poisson}}{=} \sum_{m \in \mathbb{Z}} \underbrace{\frac{1}{\sqrt{x}} e^{-\frac{\pi}{x} m^2}}_{\mathcal{F}f(m)}$$

$$= \frac{1}{\sqrt{x}} \Theta\left(\frac{1}{x}\right). \quad (*)$$

Back to ζ:

$$\Gamma(s) = \int_0^\infty x^s e^{-x} \frac{dx}{x}$$

Mellin transform of e^{-x}.

$\Gamma(s+1) = s\Gamma(s)$

$$\hat{\zeta}(2s) = \frac{1}{\pi^s}\Gamma(s)\underbrace{\zeta(2s)}_{\sum_n \frac{1}{n^{2s}}} =$$

$$= \int_0^\infty \sum_{n=1}^\infty \underbrace{\frac{x^s}{\pi^s n^{2s}}}_{\left(\frac{x}{\pi n^2}\right)^s} e^{-x}\frac{dx}{x} \overset{x \to x\pi n^2}{=}$$

$$= \text{Mellin tranform of } \sum_{n=1}^\infty e^{-\pi n^2 x}$$

$$\frac{\Theta(x)-1}{2}$$

Break $\int_0^\infty = \int_0^1 + \int_1^\infty$, replace $x \mapsto \frac{1}{x}$ in 1^{st} one

using (*)

$$\hat{\zeta}(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \frac{\Theta(x)-1}{2}\left(x^{s/2} + x^{\frac{1-s}{2}}\right)\frac{dx}{x}$$

symmetric under $s \longleftrightarrow 1-s$, converges everywhere.

Def An L-function is Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \qquad a_n \in \mathbb{C}, \quad a_n = O(n^r) \text{ some } r$$

$(\Rightarrow$ converges on $\operatorname{Re} s > r+1)$

It has an Euler product and has degree d

$$L(s) = \prod_p \frac{1}{\underbrace{F_p(p^{-s})}_{\text{local factor}}}$$

$F_p(t) \in \mathbb{C}[t]$ if degree $\leq d$, $= d$ for a.a. p.

<u>Ex</u> $\zeta(s)$ has Euler product & degree 1

(all $F_p(T) = 1 - T$)

local polynomials.

All L-fncs we will see will satisfy this, and are conjectured to

(A) Have meromorphic cont. to $\mathbb{C}$ with fin. many poles (usually none)

(B) Functional equation: $\exists$ weight k,

sign w, conductor N,

Γ-factor $\gamma(s) = \Gamma\left(\frac{s+\lambda_1}{2}\right) \cdots \Gamma\left(\frac{s+\lambda_d}{2}\right)$

such that

$$\hat{L}(s) = \left(\frac{N}{\pi^d}\right)^{s/2} \gamma(s) L(s)$$

satisfies $\hat{L}(s) = w \cdot \hat{\bar{L}}(k-s)$

$\bar{L}(s) = \sum_{n \geq 1} \frac{\overline{a_n}}{n^s}$.

(c) Riemann Hypothesis: all non-trivial zeros have $\mathrm{Re}\, s = \frac{k}{2}$.

<u>Rmks</u> • If $L(s)$ satisfies (A) + (B) [say, no poles], then as before

$$\hat{L}(s) = \int_1^\infty \left(x^{s/2} + w \cdot x^{\frac{k-s}{2}}\right) \Theta(\sqrt{N} \cdot x) \quad ;$$

$$\Theta(x) = \sum_{n=1}^\infty a_n \, \phi_{n,\gamma}(x)$$

depends only on $\gamma(s)$, decays exp. with n, e.g. $e^{-\pi n^2 x}$ for $\gamma = \Gamma(\frac{s}{2})$.

In fact,

(B) $\iff$, $$\Theta\left(\frac{1}{Nx}\right) = w \cdot \overline{\Theta}(x) \quad (**)$$

Gives a way to compute L-functions numerically (with $\sim\sqrt{N}$ terms)

"measure of arithmetic complexity of an L-function"

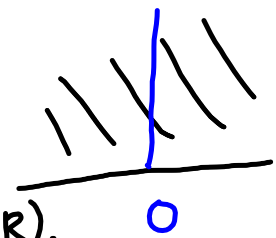
- There are fncs called "modular forms" f

  [technically, newforms of weight k, level N, w-eigenform for the Atkin-Lehner involution $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$]

  $$f : \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \} \longrightarrow \mathbb{C}$$

  such that $\Theta(x) = f(ix)$ satisfies (**) by definition.

  $\Rightarrow$ their L-fncs satisfy (A) + (B).



- 2 categories of L-fncs $L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$

(i) With a direct formula for the a_n

[e.g. $\zeta(s)$ $a_n = 1$

$L(\chi, s)$ Dirichlet characters $a_n = \chi(n)$

$\zeta_K(s)$ Dedekind, a_n = # ideals of ideals of norm n in $\mathcal{O}_K$].

$\Rightarrow$ Generally know how to prove (A) + (B).].

(ii) Only defined by an Euler product

[e.g. $L(\rho, s)$ Artin

$L(E, s)$ ell. curves

other varieties ...]

$\Rightarrow$ Never know how to prove (A) + (B), except by reducing to (i). !

§2 Dedekind ζ-functions

K number field, $[K:\mathbb{Q}] = d$

$K \cong \mathbb{Q}^d$ as a $\mathbb{Q}$-vector space

$\mathcal{O} = \mathcal{O}_K$ ring of integers, $\mathcal{O}_K \cong \mathbb{Z}^d$ as ab. group

$I \subseteq \mathcal{O}_K$ ideal, $\neq 0$

$NI = (\mathcal{O}_K : I)$, $< \infty$.

norm of an ideal

$$N(IJ) = NI \cdot NJ$$

I = unique product of prime ideals

$$I = \prod_{i=1}^{r} \mathfrak{p}_i^{n_i}$$

$\mathcal{O}/\mathfrak{p}_i$ finite integral domain $\Rightarrow$ field $\mathbb{F}_{p^r}$

$\Rightarrow \mathfrak{p}_i \subseteq (p_i)$ some primes $p_i \in \mathbb{Z}$.

In particular, take $I = (p)$ $p \in \mathbb{Z}$

$$(p) = \prod_{i=1}^{r} \mathfrak{P}_i^{e_i}$$

primes above p

e_i = ramification indices

($=1$ for a.a. p, namely $p \nmid \Delta_K$

called unramified primes p

$$f_i = \left[\mathcal{O}/\mathfrak{P}_i : \mathbb{F}_p \right]$$

residue degrees

$$[\mathcal{O}/\mathfrak{P}_i \cong \mathbb{F}_{p^f}).$$

$$N(\mathfrak{p}) = (\mathcal{O} : p\mathcal{O}) = p^d \implies$$
$$\mathbb{Z}^d \quad p\cdot\mathbb{Z}^d$$

$$d = \sum_{i=1}^{r} e_i f_i$$

[$\implies d = \sum f_i$ for unram. primes]

N.B. If $K/\mathbb{Q}$ is Galois, then $e_1 = \ldots = e_r$, $f_1 = \ldots = f_r$ ($\mathrm{Gal}(K/\mathbb{Q})$ permutes $\mathfrak{p}_i$ transitively)

$d = e \cdot f \cdot r.$

In practice,

Thm (Kummer-Dedekind) $K = \mathbb{Q}[x]/(g(x))$

$g(x) \in \mathbb{Z}(x)$, monic. Then $\Delta_K \mid \Delta_g$,

and for all $p \nmid \Delta_g$, we have $p = \prod_{i=1}^{r} \mathfrak{p}_i$

is unramified, and

$g(x) \equiv g_1 \cdots g_r \mod p$, $\deg g_i = f_i$.

<u>Def</u> The <u>Dedekind ζ-function</u> of K

$$\zeta_K(s) = \sum_{n\geq 1} \frac{a_n}{n^s} \qquad a_n = \{\#\text{ideals of norm } I \text{ in } \mathcal{O}_K\}$$

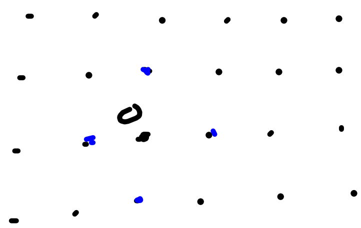
$$= \sum_{\substack{I \subseteq \mathcal{O}_K \text{ ideal} \\ I \neq 0}} \frac{1}{NI^s}$$

$$= \prod_{\substack{\mathfrak{p} \text{ prime} \\ \text{ideal} \neq 0}} \frac{1}{1 - N\mathfrak{p}^{-s}} \overset{\text{exc}}{=} \prod_{\substack{p \\ \text{prime of } \mathbb{Z}}} \frac{1}{F_p(p^{-s})}$$

$f_p \in \mathbb{Z}(x)$ degree d for $p \nmid \Delta_K$

$< d$ for $p \mid \Delta_K$.

degree d L-function.



Ex $K = \mathbb{Q}(i)$

$\mathcal{O} = \mathbb{Z}[i]$ Gaussian integers

$\mathcal{O}^\times = \{\pm 1, \pm i\}$ units

As for Riemann ζ,

$$\zeta_K(s) = \sum_{\substack{I \subseteq \mathbb{Z}[i] \\ I \neq 0}} \frac{1}{NI^s} \overset{\substack{\mathbb{Z}[i] \\ \text{PID}}}{=} \sum_{\substack{0 \neq \alpha \in \mathbb{Z}[i] \\ \text{mod } \mathbb{Z}[i]^\times}} \frac{1}{(\alpha\bar{\alpha})^s}$$

$$= \frac{1}{4} \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{(m^2+n^2)^s}$$

and same computation as before $\Rightarrow$

$$\frac{2^s}{\pi^s}\,\Gamma(s)\,\zeta_K(s) = \text{Mellin transform of } \frac{\Theta(x)-1}{4}$$

$$\Theta(x) = \sum_{m,n\in\mathbb{Z}} e^{-\pi(m^2+n^2)x}$$

$$= \sum_m e^{-\pi m^2 x} \sum_n e^{-\pi n^2 x} =$$

$$= \frac{1}{\sqrt{x}}\,\frac{1}{\sqrt{x}}\,\Theta\left(\frac{1}{x}\right).$$

$\Rightarrow$ fun. eq. for $\zeta_{\mathbb{Q}(i)}(s)$.

Poisson summation: $V = \mathbb{R}^d$, $f : V \longrightarrow \mathbb{C}$
decaying.

V^* dual vector space, $\mathcal{F}f : V^* \longrightarrow \mathbb{C}$

$$(\mathcal{F}f)(\underline{m}) = \int_V e^{-2\pi i \langle m, n \rangle} f(\underline{n})\, d\underline{n}.$$

$\Gamma \subseteq V$ rk d lattice $\Rightarrow$

$$\sum_{\underline{n} \in \Gamma} f(\underline{n}) = \frac{1}{\mathrm{vol}(V/\Gamma)} \sum_{\underline{m} \in \Gamma^*} (\mathcal{F}f)(\underline{m}).$$

Compare $\displaystyle\sum_{I \neq 0} \frac{1}{NI^s}$ to $\displaystyle\sum_{\substack{\alpha \in \mathcal{O} \\ \alpha \neq 0}} \frac{1}{N\alpha^s}$

involve $h = \#\{\text{ideals}/\text{principal ideals}\}$.

units, roots of unity

K number field, $[K:\mathbb{Q}] = d = r_1 + 2r_2$

$r_1 = \#$ real embeddings $K \hookrightarrow \mathbb{R}$

$r_2 = \#$ pairs of non-real embeddings $K \hookrightarrow \mathbb{C}$

$\mathcal{O} \hookrightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ $(\cong \mathbb{R}^d)$ lattice.

Poisson summation $\Rightarrow$

<u>Thm</u> $\zeta_K(s)$ meromorphic on $\mathbb{C}$,

simple pole at $s=1$,

residue at $s=1 = \dfrac{2^{r_1}(2\pi)^{r_2}\, h\, R}{\#\text{roots of unity in } K \times \sqrt{\Delta_K}}$

class number formula

satisfies fun. eq

$\hat{\zeta}_K(1-s) = \zeta_K(s)$.

h = class number

R = regulator (units).

MO 218759.

<u>Exc</u> If $[K:\mathbb{Q}]=n$, K Galois,

then $\underbrace{}_{}$ $\exists\infty$ primes that split completely in K

[i.e. have $e=f=1$],

[and have density $\frac{1}{n}$].